A PROBLEM OF DIOPHANTUS AND PELL NUMBERS

Andrej Dujella
Department of Mathematics, University of Zagreb, 10000 Zagreb, CROATIA

1 Introduction

The Greek mathematician Diophantus of Alexandria noted that the set $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ has the following property: the product of its any two distinct elements increased by 1 is a square of a rational number (see [3]). Fermat first found a set of four positive integers with the above property, and it was $\{1, 3, 8, 120\}$. In 1969, Davenport and Baker [2] showed that if positive integers 1, 3, 8 and d have this property then d must be 120.

Let n be an integer. A set of positive integers $\{a_1, a_2, \ldots, a_m\}$ is said to have *the property of Diophantus of order* n, symbolically $D(n)$, if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. The sets with the property $D(l^2)$ were particularly discussed in [4]. It was proved that for any integer l and any set $\{a, b\}$ with the property $D(l^2)$, where ab is not a perfect square, there exist an infinite number of sets of the form $\{a, b, c, d\}$ with the property $D(l^2)$. This result is the generalization of well known result for $l = 1$ (see [8]). The proof of this result is based on the construction of the double sequences $y_{n,m}$ and $z_{n,m}$ which are defined in [4] by second order recurrences in each indices. Solving these recurrences we obtain

$$\begin{aligned}
y_{n,m} &= \frac{l}{2\sqrt{b}}\{(\sqrt{a}+\sqrt{b})[\frac{1}{l}(k+\sqrt{ab})]^n(s+t\sqrt{ab})^m \\
&+ (\sqrt{b}-\sqrt{a})[\frac{1}{l}(k-\sqrt{ab})]^n(s-t\sqrt{ab})^m\}, \\
z_{n,m} &= \frac{l}{2\sqrt{a}}\{(\sqrt{a}+\sqrt{b})[\frac{1}{l}(k+\sqrt{ab})]^n(s+t\sqrt{ab})^m \\
&+ (\sqrt{a}-\sqrt{b})[\frac{1}{l}(k-\sqrt{ab})]^n(s-t\sqrt{ab})^m\},
\end{aligned}$$

where s and t are positive integers satisfying the Pellian equation $s^2 - abt^2 = 1$. The desired quadruples have the form $\{a, b, x_{n,m}, x_{n+1,m}\}$, where

$$x_{n,m} = (y_{n,m}^2 - l^2)/a = (z_{n,m}^2 - l^2)/b\,.$$

In [5], using the above construction, some Diophantine quadruples for the squares of Fibonacci and Lucas numbers are obtained. In [6], similar results are obtained for some classes of generalized Fibonacci numbers $w_n = w_n(a, b; p, q)$, defined as follows:

$$w_0 = a,\ w_1 = b, \quad w_n = pw_{n-1} - qw_{n-2} \quad (n \geq 2).$$

The properties of that sequence were discussed in detail in [10], [11] and [12].

In present paper we will apply the results from [6] to Pell numbers $P_n = w_n(0, 1; 2, -1)$ and Pell-Lucas numbers $Q'_n = 2Q_n = w_n(2, 2; 2, -1)$.

2 Properties of the sequence $x_{n,m}$

In this section we repeat the relevant material from [4] with some improvements.

Theorem 1 *For all integers m and n the product of any two distinct elements of the set $\{a, b, x_{n,m}, x_{n+1,m}\}$ increased by l^2 is a square of a rational number. If m is an integer and $n \in \{-1, 0, 1\}$, then $x_{n,m}$ is the integer.*

Proof: See [4]. ∎

Remark 1 From Theorem 1 it follows that if $x_{\pm 1,0}$ and $x_{\pm 2,0}$ are positive integers then the set $\{a, b, x_{\pm 1,0}, x_{\pm 2,0}\}$ has the property $D(l^2)$. Note that $x_{\pm 1,0} = a + b \pm 2k$ and $x_{\pm 2,0} = \pm 4k(k \pm a)(k \pm b)/l^2$.

Theorem 2 *If $n \in \{-1, 0, 1\}$ and $(n, m) \notin \{(-1, 0), (-1, 1), (0, -1), (0, 0), (1, -2), (1, -1)\}$, then $x_{n,m}$ is the positive integer greater than b.*

Proof: We have

$$\begin{aligned} x_{n,m+3} - x_{n,m} &= \frac{1}{a}(y_{n,m+3} + y_{n,m})(y_{n,m+3} - y_{n,m}) \\ &= \frac{1}{a}(2s-1)(y_{n,m+2} + y_{n,m+1})(2s+1)(y_{n,m+2} - y_{n,m+1}) \\ &= (4s^2 - 1)(x_{n,m+2} - x_{n,m+1})\,. \end{aligned} \tag{1}$$

