There are only finitely many Diophantine quintuples

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Abstract

A set of m positive integers is called a Diophantine m-tuple if the product of its any two distinct elements increased by 1 is a perfect square. Diophantus found a set of four positive rationals with the above property. The first Diophantine quadruple was found by Fermat (the set $\{1, 3, 8, 120\}$). Baker and Davenport proved that this particular quadruple cannot be extended to a Diophantine quintuple.

In this paper, we prove that there does not exist a Diophantine sextuple and that there are only finitely many Diophantine quintuples.

1 Introduction

A set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ is called *a Diophantine m-tuple* if $a_i \cdot a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$.

Diophantus first studied the problem of finding four numbers such that the product of any two of them increased by unity is a square. He found a set of four positive rationals with the above property: $\{1/16, 33/16, 17/4, 105/16\}$. However, the first Diophantine quadruple, $\{1, 3, 8, 120\}$, was found by Fermat. Euler was able to add the fifth positive rational, 777480/8288641, to the Fermat's set (see [6], pp. 513–520). Recently, Gibbs [13] found examples of sets of six positive rationals with the property of Diophantus.

A folklore conjecture is that there does not exist a Diophantine quintuple. The first important result concerning this conjecture was proved in 1969 by Baker and Davenport [3]. They proved that if d is a positive integer such that $\{1, 3, 8, d\}$ forms a Diophantine quadruple, then $d = 120$. This problem was stated in 1967 by Gardner [12] (see also [17]).

In 1979 Arkin, Hoggatt and Strauss [1] proved that every Diophantine triple can be extended to a Diophantine quadruple. More precisely, let $\{a, b, c\}$ be a Diophantine triple and $ab + 1 = r^2$, $ac + 1 = s^2$, $bc + 1 = t^2$, where r, s, t are positive integers. Define

$$d_+ = a + b + c + 2abc + 2rst\,.$$

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Then $\{a, b, c, d_+\}$ is a Diophantine quadruple. Indeed,

$$ad_+ + 1 = (at + rs)^2, \quad bd_+ + 1 = (bs + rt)^2, \quad cd_+ + 1 = (cr + st)^2.$$

There is a stronger version of the "Diophantine quintuple conjecture".

Conjecture 1 *If $\{a, b, c, d\}$ is a Diophantine quadruple and $d > \max\{a, b, c\}$, then $d = d_+$.*

Conjecture 1 was proved for certain Diophantine triples [16, 20] and for some parametric families of Diophantine triples [7, 8, 10]. In particular, in [10] it was proved that the pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple.

A Diophantine quadruple $D = \{a, b, c, d\}$, where $a < b < c < d$, is called *regular* if $d = d_+$. Equivalently, D is regular iff

$$(a + b - c - d)^2 = 4(ab + 1)(cd + 1) \tag{1}$$

(see [14]). The equation (1) is a quadratic equation in d. One root of this equation is d_+, and other root is

$$d_- = a + b + c + 2abc - 2rst\,.$$

It is easy to check that all "small" Diophantine quadruples are regular; e.g. there are exactly 207 quadruples with $\max\{a, b, c, d\} \leq 10^6$ and all of them are regular.

Since the number of integer points on an elliptic curve

$$y^2 = (ax + 1)(bx + 1)(cx + 1) \tag{2}$$

is finite, it follows that, for fixed a, b and c, there does not exist an infinite set of positive integers d such that a, b, c, d is a Diophantine quadruple. However, bounds for the size [2] and for the number [19] of solutions of (2) depend on a, b, c and accordingly they do not immediately yield an absolute bound for the size of such set.

The main result of the present paper is the following theorem.

Theorem 1 *There are only finitely many Diophantine quintuples.*

Moreover, this result is effective. We will prove that all Diophantine quintuples Q satisfy $\max Q < 10^{10^{26}}$. Hence we almost completely solve the problem of the existence of Diophantine quintuples. Furthermore, we prove

Theorem 2 *There does not exist a Diophantine sextuple.*

Theorems 1 and 2 improve results from [9] where we proved that there does not exist a Diophantine 9-tuple and that there are only finitely many Diophantine 8-tuples.

As in [9], we prove Conjecture 1 for a large class of Diophantine triples satisfying some gap conditions. However, in the present paper these gap conditions are much weaker than in [9]. Accordingly, the class of Diophantine triples for which we are able to prove Conjecture 1 is much larger. In fact, in an arbitrary Diophantine quadruple, we may find a triple for which we are able to prove Conjecture 1.

In the proof of Conjecture 1 for a triple $\{a,b,c\}$ we first transform the problem into solving systems of simultaneous Pellian equations. This reduces to finding intersections of binary recurrence sequences. In Section 5 we almost completely determine initial terms of these sequences, under assumption that they have nonempty intersection which induces a solution of our problem. This part is a considerable improvement of the corresponding part of [9]. This improvement is due to new "gap principles" developed in Section 4. These "gap principles" follow from the careful analysis of the elements of the binary recurrence sequences with small indices. Let us mention that in a joint paper with A. Pethő [10] we were able to determine initial terms, in a special case of triples $\{1,3,c\}$, using an inductive argument.

Applying some congruence relations we get lower bounds for solutions. In obtaining these bounds we need to assume that our triple satisfies some gap conditions like $b > 4a$ and $c > b^{2.5}$. Let us note that these conditions are much weaker then conditions used in [9], and this is due to more precise determination of the initial terms. Comparing these lower bounds with upper bounds obtained from the Baker's theory on linear forms in logarithms of algebraic numbers (a theorem of Matveev [18]) we prove Theorem 1, and comparing them with upper bounds obtained from a theorem of Bennett [5] on simultaneous approximations of algebraic numbers we prove Theorem 2. In the final steps of the proofs, we use again the above mentioned "gap principles".

2 Systems of Pellian equations

Let us fix some notation. Let $\{a,b,c\}$ be a Diophantine triple and $a < b < c$. Furthermore, let positive integers r,s,t be defined by

$$ab+1=r^2, \quad ac+1=s^2, \quad bc+1=t^2.$$

In order to extend $\{a,b,c\}$ to a Diophantine quadruple $\{a,b,c,d\}$, we have to solve the system

$$ad+1=x^2, \quad bd+1=y^2, \quad cd+1=z^2, \tag{3}$$

with positive integers x,y,z. Eliminating d from (3) we get the following system of Pellian equations

$$az^2-cx^2 = a-c\,, \tag{4}$$

$$bz^2-cy^2 = b-c\,. \tag{5}$$

In [9], Lemma 1, we proved the following lemma which describes the sets of solutions of the equations (4) and (5).

Lemma 1 *There exist positive integers i_0, j_0 and integers $z_0^{(i)}$, $x_0^{(i)}$, $z_1^{(j)}$, $y_1^{(j)}$, $i=1,\dots,i_0$, $j=1,\dots,j_0$, with the following properties:*

(i) $(z_0^{(i)},x_0^{(i)})$ *and* $(z_1^{(j)},y_1^{(j)})$ *are solutions of (4) and (5), respectively.*

(ii) $z_0^{(i)}$, $x_0^{(i)}$, $z_1^{(j)}$, $y_1^{(j)}$ *satisfy the following inequalities*

$$1 \le x_0^{(i)} \le \sqrt{\frac{a(c-a)}{2(s-1)}} < \sqrt{\frac{s+1}{2}} < 0.841\sqrt[4]{ac}\,, \tag{6}$$

$$1 \le |z_0^{(i)}| \le \sqrt{\frac{(s-1)(c-a)}{2a}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} < 0.421c\,, \tag{7}$$

$$1 \le y_1^{(j)} \le \sqrt{\frac{b(c-b)}{2(t-1)}} < \sqrt{\frac{t+1}{2}} < 0.783\sqrt[4]{bc}\,, \tag{8}$$

$$1 \le |z_1^{(j)}| \le \sqrt{\frac{(t-1)(c-b)}{2b}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}} < 0.32c\,. \tag{9}$$

(iii) If (z,x) and (z,y) are positive integer solutions of (4) and (5) respectively, then there exist $i \in \{1,\dots,i_0\}$, $j \in \{1,\dots,j_0\}$ and integers $m,n \ge 0$ such that

$$z\sqrt{a} + x\sqrt{c} = (z_0^{(i)}\sqrt{a} + x_0^{(i)}\sqrt{c})(s+\sqrt{ac})^m, \tag{10}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})(t+\sqrt{bc})^n. \tag{11}$$

Let (x,y,z) be a solution of the system (4) & (5). From (10) it follows that $z = v_m^{(i)}$ for some index i and integer $m \ge 0$, where

$$v_0^{(i)} = z_0^{(i)}, \quad v_1^{(i)} = sz_0^{(i)} + cx_0^{(i)}, \quad v_{m+2}^{(i)} = 2sv_{m+1}^{(i)} - v_m^{(i)}\,, \tag{12}$$

and from (11) we conclude that $z = w_n^{(j)}$ for some index j and integer $n \ge 0$, where

$$w_0^{(j)} = z_1^{(i)}, \quad w_1^{(j)} = tz_1^{(j)} + cy_1^{(j)}, \quad w_{n+2}^{(j)} = 2tw_{n+1}^{(j)} - w_n^{(j)}\,. \tag{13}$$

Let us consider the sequences (v_m) and (w_n) modulo $2c$. From (12) and (13) it is easily seen that

$$v_{2m}^{(i)} \equiv z_0^{(i)} \pmod{2c}, \quad v_{2m+1}^{(i)} \equiv sz_0^{(i)} + cx_0^{(i)} \pmod{2c}, \tag{14}$$

$$w_{2n}^{(j)} \equiv z_1^{(j)} \pmod{2c}, \quad w_{2n+1}^{(j)} \equiv tz_1^{(j)} + cy_1^{(j)} \pmod{2c}. \tag{15}$$

We are searching for solutions of the system (4) & (5) such that $d = (z^2-1)/c$ is an integer. Using (14) and (15), from $z = v_m = w_n$ we obtain

$$[z_0^{(i)}]^2 \equiv v_m^2 \equiv z^2 \equiv 1 \pmod{c}, \quad [z_1^{(j)}]^2 \equiv w_n^2 \equiv z^2 \equiv 1 \pmod{c}.$$

Therefore we are interested only in equations $v_m = w_n$ satisfying

$$[z_0^{(i)}]^2 \equiv [z_1^{(j)}]^2 \equiv 1 \pmod{c}.$$

We will deduce later more precise information on the initial terms $z_0^{(i)}$ and $z_1^{(j)}$ (see Section 5). Let us mention now a result of Jones [15], Theorem 8, which says that if $c < 4b$ then $|z_1^{(j)}| = 1$.

As a consequence of Lemma 1 and the relations (14) and (15), we obtain the following lemma. From now on, we will omit the superscripts (i) and (j).

