FOUR SQUARES FROM THREE NUMBERS

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ABSTRACT. We show that there are infinitely many triples of positive integers a, b, c (greater than 1) such that $ab+1$, $ac+1$, $bc+1$ and $abc+1$ are all perfect squares.

1. Introduction

A *Diophantine m-tuple* is a set of m distinct positive integers with the property that the product of any two of its distinct elements plus 1 is a perfect square. The first example of a Diophantine quadruple was found by Fermat, and it was the set $\{1, 3, 8, 120\}$. In 1969, Baker and Davenport [1] proved that Fermat's set cannot be extended to a Diophantine quintuple. There are infinitely many Diophantine quadruples. E.g., $\{k, k+2, 4k+4, 16k^3+48k^2+44k+12\}$ is a Diophantine quadruple for $k \geq 1$. In 2004, Dujella [3] proved that there is no Diophantine sextuple and that there are only finitely many Diophantine quintuples. Finally, in 2019, He, Togbé and Ziegler [6] proved that there is no Diophantine quintuple.

There are many known variants and generalizations of the notion of Diophantine m-tuples. For a survey of various generalizations and the corresponding references see Section 1.5 of the book [5].

Here we will consider a variant that was introduced in several internet forums[1], and appeared also in Section 14.5 of the book [2], where it is attributed to John Gowland. We will consider triples of positive integers a, b, c with the property that $ab + 1$, $ac + 1$, $bc + 1$ and $abc + 1$ are perfect squares. Thus, we are interested in Diophantine triples $\{a, b, c\}$ satisfying the additional property that $abc + 1$ is also a perfect square. If we allow that $a = 1$, then the problem degenerates from four conditions to only three conditions that $b + 1$, $c + 1$ and $bc + 1$ are perfect squares, or in other words that $\{1, b, c\}$ is a Diophantine triple. It is easy to see that there are infinitely many such triples, e.g. we may take $b = k^2 - 1$, $c = (k + 1)^2 - 1$ for any $k \geq 2$. Hence, we will require that a, b, c are positive integers greater than 1.

Several examples of such triples were given in Section 14.5 of [2], e.g., $(5, 7, 24)$, $(8, 45, 91)$, $(8, 105, 171)$, $(3, 133, 176)$, $(11, 105, 184)$, $(20, 84, 186)$, $(44, 102, 280)$, $(40, 119, 297)$, $(24, 301, 495)$, $(24, 477, 715)$. However, it remained an open question whether there exist infinitely many such triples. The main result of this paper gives an affirmative answer to that question.

Theorem 1. *There are infinitely many triples of positive integers a, b, c greater than 1 such that $ab + 1$, $ac + 1$, $bc + 1$ and $abc + 1$ are all perfect squares.*

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[1] `https://www.mathpages.com/home/kmath481.htm`,
`https://benvitalenum3ers.wordpress.com/2015/01/07/abc-ab1ac1bc1abc1-are-all-squares/`

2. The construction of infinitely many triples

We will search for the solutions within so-called *regular Diophantine triples*, i.e., triples $\{a,b,c\}$, such that $c = a+b+2r$, where $ab+1 = r^2$. Then $ac+1 = (a+r)^2$ and $bc+1 = (b+r)^2$, so $\{a,b,c\}$ is indeed a Diophantine triple. According to [4], most of Diophantine triples are of this form.

By studying and extending the list of known solutions, we can see that many of them have the property that a is of the form A^2+4:

$$
\begin{aligned}
&(8 = 2^2+4, 45, 91),\\
&(8 = 2^2+4, 105, 171),\\
&(20 = 4^2+4, 84, 186),\\
&(40 = 6^2+4, 119, 297),\\
&(40 = 6^2+4, 2387, 3045),\\
&(85 = 9^2+4, 672, 1235),\\
&(85 = 9^2+4, 11859, 13952),\\
&(533 = 23^2+4, 33475, 42456),\\
&(533 = 23^2+4, 509736, 543235),\\
&(1160 = 34^2+4, 165627, 194509),\\
&(1160 = 34^2+4, 2449135, 2556897),\\
&(7400 = 86^2+4, 7102165, 7568067),\\
&(7400 = 86^2+4, 101263737, 103002439),\\
&(16133 = 127^2+4, 34117191, 35617120),\\
&(16133 = 127^2+4, 482768440, 488366151).
\end{aligned}
$$