We conclude from $y_{0,1} = l(s + at) > k$ and $y_{0,-1} = l(s - at) \geq l$ that $x_{0,1} > b$ and $x_{0,-1} \geq 0$. We will prove by induction that for $m \geq 1$ it holds:

$$x_{0,m+1} \geq (4s^2 - 3)x_{0,m}\,.$$

For $m = 1$ the assertion follows from (1). Assume the assertion holds for the positive integer m. Then from (1) and $s \geq 2$ it follows that

$$\begin{aligned} x_{0,m+2} &= (4s^2 - 1)(x_{0,m+1} - x_{0,m}) + x_{0,m-1} \\ &\geq (4s^2 - 1)(1 - \frac{1}{4s^2 - 3})x_{0,m+1} \geq (4s^2 - 3)x_{0,m+1}\,. \end{aligned}$$

In the same manner we can see that for $m \le -1$ it holds: $x_{0,m-1} \ge (4s^2-3)x_{0,m}$. Since $x_{0,-2} = x_{0,1} + (4s^2-1)(x_{0,-1} - x_{0,0}) \ge x_{0,1} > b$, we conclude that $x_{0,m} > b$ for $m \notin \{-1, 0\}$.

It is easy to prove by induction that for every integer m it holds:

$$y_{1,m} = \frac{1}{l}(ky_{0,m} + az_{0,m}), \qquad y_{-1,m} = \frac{1}{l}(ky_{0,m} - az_{0,m}). \tag{2}$$

Therefore, if

$$ky_{0,m} - a|z_{0,m}| > kl\,, \tag{3}$$

then $x_{1,m}$ and $x_{-1,m}$ are integers greater than b. The condition (3) is equivalent to $x_{0,m} > w$, where

$$w = \frac{1}{l^2}[al^2 + 2k^2b + 2k^2\sqrt{l^2+b^2}]\,.$$

Since $w = a + (2 + \frac{2ab}{l^2})(b + \sqrt{l^2+b^2}) < a + 2b + b(a+2) + 2ab(2b+1) < 4ab(b+2)$, it suffices to hold

$$x_{0,m} > 4ab(b+2).$$

Note that $x_{0,2} \ge (4s^2-3)x_{0,1}$ and $x_{0,-3} \ge (4s^2-3)x_{0,-2} \ge (4s^2-3)x_{0,1}$. Furthermore,

$$\begin{aligned}(4s^2-3)x_{0,1} &> 4(s^2-1)x_{0,1} = 4abt^2\frac{l^2(s+at)^2 - l^2}{a} = 4abt^2 \cdot l^2t(2s+at+bt) \\ &> 4ab(b+2).\end{aligned}$$

Hence, if m is an integer such that $m \ge 2$ or $m \le -3$, then $x_{0,m} > w$.

Thus, we proved that if $n \in \{-1, 1\}$ and $(n,m) \notin \{(-1,-2), (-1,-1), (-1,0), (-1,1), (1,-2), (1,-1), (1,0), (1,1)\}$ then the integer $x_{n,m}$ is greater than b. But the integers $x_{1,0} = a + b + 2k$ and $x_{1,1} = (tk+s)[(at+bt+2s)k + (as+bs+2bt)]$ are obviously greater than b. Furthermore, from $y_{-1,-1} = s(k-a) - at(k-b) = k(s-at) + a(bt-s) \ge k + a > k$ we obtain $x_{-1,-1} > b$. Since $z_{-1,-1} = k(s-bt) + b(at-s) < 0$, the relation $y_{n,m-1} = sy_{n,m} - atz_{n,m}$ implies $y_{-1,-2} > y_{-1,-1} > k$ and $x_{-1,-2} > b$. This completes the proof. ■

See [4, Example 3] for the illustration of situation where $x_{n,m} = 0$ for all $(n,m) \in \{(-1,0), (-1,1), (0,-1), (0,0), (1,-2), (1,-1)\}$.

Theorem 3 *Let l be an integer and let $\{a,b\}$ be the Diophantine pair with the property $D(l^2)$. If the integer ab is not a perfect square then there exist an infinite number of Diophantine quadruples of the form $\{a,b,c,d\}$ with the property $D(l^2)$.*

Proof: We will show that the sets $\{a, b, x_{0,m}, x_{-1,m}\}$, $m \notin \{-1, 0, 1\}$, and $\{a, b, x_{0,m}, x_{1,m}\}$, $m \notin \{-2, -1, 0\}$, are Diophantine quadruples with the property $D(l^2)$.