Lemma 2

1) *If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$.*

2) *If the equation $v_{2m+1} = w_{2n}$ has a solution, then $z_0 \cdot z_1 < 0$ and $cx_0 - s|z_0| = |z_1|$. In particular, if $b > 4a$ and $c > 100a$, then this equation has no solution.*

3) *If the equation $v_{2m} = w_{2n+1}$ has a solution, then $z_0 \cdot z_1 < 0$ and $cy_1 - t|z_1| = |z_0|$.*

4) *If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $z_0 \cdot z_1 > 0$ and $cx_0 - s|z_0| = cy_1 - t|z_1|$.*

Proof. See [9], Lemma 3. ■

3 Relationships between m and n

In [9], Section 4, we proved that $v_m = w_n$ implies $m \geq n$ if $b > 4a$, $c > 100b$ and $n \geq 3$. We also proved that $m \leq \frac{3}{2}n$, provided $\{a, b, c\}$ satisfies some rather strong gap conditions.

In this section we will first prove an unconditional relationship between m and n, and then we will improve that result under various gap assumptions.

Lemma 3 *If $v_m = w_n$, then $n - 1 \leq m \leq 2n + 1$.*

Proof. We have the following estimates for v_1:

$$v_1 = sz_0 + cx_0 \geq cx_0 - s|z_0| = \frac{c^2 - ac - z_0^2}{cx_0 + s|z_0|} > \frac{c^2 - \frac{c^2}{4} - \frac{c\sqrt{c}}{2\sqrt{a}}}{2cx_0} > \frac{c}{4x_0} > \frac{c}{3.364\sqrt[4]{ac}},$$

$$v_1 < 2cx_0 < 1.682c\sqrt[4]{ac}.$$

Hence

$$\frac{c}{3.364\sqrt[4]{ac}}(2s-1)^{m-1} < v_m < 1.682c\sqrt[4]{ac}(2s)^{m-1} \quad \text{for } m \geq 1. \tag{16}$$

If $c > 4b$, then

$$w_1 \geq \frac{c^2 - bc - z_1^2}{cy_1 + b|z_1|} > \frac{c}{4y_1} > \frac{c}{3.132\sqrt[4]{bc}}.$$

If $c < 4b$ then, by [15], Theorem 8, $z_1 = \pm 1$, $y_1 = 1$ and

$$w_1 \geq c - t = a + r = s > \sqrt{ac} > \frac{c}{3.132\sqrt[4]{bc}}.$$

Furthermore, $w_1 < 2cy_1 < 1.566c\sqrt[4]{bc}$ and therefore

$$\frac{c}{3.132\sqrt[4]{bc}}(2t-1)^{n-1} < w_n < 1.566c\sqrt[4]{bc}(2t)^{n-1} \quad \text{for } n \geq 1. \tag{17}$$

We thus get

$$(2s-1)^{m-1} < 5.269\sqrt[4]{abc^2}\,(2t)^{n-1}\,. \tag{18}$$

Since

$$2s-1 = 2\sqrt{ac+1}-1 > 1.767\sqrt{ac} \quad \text{and} \quad 2t = 2\sqrt{bc+1} < 2.042\sqrt{bc}, \tag{19}$$

it follows that $(2s-1)^2 > 3.12ac > 2t$. It implies

$$(2s-1)^{m-1} < 2.635\cdot(2t)^n < (2t)^{n+0.43} < (2s-1)^{2n+0.86}$$

and $m \le 2n+1$.

On the other hand, we have

$$(2t-1)^{n-1} < 5.269\sqrt[4]{abc^2}\,(2s)^{m-1} < 5.269\sqrt[4]{abc^2}\,(2t-1)^{m-1} < (2t-1)^{m+0.489}$$

and $n \ge m+1$.

Thus we proved the lemma for $m,n \ne 0$. It remains to check that $v_0 < w_2$ and $w_0 < v_2$. Indeed,

$$w_2 = 2tw_1 - w_0 > \frac{2ct}{3.132\sqrt[4]{bc}} - \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}} > c\left(\frac{\sqrt[4]{bc}}{1.566} - \frac{1}{\sqrt{2\sqrt{bc}}}\right) > 1.093c > v_0\,,$$

$$v_2 = 2sv_1 - v_0 > \frac{2cs}{3.364\sqrt[4]{ac}} - \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} > c\left(\frac{\sqrt[4]{ac}}{1.682} - \frac{1}{\sqrt{2\sqrt{ac}}}\right) > 0.579c > w_0\,.$$

■

Lemma 4 *Assume that $c > 10^{10}$. If $v_m = w_n$ and $m,n \ge 2$, then*

1) $c > b^{4.5} \implies m \le \frac{11}{9}n + \frac{7}{9}$,

2) $c > b^{2.5} \implies m \le \frac{7}{5}n + \frac{3}{5}$,

3) $c > b^{2} \implies m \le \frac{3}{2}n + \frac{1}{2}$,

4) $c > b^{5/3} \implies m \le \frac{8}{5}n + \frac{3}{5}$.

Proof. As in the proof of Lemma 3, assuming $c > \max\{b^{5/3}, 10^{10}\}$, we have

$$v_m > \frac{0.999c}{2x_0}(2s-1)^{m-1} > \frac{c}{1.416\sqrt[4]{ac}}\,(2s-1)^{m-1},$$

$$w_n < 1.415\sqrt[4]{bc}(2t)^{n-1},$$

for $m,n \ge 1$. Hence $(2s-1)^{m-1} < 2.004\sqrt[4]{abc^2}\,(2t)^{n-1}$ and

$$1.999^{m-1}a^{(m-1)/2}c^{(m-1)/2} < 2.004^n b^{(n-1)/2+1/4}c^{(n-1)/2+1/4}a^{1/4}\,. \tag{20}$$

If $m \ge 2$ and $c > b^{\varepsilon}$, then (20) shows that at least one of the following two inequalities holds:

$$\frac{m-1}{2} < (\frac{n}{2} - \frac{1}{4})\cdot\frac{1}{\varepsilon} + \frac{n}{2}\,, \tag{21}$$

$$m-1 < 1.004n\,. \tag{22}$$

The inequality (22) implies all statements of the lemma. For $\varepsilon = 4.5$, (21) implies $m < \frac{11}{9}n + \frac{8}{9}$ and $m \le \frac{11}{9}n + \frac{7}{9}$. Similar arguments apply to all other statements of the lemma.

■

4 Gap principles

In [9], Lemma 14, we proved that if $\{a, b, c, d\}$ is a Diophantine quadruple and $a < b < c < d$, then $d \geq 4bc$. The proof was based on the fact, proved by Jones [15], Lemma 4, that $c = a + b + 2r$ or $c \geq 4ab + a + b$.

In this section we will develop a stronger and more precise gap principle by examining the equality $v_m = w_n$ for small values of m and n.

Let us note that since

$$c(4ab + 1) < d_+ < 4c(ab + 1)\,,$$

we may expect an essential improvement only if we assume that $d \neq d_+$.

Lemma 5 *Let $v_m = w_n$ and define $d = (v_m^2 - 1)/c$. If $\{0, 1, 2\} \cap \{m, n\} \neq \emptyset$, then $d < c$ or $d = d_+$.*

Proof. From the proof of Lemma 3 we have:

$$v_0 < w_2, \quad w_0 < v_2, \quad v_1 < w_3, \quad w_1 < v_4, \quad v_2 < w_4, \quad w_2 < v_6.$$

Therefore, the condition $\{0, 1, 2\} \cap \{m, n\} \neq \emptyset$ implies

$$(m, n) \in \{(0, 0), (0, 1), (1, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (2, 3), (3, 2), (4, 2), (5, 2)\}.$$

If $0 \in \{m, n\}$, then $d < c$.

If $(m, n) = (1, 1)$, then $d < c$ for $z_0 < 0$, and $d = d_+$ for $z_0 > 0$ (see [9], proof of Theorem 3).

Assume that $(m, n) = (1, 2)$. We have $v_1 = sz_0 + cx_0$, $w_2 = z_1 + 2c(bz_1 + ty_1)$. By Lemma 2, if $z_1 > 0$ then $z_0 < 0$ and $cx_0 + sz_0 = z_1$. Hence $w_2 > v_1 = w_0$. If $z_1 < 0$ then $z_0 > 0$ and

$$cx_0 - sz_0 = -z_1. \tag{23}$$

Inserting (23) into the relation $v_1 = w_2$ we obtain

$$bz_1 + ty_1 = x_0. \tag{24}$$

From (23), (24) and the system (4) & (5), we obtain $(b - a)t = z_0^2(b - a)$. Therefore, $z_0 = t$, $x_0 = r$, $z_1 = st - cr$, $y_1 = rt - bs$. It implies $v_1 = st + cr$ and

$$d = \frac{v_1^2 - 1}{c} = \frac{1}{c}\,(abc^2 + ac + bc + 1 + 2crst + abc^2 + c^2 - 1) = d_+\,.$$

Assume now that $(m, n) = (2, 1)$. In the same manner as in the case $(m, n) = (1, 2)$, we obtain $z_1 = s$, $y_1 = r$, $z_0 = st - cr$, $x_0 = rs - at$ and $d = d_+$.

Let $(m, n) = (2, 2)$. We have $v_2 = z_0 + 2c(az_0 + sx_0)$, $w_2 = z_1 + 2c(bz_1 + ty_1)$, and since $z_0 = z_1$, we obtain $az_0 + sx_0 = bz_0 + ty_1$ and

$$(b - a)(cy_1^2 - cx_0^2 + b - a) = acx_0^2 + x_0^2 - bcy_1^2 + y_1^2 - 2stx_0y_1\,.$$

Therefore $(b-a)^2 = (sy_1 - tx_0)^2$ and since $sy_1 = tx_0$, we have

$$tx_0 - sy_1 = b - a. \tag{25}$$

Furthermore,

$$(ac+1)(bx_0^2 + a - b) = a(bcx_0^2 + x_0^2 + (b-a)^2 - 2t(b-a)x_0)$$

and

$$(b-a)(x_0^2 + 2atx_0 + a^2t^2) = (b-a)(ab+1)(ac+1) = (b-a)r^2s^2.$$

Finally, $x_0 = rs - at$, $y_1 = rt - bs$, $z_0 = st - cr$. Now we obtain

$$v_2 = st - cr + 2c(ast - acr + rs^2 - ast) = st + cr$$

and $d = d_+$.

Let $(m,n) = (3,1)$. We may assume that $\{a,b,c\} \neq \{1,3,8\}$. Then $ac \geq 15$ and $bc \geq 48$. It implies $2s - 1 > 1.807\sqrt{ac}$ and $2t < 2.021\sqrt{bc}$. From (18) we obtain

$$(1.807\sqrt{ac})^2 < 5.269\sqrt[4]{abc^2} < 5.269\sqrt[4]{ac^3}\,,$$

which implies $c \leq 6$, a contradiction.

Let $(m,n) = (3,2)$. Assume that $z_0 > 0$, $z_1 < 0$. We have $v_1 > 2sz_0$, $v_2 > (4s^2-1)z_0$, $v_3 > (4s^2-1)(2s-1)z_0 > 7(ac)^{3/2}z_0$, and $w_1 < \frac{c^2}{2t|z_1|}$, $w_2 < 2tw_1 < \frac{c^2}{|z_1|}$. We have also $|z_1| > \frac{c}{4x_0}$ and therefore $w_2 < 4cx_0$. Since $x_0 < \sqrt{c}$, we obtain

$$w_2 < 4c^{3/2} < 7(ac)^{3/2}z_0 < v_3\,.$$

Assume now that $z_0 < 0$, $z_1 > 0$. Then $z_1 = sz_0 + cx_0$ and the condition $v_3 = w_2$ implies $x_0 + 2az_1 = bz_1 + ty_1 > 2bz_1$ and $x_0 > 4z_1 > \frac{c}{x_0}$, a contradiction.