Almost all of these examples follow the following pattern: a is of the form $a = A_n^2+4$, where A_n is a (two-sided) binary recursive sequence defined by

$$A_0 = 1, \quad A_1 = 6, \quad A_{n+1} = 4A_n - A_{n-1}.$$

For $n \geq 1$, the elements of the sequence A_n are: $6, 23, 86, 321, \ldots$, while for $n \leq -1$, the elements of the sequence $-A_{-n}$ are: $2, 9, 34, 127, 474, \ldots$.

Next, we study the values of r (from $ab+1 = r^2$) in observed examples. For each a, we had two triples with given property. We will give details for the second (with larger b) triples. We notice that r's have the form $r = A_n^2 R_n + A_{n+1} - 2$, where

$$R_0 = 2, \quad R_1 = 8, \quad R_n = 4R_{n-1} - R_{n-2} + 1,$$

and again we may extend the recurrence to negative indices, so for $n \leq -1$, the elements of the sequence R_{-n} are: $1, 3, 12, 46, \ldots$. (In the smaller triples, we have $r = A_n^2 R_{n-1} - A_{n-1} - 2$.)

To simplify manipulations with the above introduced recursive sequences, will we express them in the terms of the sequence

$$P_0 = 0, \quad P_1 = 1, \quad P_n = 4P_{n-1} - P_{n-2}.$$

The sequence (P_n) satisfies

$$P_n = \frac{1}{2\sqrt{3}} \left((2+\sqrt{3})^n - (2-\sqrt{3})^n \right). \tag{1}$$

Let us denote $x = P_{n+1}$, $y = P_n$. Then we have $A_n = x+2y$ and $R_n = \frac{1}{2}(5x-3y-1)$.

From (1), it follows easily that

$$x^2 - 4xy + y^2 = 1. \tag{2}$$

We will use (2) to make further expressions as homogeneous as possible in order to simplify expressions and in particular to allow factorizations. In that way, we obtain

$$\begin{aligned}
a &= 5x^2 - 12xy + 8y^2,\\
r &= \frac{17}{2}x^3 - \frac{33}{2}x^2y - \frac{5}{2}x^2 + 14xy^2 + 6xy - 7y^3 - 4y^2,\\
b &= \frac{31}{2}x^4 - \frac{55}{2}x^3y + \frac{75}{2}x^2y^2 - 25xy^3 + 8y^4 - \frac{17}{2}x^3 + \frac{33}{2}x^2y - 14xy^2 + 7y^3,\\
c &= \frac{31}{2}x^4 - \frac{55}{2}x^3y + \frac{75}{2}x^2y^2 - 25xy^3 + 8y^4 + \frac{17}{2}x^3 - \frac{33}{2}x^2y + 14xy^2 - 7y^3.
\end{aligned}$$

In order to prove Theorem 1, it remains to check that $abc+1$ is a perfect square. First we get

$$\begin{aligned}
abc =& \frac{1}{4}(3y+8x)(5x^2-12xy+8y^2)(2y^3-2xy^2-2x^2y+3x^3)\\
&\times (10y^4 - 22xy^3 + 50x^2y^2 - 39x^3y + 28x^4),
\end{aligned}$$

and then by writing $1 = (x^2-4xy+y^2)^5$ in $abc+1$, we finally obtain

$$abc + 1 = \frac{1}{4}(22y^5 - 24xy^4 - 8x^2y^3 + 84x^3y^2 - 119x^4y + 58x^5)^2,$$

which shows that $abc+1$ is indeed a perfect square. $\square$

For example, by taking $n = 4$, we have $x = 209$, $y = 56$, and we get $a = 1435208$, $r = 2347998213$, $b = 3841321681771$, $c = 3846019113405$, and $abc+1 = 4604722693427179^2$.

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