By Theorems 1 and 2, it suffices to prove that $x_{0,m} \neq x_{-1,m}$ and $x_{0,m} \neq x_{1,m}$ respectively. Let us first observe that $y_{0,m} > 0$ and (2) implies $y_{1,m} > 0$ and $y_{-1,m} > 0$. If $x_{0,m} = x_{\pm 1,m}$ then $y_{0,m}^2 = \frac{al(b-a)}{2(k-l)}$. From $y_{0,m} > k$ we obtain $(k-l)^2(2k+l) + a^2 l < 0$, which is impossible. This proves that the above sets are Diophantine quadruples. There is an infinite number of distinct quadruples between them, since $x_{0,m+1} > x_{0,m}$ for $m \geq 1$. ■

3 Diophantine quadruples and Pell numbers

In this section we construct several Diophantine quadruples represented in terms of Pell numbers P_n and Pell-Lucas numbers $Q'_n = 2Q_n$. These numbers are defined by

$$\begin{aligned} P_0 = 0,\ P_1 = 1,\quad P_{n+2} &= 2P_{n+1} + P_n, \quad n \geq 0; \\ Q_0 = 1,\ Q_1 = 1,\quad Q_{n+2} &= 2Q_{n+1} + Q_n, \quad n \geq 0. \end{aligned}$$

We will start with the analogs of the fact that the sets

$$\{n, n+2, 4(n+1), 4(n+1)(2n+1)(2n+3)\},$$

$$\{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\}$$

have the property $D(1)$ (see [13], [9], [14]).

Theorem 4 *For every positive integer n, the sets*

$$\{P_{2n},\ P_{2n+2},\ 2P_{2n},\ 4Q_{2n}P_{2n+1}Q_{2n+1}\},$$

$$\{P_{2n},\ P_{2n+2},\ 2P_{2n+2},\ 4P_{2n+1}Q_{2n+1}Q_{2n+2}\}$$

have the property $D(1)$, the sets

$$\{P_{2n},\ P_{2n+4},\ 4P_{2n+2},\ 4P_{2n+1}P_{2n+2}P_{2n+3}\},$$

$$\{P_{2n},\ P_{2n+4},\ 8P_{2n+2},\ 4Q_{2n+1}P_{2n+2}Q_{2n+3}\}$$

have the property $D(4)$ and the set

$$\{P_{2n},\ P_{2n+8},\ 36P_{2n+4},\ P_{2n+2}P_{2n+4}P_{2n+6}\}$$

has the property $D(144)$.

Since $P_{2n} = 2w_n(0,1;6,1)$, we obtain from [6, (9)] the identity:

$$P_{2n-4}P_{2n-2}P_{2n+2}P_{2n+4} + [(12 \pm 2)P_{2n}]^2 = (P_{2n}^2 \pm 24)^2,$$

and the following theorem can be proved using the construction from Remark 1 (see [6, Theorem 5]).

Theorem 5 *For every integer $n \geq 3$, the set*

$$\{P_{2n-4}P_{2n-2}, P_{2n+2}P_{2n+4}, 196P_{2n}^2, 4P_{2n-3}P_{2n+3}(P_{2n}^2+24)\}$$

has the property $D(196P_{2n}^2)$, and the set

$$\{P_{2n-4}P_{2n-2}, P_{2n+2}P_{2n+4}, 200P_{2n}^2, 4Q_{2n-3}Q_{2n+3}(P_{2n}^2-24)\}$$

has the property $D(100P_{2n}^2)$.

If we have the pair of identities of the form: $ab+l^2=k^2$ and $s^2-abt^2=1$, then we can construct the sequence $x_{n,m}$ and obtain an infinite number of Diophantine quadruples with the property $D(l^2)$. There are several pairs of identities for Pell and Pell-Lucas numbers which have the above form. For example,

$$P_{n-1}P_{n+1}+P_n^2 = Q_n^2, \tag{4}$$
$$(P_n^2+P_{n-1}P_{n+1})^2-4P_{n-1}P_{n+1}P_n^2 = 1 \tag{5}$$

and

$$Q_{n-1}Q_{n+1}+Q_n^2 = 4P_n^2, \tag{6}$$
$$4P_n^4-Q_{n-1}Q_{n+1}Q_n^2 = 1. \tag{7}$$

Applying the construction of section 2 to these pairs of identities we get

Theorem 6 *For every positive integer $n \geq 2$, the sets*

$$\{P_{n-1},\, P_{n+1},\, 4Q_{n-1}P_n^3Q_n,\, 4P_n^3Q_nQ_{n+1}\}, \tag{8}$$

$$\{P_{n-1},\, P_{n+1},\, 4P_n^3Q_nQ_{n+1},\, 4Q_nP_{n+1}Q_{n+1}(P_{n+1}^2-P_nP_{n+1}-P_n^2)\} \tag{9}$$

have the property $D(P_n^2)$, and the sets

$$\{Q_{n-1},\, Q_{n+1},\, 4P_{n-1}P_nQ_n^3,\, 4P_nQ_n^3P_{n+1}\}, \tag{10}$$

$$\{Q_{n-1},\, Q_{n+1},\, 4P_nQ_n^3P_{n+1},\, 4P_nP_{n+1}Q_{n+1}(P_nP_{n+2}-P_{n-1}P_{n+1})\} \tag{11}$$

have the property $D(Q_n^2)$.