Let $(m,n) = (2,3)$. If $z_0 > 0$, $z_1 < 0$, then $v_2 = w_3$ implies $y_1 + 2bz_0 = az_0 + sx_0 < 2az_0 + \frac{c}{z_0}$. Therefore $z_0^2 < \frac{c}{4}$. On the other hand, $z_0^2 \equiv 1 \pmod{c}$. Hence, $z_0 = 1$. But, if $c > 4b$ then $z_0 > \frac{c}{4y_1} > 1$. If $c < 4b$ then $z_1 = -1$, $y_1 = 1$, and Lemma 2 implies $c = b+2$, which contradicts the fact that $c = a + b + 2r$.

Assume that $z_0 < 0$, $z_1 > 0$. As in the case $(m,n) = (3,2)$, we have $v_2 < \frac{c^2}{|z_0|}$ and $w_3 > 7.3(bc)^{3/2}z_1$. If $c > 4b$ then $|z_0| > \frac{c}{4y_1}$ and since $y_1 < \sqrt{c}$, we obtain

$$v_2 < 4c^{3/2} < 7.3(bc)^{3/2}z_1 < w_3\,.$$

If $c < 4b$ then

$$w_3 > 7.3b^{1.5}c^{1.5} > 0.9c^3 > c^2 > v_2\,.$$

Let $(m,n) = (4,2)$. We have $v_4 = z_0 + 4c(2az_0 + sx_0) + 8ac^2(az_0 + sx_0)$, and the relation $v_4 = w_2$ implies

$$bz_0 + ty_1 = 4az_0 + 2sx_0 + 4ac(az_0 + sx_0). \tag{26}$$

It holds $ty_1 - b|z_0| < \frac{c}{|z_0|}$. Hence, if $z_0 > 0$ then the left hand side of (26) is $\le 2bz_0 + \frac{c}{z_0} < 3cz_0$, while the right hand side is $> 4a^2cz_0 > 3cz_0$. If $z_0 < 0$, then the left hand side of (26) is $\le \frac{c}{|z_0|} \le c$, while the right hand side is $\ge 4ac - 4a|z_0| > ac \ge c$.

Let $(m,n) = (5,2)$. It follows from (18) that

$$(1.807\sqrt{ac})^4 < 5.269\sqrt{bc} \cdot 2.021\sqrt{bc}$$

and $10.66a^2c^2 < 10.65bc$, a contradiction. ■

We now can prove the following gap principal, which we will improve again in Proposition 1 below.

Lemma 6 *If $\{a,b,c,d\}$ is a Diophantine quadruple and $a < b < c < d$, then $d = d_+$ or $d \ge 1.16\, c^{2.5} b^{1.5}$.*

Proof. By Lemma 5, if $d \ne d_+$ then $m \ge 3$ and $n \ge 3$. From (17) it follows that

$$w_3 > (2t-1)^2 \cdot \frac{c}{3.132\sqrt[4]{bc}} > \frac{81}{24 \cdot 3.132}\sqrt[4]{b^3c^3} \cdot c$$

and

$$d \ge \frac{1.161b^{1.5}c^{3.5} - 1}{c} > 1.16c^{2.5}b^{1.5}\,.$$

■

Using the gap principle from Lemma 6 we can prove the following lemma.

Lemma 7 *Under the notation from above, we have $v_3 \ne w_3$.*

Proof. If $z_0, z_1 > 0$ then define $z' := cx_0 - sz_0 = cy_1 - tz_1$, and if $z_0, z_1 < 0$ then define $z' := cx_0 + sz_0 = cy_1 + tz_1$. Define also $d_0 = (z'^2 - 1)/c$. Then d_0 is an integer. Furthermore, $cd_0 + 1 = z'^2$,

$$\begin{aligned} ad_0 + 1 &= \frac{1}{c}(ac^2x_0^2 \mp 2acsx_0z_0 + a^2cz_0^2 + az_0^2 - a + c) \\ &= (acx_0^2 \mp 2asx_0z_0 + a^2z_0^2 + x_0^2) = (sx_0 \mp az_0)^2, \end{aligned}$$

$$bd_0 + 1 = \frac{1}{c}(bc^2y_1^2 \mp 2bcty_1z_1 + b^2cz_1^2 - b + c) = (ty_1 \mp bz_1)^2.$$

We have

$$|z'| = \frac{c^2 - ac - z_0^2}{cx_0 + s|z_0|} > \frac{c}{3.364\sqrt[4]{ac}} \quad \text{and} \quad |z'| < c.$$

Therefore

$$d_0 > \frac{0.088\frac{c\sqrt{c}}{\sqrt{a}} - 1}{c} > 0.043\frac{\sqrt{c}}{\sqrt{a}} > 0\,,$$

and thus the set $\{a, b, c, d_0\}$ forms a Diophantine quadruple. Since $d_0 < c$, by Lemma 6, we have two possibilities: the quadruple $\{a, b, c, d_0\}$ is regular or

$$c > 1.16 d_0^{2.5} b^{1.5}\,. \tag{27}$$

Assume that $\{a, b, c, d_0\}$ is regular, i.e. $d_0 = d_-$. Then $z' = cr - st$. From $c(x_0 - r) = s(|z_0| - t)$ and $\gcd(c, s) = 1$ it follows that $|z_0| \equiv t \pmod{c}$, and since $|z_0| < c$, $t < c$, we conclude that $|z_0| = t$ and $x_0 = r$. We can proceed analogously to prove that $|z_1| = s$ and $y_1 = r$.

The condition $v_3 = w_3$ implies

$$sz_0 + 3cx_0 + 4ac(cx_0 + sz_0) = tz_1 + 3cy_1 + 4bc(cy_1 + tz_1)$$

and we obtain $a = b$, a contradiction.

Hence we may assume that the quadruple $\{a, b, c, d_0\}$ is not regular. But it means that $c > 10^6$, which implies

$$|z'| = \frac{c^2 - ac - z_0^2}{cx_0 + s|z_0|} > \frac{0.749c}{2x_0} > \frac{0.749c}{1.415\sqrt[4]{ac}} > 0.529\frac{c}{\sqrt[4]{ac}}$$

and

$$d_0 > \frac{0.2798\frac{c\sqrt{c}}{\sqrt{a}} - 1}{c} > 0.2797\frac{\sqrt{c}}{\sqrt{a}}\,.$$

Thus from (27) we obtain $c > 0.0479 c^{1.25} a^{-1.25} b^{1.5} > 0.0479 c^{1.25} a^{0.25}$ and $ac \le 189958$, which contradicts the assumption that $c > 10^6$. ∎

Now we can prove the following strong gap principle, which is the main improvement to [9] and which we will use several times later. Observe that especially the dependence on c is much better than in the gap principle in [9].

Proposition 1 *If $\{a, b, c, d\}$ is a Diophantine quadruple and $a < b < c < d$, then $d = d_+$ or $d > 2.695\, c^{3.5} a^{2.5}$.*

Proof. From Lemmas 5 and 7 it follows that $m \ge 4$ or $n \ge 4$. By (16), we have

$$v_4 \ge \frac{c}{3.364\sqrt[4]{ac}}\,(2s-1)^3 \ge \frac{125}{8\sqrt{8}\cdot 3.36}\,\sqrt[4]{a^5c^5}\cdot c\,.$$

Therefore, if $m \ge 4$ then

$$d \ge \frac{2.696 a^{2.5} c^{4.5} - 1}{c} > 2.695 c^{3.5} a^{2.5}\,.$$

Similarly, from (17) it follows

$$w_4 \ge \frac{c}{3.132\sqrt[4]{bc}}\,(2t-1)^3 \ge \frac{729}{24\sqrt{24}\cdot 3.132}\,\sqrt[4]{b^5c^5}\cdot c\,,$$

and for $n \ge 4$ we obtain

$$d \ge \frac{3.919 b^{2.5} c^{4.5} - 1}{c} > 3.918 c^{3.5} b^{2.5} > 2.695 c^{3.5} a^{2.5}\,.$$

∎

Corollary 1 *If $\{a,b,c,d,e\}$ is a Diophantine quintuple and $a<b<c<d<e$, then $e > 2.695\, d^{3.5}b^{2.5}$.*

Proof. Assume that $\{b,c,d,e\}$ is a regular Diophantine quadruple. Then $e \le 4d(bc+1) < d^3$. The quadruple $\{a,c,d,e\}$ is not regular and, by Proposition 1, we have $e > 2.695d^{3.5}a^{2.5} > d^3$.

Therefore the quadruple $\{b,c,d,e\}$ is not regular and hence $e > 2.695d^{3.5}b^{2.5}$. ■

5 Determination of the initial terms

Using the gap principles developed in the previous section we will improve Lemma 2 and obtain more specific information on the initial terms of the sequences (v_m) and (w_n).

Lemma 8

1) *If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$. Furthermore, $|z_0| = 1$ or $|z_0| = cr - st$ or $|z_0| < \min\{0.869\, a^{-5/14}c^{9/14},\ 0.972\, b^{-0.3}c^{0.7}\}$.*

2) *If the equation $v_{2m+1} = w_{2n}$ has a solution, then $|z_0| = t$, $|z_1| = cr - st$ and $z_0z_1 < 0$.*

3) *If the equation $v_{2m} = w_{2n+1}$ has a solution, then $|z_0| = cr - st$, $|z_1| = s$ and $z_0z_1 < 0$.*

4) *If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $|z_0| = t$, $|z_1| = s$ and $z_0z_1 > 0$.*

Proof.

1) From Lemma 2 we have $z_0 = z_1$. Define $d_0 = (z_0^2 - 1)/c$. Then d_0 is an integer and

$$cd_0 + 1 = z_0^2, \quad ad_0 + 1 = \frac{1}{c}(az_0^2 - a + c) = x_0^2, \quad bd_0 + 1 = \frac{1}{c}(bz_1^2 - b + c) = y_1^2.$$

Hence, we have three possibilities: $d_0 = 0$ or $\{a,b,c,d\}$ is a regular Diophantine quadruple or $\{a,b,c,d\}$ is an irregular Diophantine quadruple. If $d_0 = 0$ then $|z_0| = 1$. If the quadruple $\{a,b,c,d\}$ is regular, then (since $d_0 < c$) we have $d_0 = d_-$ and $|z_0| = cr - st$. Otherwise we may apply Proposition 1 to obtain $c \ge 2.695d_0^{3.5}a^{2.5}$. Since $|z_0| \ne 1$, we have $z_0^2 \ge c+1$. We may assume that $c > 10^6$ and therefore

$$d_0 = \frac{z_0^2 - 1}{c} \ge \frac{z_0^2}{c}\Big(1 - \frac{1}{c+1}\Big) > 0.999\frac{z_0^2}{c}\,.$$

Hence we obtain $c^{4.5} > 2.685|z_0^7|a^{2.5}$ and $|z_0| < 0.869a^{-5/14}c^{9/14}$.

Analogously, from Lemma 6, we obtain $|z_0| < 0.972b^{-0.3}c^{0.7}$.

2) Let $z' = z_1 = cx_0 + sz_0$ if $z_1 > 0$, and $z' = -z_1 = cx_0 - sz_0$ if $z_1 < 0$. Define $d_0 = (z'^2 - 1)/c$. Then d_0 is an integer and

$$cd_0 + 1 = z'^2, \quad ad_0 + 1 = (sx_0 \pm az_0)^2, \quad bd_0 + 1 = y_1^2.$$

In the proof of Lemma 7 it is shown that $d_0 > 0$ and $d_0 < c$. Therefore $\{a,b,c,d\}$ is a Diophantine quadruple, and by the proof of Lemma 7, it must be regular. It means that $d_0 = d_-$ and $|z'| = cr - st$. It implies $|z_1| = cr - st$ and $|z_0| = t$.

3) Let $z' = z_0 = cy_1 + tz_1$ if $z_0 > 0$, and $z' = -z_0 = cy_1 - tz_1$ if $z_0 < 0$, and define $d_0 = (z'^2 - 1)/c$. Then

$$cd_0 + 1 = z'^2, \quad ad_0 + 1 = x_0^2, \quad bd_0 + 1 = (ty_1 \pm bz_1)^2$$

and $0 < d_0 < c$. If the quadruple $\{a, b, d_0, c\}$ is not regular, then from Lemma 6 we have

$$c \geq 1.16 d_0^{2.5} b^{1.5} . \tag{28}$$

We can assume that $c > 10^6$. If $c > 4b$ then

$$|z'| = \frac{c^2 - bc - z_1^2}{cy_1 + t|z_1|} > \frac{0.749c}{2y_1} > \frac{0.749c}{1.415\sqrt[4]{bc}} > 0.529\frac{c}{\sqrt[4]{bc}} ,$$

and if $c < 4b$ then

$$|z'| > \sqrt{ac} > 0.529\frac{c}{\sqrt[4]{bc}} .$$

Therefore,

$$d_0 > \frac{0.2798\frac{c\sqrt{c}}{\sqrt{b}} - 1}{c} > 0.279\frac{\sqrt{c}}{\sqrt{b}} .$$

Now (28) implies $c > \frac{1}{20.97}c^{5/4}b^{1/4}$ and $bc \leq 193372$, a contradiction.

It follows that the quadruple $\{a, b, c, d\}$ is regular, i.e. $d_0 = d_-$ and $|z'| = cr - st$. Hence $|z_0| = cr - st$ and $c(y_1 - r) = t(|z_1| - s)$. Since $\gcd(t, c) = 1$ we have $|z_1| \equiv s \pmod{c}$, which implies $|z_1| = s$.

4) Let $z' = cx_0 - s|z_0| = cy_1 - t|z_1|$ and $d_0 = (z'^2 - 1)/c$. In the proof of Lemma 7 we have shown that $\{a, b, d_0, c\}$ is a regular Diophantine quadruple, and that this fact implies $|z_0| = t$ and $|z_1| = s$. ■

6 Standard Diophantine triples

In [9] we proved Conjecture 1 for triples satisfying some gap conditions like $b > 4a$ and $c > \max\{b^{13}, 10^{20}\}$. In Section 7 we will prove Conjecture 1 under certain weaker assumptions. This results will suffice for proving Theorem 1 since we will show that every Diophantine quadruple contains a triple which satisfies some of our gap assumptions.

Definition 1 *Let $\{a, b, c\}$ be a Diophantine triple and $a < b < c$. We call $\{a, b, c\}$ a Diophantine triple of*

- the first kind *if $c > b^{4.5}$,*
- the second kind *if $b > 4a$ and $c > b^{2.5}$,*
- the third kind *if $b > 12a$ and $b^{5/3} < c < b^2$,*
- the fourth kind *if $b > 4a$ and $b^2 \leq c < 6ab^2$.*

A triple $\{a, b, c\}$ is called standard *if it is a Diophantine triple of the first, second, third or fourth kind.*

The Diophantine triples of the first and the second kind appear naturally when we try to modify results from [9] using Lemma 8. They correspond to triples with properties $b > 4a$, $c > b^{13}$ and $b > 4a$, $c > b^5$, considered in [9]. On the other hand, triples of the third and the fourth kind come from the analysis what kind of triples a regular Diophantine quadruple may contain. Note also that these four cases are not mutually exclusive.

We now use the improved gap principle (Proposition 1) again to show that the set of all standard Diophantine triples is large.

Proposition 2 *Every Diophantine quadruple contains a standard triple.*

Proof. Let $\{a, b, c, d\}$ be a Diophantine quadruple. If it is not regular, then by Proposition 1 and [9], Lemma 14, we have $d > c^{3.5}$ and $c > 4a$. Hence $\{a, c, d\}$ is a triple of the second kind.

Assume that $\{a, b, c, d\}$ is a regular quadruple. Then

$$c(4ab + 1) < d < 4c(ab + 1).$$

If $b > 4a$ and $c \geq b^{1.5}$, then $d > b^{2.5}$ and we see that $\{a, b, d\}$ is a triple of the second kind.

If $b > 4a$ and $c < b^{1.5}$, then we have two possibilities: if $c = a + b + 2r$ then $c < 4b$, $c^2 < d < 4bc^2$ and therefore $\{b, c, d\}$ is a triple of the fourth kind; if $c \geq 4ab + a + b$ then $d < c^2$, $d > bc > c^{5/3}$ and it follows that $\{a, c, d\}$ is a triple of the third kind.

We may now assume that $b < 4a$. By [15], Theorem 8, we have $c = c_k$ $(k \geq 1)$ or $c = \bar{c}_k$ $(k \geq 2)$, where the sequences (c_k) and $(\bar{c}_k)$ are defined by

$$c_0 = 0, \quad c_1 = a + b + 2r, \quad c_k = (4ab + 2)c_{k-1} - c_{k-2} + 2(a + b),$$

$$\bar{c}_0 = 0, \quad \bar{c}_1 = a + b - 2r, \quad \bar{c}_k = (4ab + 2)\bar{c}_{k-1} - \bar{c}_{k-2} + 2(a + b).$$

If $c > b^{2.5}$ then $d > 4abc > b^{4.5}$ and we see that $\{a, b, d\}$ is a triple of the first kind.

Since $c_2 = 4r(a+r)(b+r) > 4ab^2 > b^3$, the condition $c \leq b^{2.5}$ implies $c = c_1$ or $c = \bar{c}_2$.

If $c = c_1$ then

$$c \leq a + b + \frac{4}{\sqrt{3}}\sqrt{ab} = \sqrt{ab}\Big(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} + \frac{4}{\sqrt{3}}\Big) \leq \sqrt{ab}(0.5 + 2 + \frac{4}{\sqrt{3}}) < 4.81\sqrt{ab}$$

and $c \geq 4r$. Hence $d > 4abc > 0.172c^3 > c^2$ and $d < 4cr^2 \leq c^2 r < 2ac^2$. Therefore the triple $\{a, c, d\}$ is of the fourth kind.

If $c \leq b^{2.5}$ and $c = \bar{c}_2 \geq 4ab + 2a + 2b$ (we may assume $b > a + 2$, since otherwise $\bar{c}_2 = c_1$), then $d < c^2$ and $d > 4abc > b^2 c > c^{1.8} > c^{5/3}$. Therefore the triple $\{a, c, d\}$ is of the third kind. ■

7 Lower bounds for solutions

The main tool in obtaining lower bounds for m and n satisfying $v_m = w_n$ $(m, n > 2)$ is the congruence method introduced in the joint paper of the author with A. Pethő [10].

Lemma 9

1) $v_{2m} \equiv z_0 + 2c(az_0m^2 + sx_0m) \pmod{8c^2}$

2) $v_{2m+1} \equiv sz_0 + c[2asz_0m(m+1) + x_0(2m+1)] \pmod{4c^2}$

3) $w_{2n} \equiv z_1 + 2c(bz_1n^2 + ty_1n) \pmod{8c^2}$

4) $w_{2n+1} \equiv tz_1 + c[2btz_1n(n+1) + y_1(2n+1)] \pmod{4c^2}$

Proof. See [9], Lemma 4. ■

If $v_m = w_n$ then, of course, $v_m \equiv w_n \pmod{4c^2}$ and we can use Lemma 9 to obtain some congruences modulo c. However, if a, b, m and n are small compared with c, then these congruences are actually equations. It should be possible to prove that these new equations are in contradiction with the starting equations $v_m = w_n$. This will imply that m and n cannot be too small. We will prove a lower bound for n (and therefore also for m by Lemma 3) depending on c and we will do this separately for Diophantine triples of the first, second, third and fourth kind in the following four lemmas.

Lemma 10 *Let $\{a, b, c\}$ be a Diophantine triple of the first kind and $c > 10^{100}$. If $v_m = w_n$ and $n > 2$, then $n > c^{0.01}$.*

Proof.
Assume that $n \le c^{0.01}$. By Lemma 17, we have $\max\{\lfloor m/2\rfloor+1,\ \lfloor n/2\rfloor+1\} < n \le c^{0.01}$.
According to Lemma 8, we will consider six cases.
1.1) $v_{2m} = w_{2n}, \quad |z_0| = 1$
From Lemma 9 we have

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{4c}. \tag{29}$$

Since $c > b^{4.5}$, we have $am^2 < c^{0.243} < c$, $sm < c^{0.623} < c$, $bn^2 < c^{0.243} < c$, $tn < c^{0.623} < c$. Therefore we may replace $\equiv$ by $=$ in (29):

$$\pm am^2 + sm = \pm bn^2 + tn. \tag{30}$$

From (30), squaring twice, we obtain

$$[(am^2 - bn^2)^2 - m^2 - n^2]^2 \equiv 4m^2n^2 \pmod{c}.$$

Since $4m^2n^2 < c^{0.047} < c$ and $[(am^2 - bn^2)^2 - m^2 - n^2]^2 < c^{0.969} < c$, we therefore have $[(am^2 - bn^2)^2 - m^2 - n^2]^2 = 4m^2n^2$, and

$$am^2 - bn^2 = \pm m \pm n. \tag{31}$$

From (30) and (31) we obtain

$$m(s \pm 1) = n(t \pm 1). \tag{32}$$

Inserting (32) into (30) we obtain

$$n = \frac{(s \pm 1)[t(s \pm 1) - (t \pm 1)s]}{\pm[a(t \pm 1)^2 - b(s \pm 1)^2]} = \frac{(s \pm 1)(\pm t \mp s)}{\pm(\pm 2at + 2a \pm 2bs - 2b)}\,. \tag{33}$$

Since

$$|(s\pm 1)(\pm t\mp s)| \geq (s-1)(t-s) = \frac{(s-1)c(b-a)}{t+s} > \frac{2c(s-1)}{2\sqrt{bc}} > c\cdot\frac{\sqrt{a}}{2\sqrt{b}}$$

and $|\pm 2at+2a\mp 2bs-2b| \leq 4bs+4b < 6b\sqrt{ac}$, (33) implies that $n > \frac{\sqrt{c}}{12\sqrt{b}} > c^{0.377}$, a contradiction.

1.2) $v_{2m} = w_{2n}, \quad |z_0| = cr - st$

We may assume that $b \geq 4$. Then we have

$$|z_0| = |z_1| = \frac{c^2-ac-bc-1}{cr+st} > \frac{\frac{64}{63}c^2}{2rc} > \frac{c}{2.155\sqrt{ab}},$$

and from $|z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}$ it follows $c < 5.4a^2b < 5.4b^3 < b^{4.5}$, a contradiction.

1.3) $v_{2m} = w_{2n}, \quad |z_0| \neq 1,\ cr-st$

From Lemma 9 we obtain

$$az_0m^2 + sx_0m \equiv bz_0n^2 + ty_1n \pmod{4c}. \tag{34}$$

By Lemma 8, we have

$$|az_0m^2| < c^{0.806} < c, \quad |sx_0m| < (a|z_0| + \frac{c}{|z_0|})m < c^{0.8} < c,$$

$$|bz_0n^2| < c^{0.876} < c, \quad |ty_1n| < (b|z_1| + \frac{c}{|z_1|})n < c^{0.87} < c.$$

Therefore,

$$az_0m^2 + sx_0m = bz_0n^2 + ty_1n. \tag{35}$$

Assume now that $b > 4a$. Since $|z_0| \neq 1$, we have $z_0^2 \geq \max\{c+1, 3c/a\}$. This implies

$$0 \leq \frac{sx_0}{a|z_0|} - 1 = \frac{x_0^2+ac-a^2}{a|z_0|(sx_0+a|z_0|)} \leq \frac{1.001ac}{2a^2z_0^2} < 0.1669,$$

$$0 \leq \frac{ty_1}{b|z_1|} - 1 = \frac{y_1^2+bc-b^2}{b|z_1|(ty_1+b|z_1|)} \leq \frac{1.001bc}{2b^2z_0^2} < 0.0418.$$

If $z_0 > 1$ then, by (35), we have $az_0m(m+1.1669) \geq bz_0n(n+1)$, and since $m,n \geq 2$, we obtain $a\cdot 1.5835m^2 > bn^2$ and $\frac{m}{n} \geq 1.589$. But now Lemma 4 implies $n \leq 1$, a contradiction.

If $z_0 < -1$ then (35) implies $a|z_0|m(m-1)) \geq b|z_0|n(n-1.0418)$. Hence, by Lemma 4, we conclude that

$$4n(n-1.0418) < \Big(\frac{11}{9}n + \frac{7}{18}\Big)\Big(\frac{11}{9}n - \frac{11}{18}\Big) < 1.4939n^2 - 0.2716n - 0.2376,$$

which implies $2.5061n^2 - 3.8956n + 0.2376 < 0$ and $n \leq 1$, a contradiction.

Assume now that $b < 4a$. Squaring twice the relation (35) we obtain

$$[(am^2 - bn^2)^2 - x_0^2m^2 - y_1^2n^2]^2 \equiv 4x_0^2y_1^2m^2n^2 \pmod{c}. \tag{36}$$

By Proposition 1, we have

$$y_1^2 < \frac{b}{c}z_0^2 + 1 < \frac{4a}{c} \cdot a^{-5/7}c^{9/7} \cdot 0.754 + 1 < 3.018(ac)^{2/7} < c^{0.364}.$$

This implies that the both sides of the congruence (36) are less than c. Indeed, the left hand side is bounded above by

$$\max\{c^{\frac{8}{9}+0.08}, c^{\frac{8}{9}+0.08}, c^{0.728+0.006+0.04}\} < c^{0.969} < c,$$

while $4x_0^2y_1^2m^2n^2 < c^{0.006+0.728+0.04} = c^{0.774} < c$. Therefore we have an equality in (36), and this implies

$$am^2 - bn^2 = \pm x_0m \pm y_1n. \tag{37}$$

From (35) and (37) we obtain $x_0m(s \pm 1) = y_1n(t \pm 1)$ and $n = A/B$, where

$$A = x_0^2y_1(s \pm 1)(\pm t \mp s), \quad B = z_0[abc(y_1^2 - x_0^2) + 2(a-b) \pm 2aty_1^2 \mp 2bsx_0^2].$$

We have the following estimates

$$|A| \le x_0^2y_1(s+1)(t+s) < 2.005x_0^2y_1c\sqrt{ab}\,,$$

$$\begin{aligned} |B| &> |z_0|[abc(2y_1 - 1) - 2b - 4aty_1^2 - 2s(b-a)] \\ &> |z_0|y_1abc\Big[2 - \frac{1}{y_1} - \frac{2}{acy_1} - \frac{4ty_1}{bc} - \frac{2s}{acy_1}\Big] > 1.569|z_0|y_1abc. \end{aligned}$$

These estimates yield

$$n < \frac{2.005x_0^2y_1c\sqrt{ab}}{1.569|z_0|y_1abc} < 1.278\frac{x_0^2}{|z_0|\sqrt{ab}} < 1.278\frac{az_0^2}{0.999|z_0|c\sqrt{ab}} < 1.28\frac{|z_0|}{c} < c^{-1/4} < 1,$$

a contradiction.

2) $v_{2m+1} = w_{2n}$

The impossibility of this case is proven in **1.2)**.

3) $v_{2m} = w_{2n+1}$

By Lemma 8 and **1.2)**, $|z_0| > \frac{c}{2.155\sqrt{ab}}$, and from $|z_0| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}$ it follows $c < 5.4ab^2 < 5.4b^3 < b^{4.5}$, a contradiction.

4) $v_{2m+1} = w_{2n+1}$

From Lemmas 8 and 9 we obtain

$$\pm astm(m+1) + rm' \equiv \pm bstn(n+1) + rn' \pmod{2c}, \tag{38}$$

where $m' = m$, $n' = n$ if $z_0 < 0$, and $m' = m+1$, $n' = n+1$ if $z_0 > 0$. Multiplying (38) by s and t, respectively, we obtain

$$\pm atm(m+1) + rsm' \equiv \pm btn(n+1) + rsn' \pmod{2c}, \tag{39}$$

$$\pm asm(m+1) + rtm' \equiv \pm bsn(n+1) + rtn' \pmod{2c}. \tag{40}$$

We have $|btn(n+1)| < c^{0.855} < c, \quad rtn' < c^{0.845} < c$. Therefore we have equalities in (39) and (40). This implies $rm' = rn'$ and $am(m+1) = bn(n+1)$, and finally $m = n = 0$. ∎

Lemma 11 *Let $\{a, b, c\}$ be a Diophantine triple of the second kind and $c > 10^{100}$. If $v_m = w_n$ and $n > 2$, then $n > c^{0.04}$.*

Proof. Assume that $n \leq c^{0.04}$. Then $\max\{\lfloor m/2 \rfloor + 1, \lfloor n/2 \rfloor + 1\} < n \leq c^{0.04}$.

1.1) $v_{2m} = w_{2n}, \quad |z_0| = 1$

Since $am^2 < c^{0.408}$, $sm < c^{0.701}$, $bn^2 < c^{0.408}$ and $tn < c^{0.701}$, (29) implies (30). Furthermore

$$\frac{am^2}{sm} < \frac{m\sqrt{a}}{\sqrt{c}} < c^{-0.06} < 0.001, \quad \frac{bn^2}{tn} < \frac{n\sqrt{b}}{\sqrt{c}} < c^{-0.06} < 0.001.$$

Hence

$$\frac{m}{n} \geq \frac{0.999}{1.001} \cdot \frac{t}{s} > \frac{0.999}{1.001^2}\sqrt{\frac{b}{a}} > 1.99,$$

contrary to Lemma 4.

1.2) $v_{2m} = w_{2n}, \quad |z_0| = cr - st$

We have

$$|z_0| = |z_1| = \frac{c^2 - ac - bc - 1}{cr + st} > \frac{c^2 - \frac{5}{4}c^{1.4}}{2rc} > \frac{c - \frac{5}{4}c^{0.4}}{2.13\sqrt{ab}},$$

and from $|z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}$ it follows $c < 5.25a^2b < 1.32ab^2$. Hence $\{a, b, c\}$ is a Diophantine triple of the fourth kind, and this case will be treated in Lemma 13.

1.3) $v_{2m} = w_{2n}, \quad |z_0| \neq 1, \, cr - st$

By Proposition 1, we have $c > b^{3.5}$, and Lemma 8 implies

$$|az_0m^2| < c^{0.907} < c, \quad |sx_0m| < c^{0.871} < c, \quad |bz_0n^2| < c^{0.98} < c, \quad |ty_1n| < c^{0.944} < c.$$

Therefore, equation (35) holds again. As in Lemma 10, we obtain a contradiction with Lemma 4.

2) $v_{2m+1} = w_{2n}$

This case is impossible by Lemma 2. Namely, if $c \leq 100a$ then $a \geq 10^{98}$ and $c \leq a^{1.03}$, a contradiction.

3) $v_{2m} = w_{2n+1}$

As in **1.2)**, we obtain $c < 5.25ab^2$ and again $\{a, b, c\}$ is a triple of the fourth kind.

4) $v_{2m+1} = w_{2n+1}$

Relation (38) implies

$$[am(m+1) - bn(n+1)]^2 \equiv r^2(n-m)^2 \pmod{2c}.$$

Since $a^2m^2(m+1)^2 < c^{0.96}$, $b^2n^2(n+1)^2 < c^{0.96}$, and $r^2m^2 < c^{0.48}$, we obtain

$$bn(n+1) - am(m+1) = \pm r(m-n), \tag{41}$$

which implies

$$\frac{2n+1}{2m+1} - \sqrt{\frac{a}{b}} = \frac{\pm 4r(m-n) + b - a}{[(2n+1)\sqrt{b} + (2m+1)\sqrt{a}]\sqrt{b}(2m+1)}. \tag{42}$$

By Lemma 4, the right hand side of (42) is

$$\le \frac{4r(m-n)}{b(2n+1)(2m+1)} + \frac{b}{b(2n+1)(2m+1)} < \frac{4\cdot\frac{b}{2}\cdot\frac{4n+5}{10}}{b(2n+1)(2m+1)} + \frac{1}{15} < \frac{3}{25} + \frac{1}{15} = \frac{14}{75}.$$

Therefore,

$$\frac{2n+1}{2m+1} < \frac{103}{150} < 0.687\,. \tag{43}$$

If $n = 1$ then $m = 1$ (by Lemma 4), contrary to (43). If $n = 2$ then $m = 2$ or $m = 3$, which both contradict the relation (43).

Hence we may assume that $n \ge 3$, $m \ge 3$, and now (42) implies $\frac{2n+1}{2m+1} < 0.59$. On the other hand, from Lemma 4 we see that $\frac{2n+1}{2m+1} \ge \frac{5}{7} - \frac{3}{7(2m+1)} \ge 0.653$, a contradiction. ∎

Lemma 12 *Let* $\{a, b, c\}$ *be a Diophantine triple of the third kind and* $c > 10^{100}$. *If* $v_m = w_n$ *and* $n > 2$, *then* $n > c^{0.15}$.

Proof. Assume that $n \le c^{0.15}$. Then $\max\{\lfloor m/2\rfloor + 1, \lfloor n/2\rfloor + 1\} < n \le c^{0.15}$.

1.1) $v_{2m} = w_{2n}$, $|z_0| = 1$

Since $am^2 < c^{0.9}$, $sm < c^{0.951}$, $bn^2 < c^{0.9}$ and $tn < c^{0.951} < c$, the proof is identical to that of Lemma 11.

1.2) $v_{2m} = w_{2n}$, $|z_0| = cr - st$

From (34) we have

$$\pm astm(m\mp 1) + rm \equiv \pm bstn(n\mp 1) + rn \pmod{c}. \tag{44}$$

Multiplying (44) by $2st$ we obtain

$$\pm 2[am(m\mp 1) - bn(n\mp 1)] \equiv 2rst(n-m) \pmod{2c}. \tag{45}$$

Let α be the absolutely least residue of $2rst$ modulo $2c$, and let $A = (2rst-2cr^2+c)(st+cr)$. Then $|\alpha|\cdot(st+cr) \le |A|$ and

$$A = 2acr + 2bcr + 2r + cst - c^2r < 2acr + 2bcr + 2r.$$

Since $cr-st = \frac{c^2-bc-ac-1}{cr+st} < \frac{c^2}{2c\sqrt{ab}} < b\sqrt{b} < br$, we obtain $|A| < 2r(ac+bc+1) < \frac{13}{6}bcr$ and $|\alpha| < \frac{13bcr}{12c\sqrt{ab}} < 1.15b$. We have $2am(m\mp1) < c^{0.904}$, $2bn(n\mp1) < c^{0.904}$, $|\alpha(m-n)| < c^{0.651}$. Therefore

$$\pm 2[am(m\mp 1) - bn(n\pm 1)] = \alpha(n-m),$$

and it implies

$$\frac{2n\mp 1}{2m\mp 1} - \sqrt{\frac{a}{b}} = \frac{\pm 2\alpha(m-n) + b - a}{[(2n\mp 1)\sqrt{b} + (2m\mp 1)\sqrt{a}]\sqrt{b}(2m\mp 1)}\,. \tag{46}$$

By Lemma 4, the right hand side of (46) is

$$\le \frac{1.15b\cdot\frac{6n+3}{5}}{b(2n-1)(2m-1)} + \frac{1}{(2n-1)(2m-1)} < 0.495\,,$$

and therefore

$$\frac{2n \mp 1}{2m \mp 1} < 0.784\,. \tag{47}$$

If $n = 2$ then Lemma 4 implies $m = 2$ or 3. The case $m = 2$ contradicts the inequality (47), while for $m = 3$ relation (46) implies $\frac{2n\mp 1}{2m\mp 1} < 0.586$. But for $n = 2$, $m = 3$ we have $\frac{2n\mp 1}{2m\mp 1} \in \{\frac{3}{5}, \frac{5}{7}\}$, a contradiction.

Hence $n \geq 3$, and since $n = m = 3$ contradicts (47), we have also $m \geq 4$. Now, from (46) we obtain $\frac{2n\mp 1}{2m\mp 1} < 0.456$. But for $m \geq 4$ we have $\frac{2m+1}{2n+1} > \frac{2n-1}{2m-1} > \frac{5}{8} - \frac{3}{4(2m-1)} > 0.517$, a contradiction.

1.3) $v_{2m} = w_{2n}$, $|z_0| \neq 1$, $cr - st$

Proposition 1 implies $c > b^{3.5}$ and therefore this case is impossible.

2) $v_{2m+1} = w_{2n}$

The impossibility of this case is shown in Lemma 11.

3) $v_{2m} = w_{2n+1}$

From Lemmas 8 and 9 it follows that

$$\pm 2astm(m \mp 1) + r(2m \pm 1) \equiv \pm 2bstn(n+1) + r(2n+1) \pmod{4c}$$

and

$$\pm 2[am(m \mp 1) - bn(n+1)] \equiv 2rst(n - m + \delta) \pmod{2c}, \tag{48}$$

where $\delta \in \{0, 1\}$.

Let α be defined as in **1.2)**. As in **1.2)**, we obtain

$$\pm 2[am(m \mp 1) - bn(n+1)] = \alpha(n - m + \delta)$$

and

$$\frac{2n+1}{2m \mp 1} - \sqrt{\frac{a}{b}} = \frac{\pm 2\alpha(m - n - \delta) + b - a}{[(2n+1)\sqrt{b} + (2m \mp 1)\sqrt{a}]\sqrt{b}(2m \mp 1)}\,. \tag{49}$$

The right hand side of (49) is

$$\leq \frac{1.15b \cdot \frac{6n+11}{5}}{b(2n-1)(2m \mp 1)} + \frac{1}{(2n-1)(2m \mp 1)}\,,$$

and therefore

$$\frac{2n+1}{2m-1} < 0.861\,, \qquad \frac{2n+1}{2m+1} < 0.617\,, \tag{50}$$

respectively.

If $n = 1$, then by Lemmas 4 and 7 we have $m = 2$. This is clearly impossible if we have $\frac{2n+1}{2m-1}$ on the left hand side of (49), while for $\frac{2n+1}{2m+1}$ we obtain $\frac{3}{5} < 0.509$, a contradiction.

If $n = 2$ then, by Lemma 17, we have $m = 2$, 3 or 4, and (50) implies $m = 4$. Since $m = n = 3$ is also impossible by (50), we conclude that $n \geq 2$ and $m \geq 4$. This implies $\frac{2n+1}{2m-1} < 0.469$, $\frac{2n+1}{2m+1} < 0.429$, respectively. On the other hand, we have $\frac{2n+1}{2m-1} > \frac{2n+1}{2m+1} \geq \frac{5}{8} - \frac{1}{2m+1} > 0.513$, a contradiction.

4) $v_{2m+1} = w_{2n+1}$

Let α be defined as in **1.2)**. From (38) we obtain

$$\pm 2[am(m+1) - bn(n+1)] = \alpha(n-m),$$

which yields

$$\frac{2n+1}{2m+1} - \sqrt{\frac{a}{b}} = \frac{\pm 2\alpha(m-n) + b - a}{[(2n+1)\sqrt{b} + (2m+1)\sqrt{a}]\sqrt{b}(2m+1)}. \tag{51}$$

From (51) it follows that $\frac{2n+1}{2m+1} < 0.494$. But Lemma 4 implies $\frac{2n+1}{2m+1} \geq \frac{5}{8} - \frac{3}{8(2m+1)} \geq 0.55$, a contradiction. ■

Lemma 13 *Let $\{a, b, c\}$ be a Diophantine triple of the fourth kind and $c > 10^{100}$. If $v_m = w_n$ and $n > 2$, then $n > c^{0.2}$.*

Proof. Assume that $n \leq c^{0.2}$. Then $\max\{\lfloor m/2 \rfloor + 1,\ \lfloor n/2 \rfloor + 1\} < n \leq c^{0.2}$.
1.1) $v_{2m} = w_{2n}, \quad |z_0| = 1$
The proof is identical to that of Lemma 11.
1.2) $v_{2m} = w_{2n}, \quad |z_0| = cr - st$
The first part of the proof is identical to that of Lemma 12. In particular, the relation (45) is valid. Estimating $cr - st$, we get

$$cr - st < \frac{c}{2\sqrt{ab}} < \frac{6ab^2}{2\sqrt{ab}} < 3b\sqrt{ab} < 3br.$$

Therefore $|A| < 2r(ac + bc + 1) < 2.5bcr$ and $|\alpha| < \frac{2.5bcr}{2c\sqrt{ab}} < 1.33b$. Note that

$$\alpha^2 \equiv 4r^2s^2t^2 \equiv 4r^2 \pmod{2c},$$

$4r^2 \leq 4.5ab < 1.125b^2 < 2c$ and $\alpha^2 < 1.77b^2 < 2c$. Hence, $\alpha^2 = 4r^2$ and $\alpha = \pm 2r$.
Since $am(m+1) < c^{0.9}$, $bn(n+1) < c^{0.9}$, $r(m-n) < c^{0.7}$, we have

$$bn(n+1) - am(m+1) = \pm r(m-n). \tag{52}$$

Inserting $\alpha = \pm 2r$ in (46) and using Lemma 4 and the estimate $|\alpha| = 2r < \sqrt{b^2+4} < 1.001b$, we obtain

$$\frac{2n \mp 1}{2m \mp 1} < \frac{1}{2} + \frac{1.001n + 1}{(2n-1)(2m-1)} < 0.834. \tag{53}$$

If $(m, n) = (3, 2)$, then (52) implies $6b - 12a = \pm r$ and $(4b - 9a)(9b - 16a) = 1$, which is clearly impossible for $b > 4a$.
Hence $n \geq 3$, $m \geq 4$ and (53) yields $\frac{2n\mp 1}{2m \mp 1} < 0.615$. On the other hand, $\frac{2n+1}{2m+1} > \frac{2n-1}{2m-1} \geq \frac{2}{3} - \frac{1}{3(2m-1)} > 0.619$, a contradiction.
1.3) $v_{2m} = w_{2n}, \quad |z_0| \neq 1,\ cr - st$
Proposition 1 implies $c > b^{3.5}$, and we have $c < 6ab^2 < 1.5b^3$. Therefore, this case is impossible.
2) $v_{2m+1} = w_{2n}$

The impossibility of this case is shown in Lemma 11.

3) $v_{2m} = w_{2n+1}$

The first part of the proof is the same as in Lemma 12. As in **1.2)**, we conclude that $\alpha = \pm r$. We have

$$am(m \mp 1) - bn(n+1)] = \pm r(m - n - \delta) \tag{54}$$

and

$$\frac{2n+1}{2m \mp 1} - \sqrt{\frac{a}{b}} = \frac{\pm 4r(m-n-\delta) + b - a}{[(2n+1)\sqrt{b} + (2m \mp 1)\sqrt{a}]\sqrt{b}(2m \mp 1)}. \tag{55}$$

The relation (55) implies

$$\begin{aligned} \frac{2n+1}{2m \pm 1} &< \frac{1}{2} + \frac{2r(n+2-2\delta)}{b(2n+1)(2m \mp 1) + 1.001r(2m \mp 1)^2} + \frac{1}{(2n+1)(2m \mp 1)} \\ &< \frac{1}{2} + \frac{1.001(n+2-2\delta)+1}{(2n+1)(2m \mp 1)} < 0.945\,. \end{aligned} \tag{56}$$

If $n = 1$, then $m = 2$ and we must have the sign $+$ in (56). But then (54) implies $6a - 2b = \pm r$ and $(9a - 4b)(4a - b) = \pm 1$, a contradiction.

If $(m, n) = (3, 2)$, then we obtain $12a - 6b = \pm r$, which has no solution by **1.2)**. If $(m, n) = (4, 2)$, we have two possibilities: $12a - 6b = \pm r$ or $20a - 6b = \pm 2r$. We have to consider only the second possibility, and it implies $(25a - 9b)(4a - b) = 1$, which has no integer solution.

Hence $n \geq 3$, $m \geq 4$ and $\frac{2n+1}{2m \mp 1} < 0.623$. We see that $(m, n) \neq (4, 3)$, which implies $m \geq 5$ and $\frac{2n+1}{2m+1} < 0.55$, $\frac{2n+1}{2m-1} < 0.545$, respectively. On the other hand, $\frac{2n+1}{2m-1} > \frac{2n+1}{2m+1} \geq \frac{2}{3} - \frac{1}{2m+1} > 0.576$, a contradiction.

4) $v_{2m+1} = w_{2n+1}$

As in the proof of Lemmas 11 and 12, using $\alpha = \pm 2r$, we obtain (42). It implies

$$\frac{2n+1}{2m+1} < \frac{1}{2} + \frac{1.001(n+1)+1}{(2n+1)(2m+1)} < 0.834\,. \tag{57}$$

If $(m, n) = (2, 1)$, then (41) implies $2b - 6a = \pm r$, which is impossible (see **2)**). Hence we have $n \geq 2$, $m \geq 3$, and (57) actually gives $\frac{2n+1}{2m+1} < 0.615$. On the other hand $\frac{2n+1}{2m+1} \geq \frac{2}{3} - \frac{1}{3(2m+1)} > 0.619$, a contradiction. ■

8 Linear forms in three logarithms

Solving recurrences (12) and (13) we obtain

$$v_m = \frac{1}{2\sqrt{a}}[(z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m + (z_0\sqrt{a} - x_0\sqrt{c})(s - \sqrt{ac})^m], \tag{58}$$

$$w_n = \frac{1}{2\sqrt{b}}[(z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n + (z_1\sqrt{b} - y_1\sqrt{c})(t - \sqrt{bc})^n]. \tag{59}$$

Using standard techniques (see e.g. [3, 11]) we may transform the equation $v_m = w_n$ into an inequality for a linear form in three logarithms of algebraic numbers. In [9], Lemma

5, we proved that (assuming $c > 4b$, but this assumption can be replaced by $c > b + \sqrt{c}$, which is satisfied for any triple $\{a, b, c\}$) if $m, n \neq 0$, then

$$0 < m\log(s+\sqrt{ac}) - n\log(t+\sqrt{bc}) + \log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})} < \frac{8}{3}ac(s+\sqrt{ac})^{-2m}. \tag{60}$$

Thus, we have everything ready for the applications of the Baker's theory of linear forms in logarithms of algebraic numbers. We will use Matveev's result ([18], Theorem 2.1), which is quoted below with some restrictions and simplifications.

Lemma 14 ([18]) *Let Λ be a linear form in logarithms of l multiplicatively independent totally real algebraic numbers $\alpha_1, \ldots, \alpha_l$ with rational integer coefficients $b_1, \ldots, b_l$ ($b_l \neq 0$). Let $h(\alpha_j)$ denotes the absolute logarithmic height of α_j, $1 \leq j \leq l$. Define the numbers D, A_j, $1 \leq j \leq l$, and B by $D = [\mathbb{Q}(\alpha_1, \ldots, \alpha_l) : \mathbb{Q}]$, $A_j = \max\{D\,h(\alpha_j), |\log\alpha_j|\}$, $B = \max\{1, \max\{\frac{|b_j|A_j}{A_l} : 1 \leq j \leq l\}\}$. Then*

$$\log\Lambda > -C(l)C_0W_0D^2\Omega\,, \tag{61}$$

where $C(l) = \frac{8}{(l-1)!}(l+2)(2l+3)(4e(l+1))^{l+1}$, $C_0 = \log(e^{4.4l+7}l^{5.5}D^2\log(eD))$, $W_0 = \log(1.5eBD\log(eD))$, $\Omega = A_1\cdots A_l$.

Proposition 3 *Assume that $c > \max\{b^{5/3}, 10^{100}\}$. If $v_m = w_n$, then*

$$\frac{m}{\log(31.3(m+1))} < 3.826\cdot 10^{12}\log^2 c\,. \tag{62}$$

Proof. We apply Lemma 14 to the form (60). We have $l = 3$, $D = 4$, $\alpha_1 = s + \sqrt{ac}$, $\alpha_2 = t + \sqrt{bc}$,

$$\alpha_3 = \frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})}\,.$$

Furthermore, $A_1 = 2\log\alpha_1 < 1.608\log c$, $A_2 = 2\log\alpha_2 < 1.608\log c$. The conjugates of α_3 are

$$\frac{\sqrt{b}(z_0\sqrt{a}\pm x_0\sqrt{c})}{\sqrt{a}(z_1\sqrt{b}\pm y_1\sqrt{c})}\,,$$

and the leading coefficient of the minimal polynomial of α_3 is $a_0 = a^2(c-b)^2$. We proceed with the following estimates:

$$\frac{\sqrt{b}(x_0\sqrt{c}+|z_0|\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+|z_1|\sqrt{b})} < \frac{\sqrt{b}\cdot 2x_0\sqrt{c}}{\sqrt{a}\cdot y_1\sqrt{c}} < 1.415\sqrt[4]{\frac{b^2c}{a}}\,,$$

$$\frac{\sqrt{b}(x_0\sqrt{c}+|z_0|\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}-|z_1|\sqrt{b})} < \frac{\sqrt{b}\cdot 2x_0\sqrt{c}\cdot 2y_1\sqrt{c}}{\sqrt{a}(c-b)} < \frac{2.001c\sqrt{b}\sqrt[4]{abc^2}}{0.999\sqrt{a}\cdot c} < 2.004\sqrt[4]{\frac{b^3c^2}{a}}\,,$$

$$\frac{\sqrt{b}(x_0\sqrt{c}-|z_0|\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+|z_1|\sqrt{b})} < \frac{\sqrt{b}(c-a)}{\sqrt{a}\cdot x_0\sqrt{c}\cdot y_1\sqrt{c}} < \sqrt{\frac{b}{a}}\,,$$

$$\frac{\sqrt{b}(x_0\sqrt{c}-|z_0|\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}-|z_1|\sqrt{b})} < \frac{\sqrt{b}(c-a)\cdot 2y_1\sqrt{c}}{\sqrt{a}(c-b)\cdot x_0\sqrt{c}} < 1.415\sqrt[4]{\frac{b^3c}{a^2}}\,.$$

Therefore, $A_3 = 4h(\alpha_3) < \log(4.013\,a^{1/2}b^{5/2}c^3) < 4.804\log c$. We also have $A_3 \geq \log(a^2(c-b)^2) \geq 1.9999\log c$. Since $\max\{m,n\} \in \{m, m+1\}$ (by Lemma 3), we conclude that $B < 0.8041(m+1)$.

We may assume that $m > 10^{12}$. Therefore we have

$$\log\frac{8}{3}ac(s+\sqrt{ac})^{-2m} < -0.9999\,m\log c\,.$$

It is clear that α_1, α_2 and α_3 are multiplicatively independent and totally real. Hence, we may apply Lemma 14. Putting all the above estimates in (61), we obtain

$$0.9999\,m\log c < 3.8255\cdot 10^{12}\cdot\log^3 c\cdot\log(31.3(m+1))$$

and

$$\frac{m}{\log(31.3(m+1))} < 3.826\cdot 10^{12}\log^2 c\,.$$

■

Note that the assumption $c > \max\{b^{5/3}, 10^{100}\}$ is not essential. It has an effect only on the constant on the right hand side of (62) (see [9], Section 10). Baker's method can be applied without any gap assumption. However, gap assumptions are necessary for obtaining lower bounds for solutions using the "congruence method" of Section 7.

We now compare the lower bounds from Section 7 with the upper bound from Proposition 3 to prove Conjecture 1 for all standard triples with c large enough.

Proposition 4 *Let $\{a,b,c\}$ be a standard Diophantine triple and $c \geq 10^{2171}$. If $\{a,b,c,d\}$ is a Diophantine quadruple and $d > c$, then $d = d_+$.*

Proof. Let $ad+1 = x^2$, $bd+1 = y^2$, $cd+1 = z^2$. Then there exist integers $m, n \geq 0$ such that

$$z = v_m = w_n\,,$$

where the sequences (v_m) and (w_n) are defined by (12) and (13).

Assume that $d \neq d_+$. Then from Lemma 5 it follows that $m \geq 3$ and $n \geq 3$. Hence we may apply Lemmas 10 – 13. We get that in all cases $n > c^{0.01}$. Now, by Lemma 3, $m+1 \geq n > c^{0.01}$. If we put this in (62), we obtain

$$\frac{m}{\log(31.3(m+1))\log^2(m+1)} < 3.826\cdot 10^{16}\,,$$

which implies $m < 5.108\cdot 10^{21}$, and finally $c < (m+1)^{100} < 10^{2171}$. ■

Corollary 2 *Let $\{a,b,c\}$ be a standard Diophantine triple such that $c < 2.695\,b^{3.5}$ and $c \geq 10^{2171}$. If $\{a,b,c,d\}$ is a Diophantine quadruple, then $d = d_-$ or $d = d_+$.*

Proof. By Proposition 4, we may assume that $d < c$. Since $c < 2.695b^{3.5}$, Proposition 1 implies that $\{a, b, d, c\}$ is a regular quadruple. From the proof of Lemma 5 it follows that $(m, n) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Assume that $(m, n) = (0, 0)$. By Lemma 8 and the regularity of $\{a, b, d, c\}$, we conclude that $|z_0| = 1$ or $|z_0| = cr - st$. Since $d > 0$, we have $|z_0| = cr - st$ and $d = (z_0^2 - 1)/c = d_-$.

If $(m, n) = (0, 1)$ then Lemma 8 implies $|z_0| = cr - st$ and $d = d_-$. Analogously, if $(m, n) = (1, 0)$ then $|z_1| = cr - st$ and $d = d_-$.

Assume finally that $(m, n) = (1, 1)$. Then $z_0, z_1 < 0$ and Lemma 8 implies $z_0 = -t$, $z_1 = -s$. We have $v_1 = cr - st$ and $d = (v_1^2 - 1)/c = d_-$. ■

Corollary 3 *Let $\{a, b, c\}$ be a Diophantine triple of the third or fourth kind and $c \geq 10^{2171}$. If $\{a, b, c, d\}$ is a Diophantine quadruple, then $d = d_-$ or $d = d_+$.*

Proof. The statement follows directly from Corollary 2 since $6ab^2 < 2.695b^{3.5}$ for any Diophantine pair $\{a, b\}$. ■

9 Proof of Theorem 1

Let $\{a, b, c, d, e\}$ be a Diophantine quintuple and $a < b < c < d < e$. Consider the quadruple $\{a, b, c, d\}$. By Proposition 2, it contains a standard triple, say $\{A, B, C\}$, $A < B < C$.

If $d \geq 10^{2171}$ then we may apply Proposition 4 on the triple $\{A, B, C\}$. We conclude that

$$e = A + B + C + 2ABC + 2\sqrt{(AB + 1)(AC + 1)(BC + 1)} < 4d(bc + 1) < d^3.$$

On the other hand, by Corollary 1, we have

$$e > 2.695d^{3.5}b^{2.5} > d^3,$$

a contradiction. Hence $d < 10^{2171}$.

Consider now the quadruple $\{A, B, C, e\}$. We have $C \leq d < 10^{2171}$. Let (V_m) and (W_n) be the sequences defined in the same manner as (v_m) and (w_n), using A, B, C instead of a, b, c. Let $e \cdot C + 1 = V_m^2$. By Proposition 3,

$$\frac{m}{\log(31.3(m + 1))} < 9.561 \cdot 10^{19}$$

and $m < 5.109 \cdot 10^{21}$.

From (16) we obtain

$$V_m < 1.7C\sqrt[4]{AC}(2\sqrt{AC + 1})^{m-1} < 2^m \cdot d^{m+0.5}.$$

Therefore

$$\log_{10} V_m < m \log_{10} 2 + (m + 0.5) \cdot 2171 < 1.1094 \cdot 10^{25}$$

and

$$\log_{10} e < 2 \log_{10} V_m < 2.2188 \cdot 10^{25}.$$

Hence $e < 10^{10^{26}}$. ■

Corollary 4 *If $\{a, b, c, d, e\}$ is a Diophantine quintuple and $a < b < c < d < e$, then $d < 10^{2171}$ and $e < 10^{10^{26}}$.*

Remark 1 We can use the theorem of Baker and Wüstholz [4] instead of the theorem of Matveev [18] in the proof of Theorem 1. In that way we obtain slightly larger constants in Corollary 4, namely, $d < 10^{2411}$ and $e < 10^{10^{28}}$.

10 There does not exist a Diophantine sextuple

Since the bound for the size of elements of a Diophantine quintuple from Corollary 4 is huge, it is computationally infeasible to check whether there exist any Diophantine quintuple. However, using a theorem of Bennett [5], Theorem 3.2, on simultaneous approximations of algebraic numbers, instead of the theorem of Matveev (or Baker and Wüstholz), we are able to prove that there is no Diophantine sextuple.

Lemma 15 ([5]) *If a_i, p_i, q and N are integers for $0 \le i \le 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \le j \le 2$, q nonzero and $N > M^9$, where $M = \max_{0\le i\le 2}\{|a_i|\}$, then we have*

$$\max_{0\le i\le 2}\left\{\left|\sqrt{1+\frac{a_i}{N}}-\frac{p_i}{q}\right|\right\} > (130N\gamma)^{-1}q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(33N\gamma)}{\log\left(1.7N^2\prod_{0\le i<j\le 2}(a_i-a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1} & \text{if } a_2-a_1 \ge a_1-a_0, \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0} & \text{if } a_2-a_1 < a_1-a_0. \end{cases}$$

We will apply Lemma 15 to the numbers $\theta_1 = \frac{s}{a}\sqrt{\frac{a}{c}}$ and $\theta_2 = \frac{t}{b}\sqrt{\frac{b}{c}}$. We have

$$\begin{aligned} \theta_1 &= \sqrt{\frac{(ac+1)a}{a^2c}} = \sqrt{1+\frac{1}{ac}} = \sqrt{1+\frac{b}{abc}}, \\ \theta_2 &= \sqrt{1+\frac{1}{bc}} = \sqrt{1+\frac{a}{abc}}. \end{aligned}$$

By [9], Lemma 12, it holds

$$\max\left(|\theta_1 - \frac{sbx}{abz}|, |\theta_2 - \frac{tay}{abz}|\right) < \frac{c}{2a}\cdot z^{-2}. \tag{63}$$

In order to apply Lemma 15, we have to assume that there is a big gap between b and c. In the following two lemmas we will show that, if we assume that b and c satisfy some strong gap conditions, the upper and lower bounds obtained in Sections 7 and 8 can be significantly improved. We will show in the proof of Theorem 2 that this strong gap conditions are satisfied by the second and the fifth element of a Diophantine sextuple.

Lemma 16 *Let $\{a,b,c,d\}$, $a<b<c<d$, be a Diophantine quadruple.*

1) *If $c > 344.9\,b^{9.5}a^{3.5}$, then $d < c^{27.62}$.*

2) *If $c > 296.4\,b^{11.6}a^{1.4}$, then $d < c^{21.47}$.*

Proof. Let $ad+1=x^2$, $bd+1=y^2$, $cd+1=z^2$. We apply Lemma 15 with $a_0=0$, $a_1=a$, $a_2=b$, $N=abc$, $M=b$, $q=abz$, $p_1=sbx$, $p_2=tay$. As in the proof of [9], Corollary 1, we obtain

$$\log z < \frac{\log(32.5a^2b^6c^2)\log(1.7c^2(b-a)^{-2})}{\log(\frac{1.7c}{16.5ab^4(b-a)^2})}\,. \tag{64}$$

Assume that $c > 344.9b^{9.5}a^{3.5}$. We have

$$32.5a^2b^6c^2 < c^{2+\frac{6}{9.5}},\quad 1.7c^2(b-a)^{-2} < c^2,\quad \frac{1.7c}{16.5ab^4(b-a)^2} > c^{1-\frac{6}{9.5}}.$$

Inserting these estimates in (64), we obtain

$$\log z < \frac{2\cdot 2.632\log^2 c}{0.368\log c} < 14.31\log c\,.$$

Hence,

$$z < c^{14.31} \tag{65}$$

and $d=\frac{z^2-1}{c} < c^{27.62}$.

Assume now that $c > 296.4b^{11.6}a^{1.4}$. Then we have

$$32.5a^2b^6c^2 < 32.5a^{0.9}b^{7.1}c^2 < c^{2+\frac{7.1}{11.6}} < c^{2.613},\quad 1.7c^2(b-a)^{-2} < c^2,$$

$$\frac{1.7c}{16.5ab^6} > \frac{1.7c}{16.5a^{0.7}b^{6.3}} > c^{1-\frac{6.3}{11.6}} > c^{0.456},$$

and from (64) we obtain

$$z < c^{11.47} \tag{66}$$

and $d < c^{21.47}$. ■

Lemma 17 *Let $\{a,b,c\}$ be a Diophantine triple. Assume that $v_m=w_n$ and $n>2$.*

1) *If $c > \max\{b^{11.6}, 2.97\cdot 10^{16}\}$, then $n > c^{0.0815}$.*

2) *If $b>4a$ and $c > \max\{b^{9.5}, 1.5\cdot 10^{13}\}$, then $n > c^{0.11}$.*

Proof.

1) The proof is completely analogous to the proof of Lemma 10.

2) The proof is completely analogous to the proof of Lemma 11. ■

Proposition 5 *If $\{a,b,c,d\}$ is a Diophantine quadruple such that $c > \max\{296.4\,b^{11.6}a^{1.4}, 2.97\cdot 10^{16}\}$ or $b>4a$ and $c > \max\{334.9\,b^{9.5}a^{3.5}, 1.5\cdot 10^{13}\}$, and $d>c$, then $d=d_+$.*

Proof. Assume that $d \neq d_+$. Then $n \geq 3$ and we may apply Lemma 17.

If $c > \max\{296.4b^{11.6}a^{1.4}, 2.97 \cdot 10^{16}\}$, then $n > c^{0.0815}$. On the other hand, from (17) we have

$$z = w_n > \frac{c}{3.132\sqrt[4]{bc}}(1.999\sqrt{bc})^{n-1} > c^{\frac{n}{2}+\frac{1}{4}}.$$

From (66) we conclude that $n \leq 22$ and it implies that $c < 12.97 \cdot 10^{16}$, a contradiction.

If $b > 4a$ and $c > \max\{334.9b^{9.5}a^{3.5}, 1.5 \cdot 10^{13}\}$, then $n > c^{0.11}$. From (65) it follows that $n \leq 28$ and $c < 1.5 \cdot 10^{13}$, a contradiction. ■

Proof of Theorem 2: Let $\{a, b, c, d, e, f\}$, $a < b < c < d < e < f$, be a Diophantine sextuple. By Corollary 1, we have

$$e > 2.695d^{3.5}b^{2.5} > 2.695(4abc)^{3.5}b^{2.5} > 344.9b^{9.5}a^{3.5}.$$

If $b < 4a$ then $c \geq a + b + 2r > \frac{9}{4}b$, and we obtain

$$e > 2.695 \cdot 2.25^{3.5}b^{9.5}(4a)^{1.4}b^{2.1} > 296.4b^{11.6}a^{1.4}.$$

Assume now that

$$e > 2.97 \cdot 10^{16} \quad \text{or} \quad \Big(b < 4a \quad \text{and} \quad e > 1.5 \cdot 10^{13} \Big). \tag{67}$$

Then we may apply Proposition 5. We conclude that the quadruple $\{a, b, e, f\}$ is regular. It implies that $f \leq 4e(ab+1) < e^3$. On the other hand, from Corollary 1 we have $f > e^{3.5}$, a contradiction.

It remains to consider the case when the conditions (67) are not satisfied. If $e \leq 2.97 \cdot 10^{16}$ then $d < 4 \cdot 10^4$, and it is easy to find all quadruples satisfying $2.695d^{3.5}b^{2.5} \leq 2.97 \cdot 10^{16}$ or $b < 4a$ and $2.695d^{3.5}b^{2.5} \leq 1.5 \cdot 10^{13}$. There are exactly 10 such quadruples: $\{1, 3, 8, 120\}$, $\{1, 3, 120, 1680\}$, $\{1, 8, 15, 528\}$, $\{2, 4, 12, 420\}$, $\{3, 5, 16, 1008\}$, $\{3, 8, 21, 2080\}$, $\{4, 6, 20, 1980\}$, $\{4, 12, 30, 5852\}$, $\{5, 7, 24, 3432\}$ and $\{6, 8, 28, 5460\}$. However, we have already mentioned that Baker and Davenport [3] proved that $\{1, 3, 8\}$ cannot be extended to a quintuple. The same result was proved for the triples $\{1, 8, 15\}$, $\{1, 3, 120\}$ in [16], and for all triples of the forms $\{k-1, k+1, 4k\}$ and $\{F_{2k}, F_{2k+2}, F_{2k+4}\}$ (F_n denotes the n^{th} Fibonacci number) in [7, 8].

Therefore, it suffices to show that $\{4, 12, 30, 5852\}$ cannot be extended to a Diophantine sextuple. But it is easy to prove, using the original Baker-Davenport method, that if d is a positive integer such that $\{4, 12, 30, d\}$ is a Diophantine quadruple, then $d = 5852$. First of all, in this case we have $z_0 = z_1 = \pm 1$. If we apply Lemma 14 on the form (60) with $(a, b, c) = (4, 12, 30)$, we obtain $m < 2 \cdot 10^{15}$. Now we may apply the Baker-Davenport reduction method [3] (see also [10], Lemma 5). In the first step of the reduction we obtain $m \leq 7$. The second step gives $m \leq 1$ if $z_0 = 1$, and $m \leq 2$ if $z_0 = -1$, which proves that $d = 5852$. Namely, $d = 5852$ corresponds to $z_0 = -1$ and $m = n = 2$. ■

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