Theorem 7 *For every positive integer $n \geq 3$, the sets*

$$\{P_{n-2},\, P_{n+2},\, 4P_{n-1}P_{n+1}Q_n^2P_{n+1},\, 4Q_{n-1}P_nQ_n^2Q_{n+1}\},$$

$$\{P_{n-2},\, P_{n+2},\, 4Q_{n-1}P_nQ_n^2Q_{n+1},\, 16Q_{n-1}P_nQ_{n+1}(2P_n^2-P_{n-1}P_{n+1})\}$$

have the property $D(4Q_n^2)$.

Proof: The proof is based of the following identities:

$$P_{n-2}P_{n+2}+4Q_n^2 = 9P_n^2, \tag{12}$$
$$(3P_n^2-2P_{n-1}P_{n+1})^2-P_{n-2}P_{n+2}P_n^2 = 4. \tag{13}$$

Dividing both sides of the identity (13) by 4, we can set $a=P_{n-2}$, $b=P_{n+2}$, $l=2Q_n$, $k=3P_n$, $s=\frac{1}{2}(3P_n^2-2P_{n-1}P_{n+1})$ and $t=\frac{1}{2}P_n$.

We have

$$y_{0,0}=z_{0,0}=3P_n,\ \ y_{1,0}=3P_n+P_{n-2},\ \ z_{1,0}=3P_n+P_{n+2},$$
$$y_{-1,0}=3P_n-P_{n-2},\ \ z_{-1,0}=3P_n-P_{n+2}.$$

To simplify notation, we write $P_{n+1}=A$, $P_n=B$. This gives

$$(A^2-2AB-B^2)^2=1\,. \tag{14}$$

We now have

$$\begin{aligned} y_{0,1} &= sy_{0,0}+atz_{0,0}=2(A-B)(4B^2+AB-A^2) \\ y_{1,1} &= 2(A^3-7A^2b+7AB^2+11B^3) \\ y_{-1,1} &= 2(A-B)(3AB-A^2-B^2)\,, \end{aligned}$$

and, by (14), we get

$$\begin{aligned} x_{0,1} &= [{y_{0,1}}^2-l^2(A^2-2AB-B^2)^2]/a=4B(A-B)^2(A+B)(3B-A) \\ &= 4P_nQ_n^2Q_{n+1}Q_{n-1} \\ x_{1,1} &= 16B(A+B)(3B-A)(2B^2+2AB-A^2) \\ &= 16P_nQ_{n+1}Q_{n-1}(2P_n^2-P_{n-1}P_{n+1}) \\ x_{-1,1} &= 4AB(A-2B)(A-B)^2 \\ &= 4P_nP_{n+1}P_{n-1}Q_n^2\,, \end{aligned}$$

which proves the theorem. ■

4 Diophantine quintuples

It was proved in [7] that for every Diophantine quadruple $\{x_1,x_2,x_3,x_4\}$ with the property $D(l^2)$ such that $x_1x_2x_3x_4\neq l^4$, there exist a positive rational number x_5 with the property that $x_ix_5+l^2$ is a square of a rational number for $i=1,2,3,4$. This construction generalizes that of [1]. However, on the quadruples in this paper these two constructions coincide. We proceed with an example.

Example 1 From (6) it follows that the set $\{1, Q_{n-1}P_n^2Q_{n+1}\}$ has the property $D(P_n^2Q_n^2)$. For $a=1$, $b=Q_{n-1}P_n^2Q_{n+1}$, $k=2P_n^2$ and $l=P_nQ_n$ we get $x_{1,0}=P_{n-1}Q_n^2P_{n+1}$, $x_{2,0}=8P_n^2Q_n^2$. The constructions from [1] and [7] on the set $\{a,b,x_{1,0},x_{2,0}\}$ give the rational number $\frac{6P_{2n}^2(Q_{2n}^2-4)}{(Q_{2n}^2-10)^2}$. Hence, for every integer $n \geq 2$ the set

$$\{(Q_{2n}^2-10)^2, Q_{n-1}P_n^2Q_{n+1}(Q_{2n}^2-10)^2, P_{n-1}Q_n^2P_{n+1}(Q_{2n}^2-10)^2, \\ 2P_{2n}^2(Q_{2n}^2-10)^2, 6P_{2n}^2(Q_{2n}^2-4)\} \tag{15}$$

is the Diophantine quintuple with the property $D(P_n^2Q_n^2(Q_{2n}^2-10)^4)$. From (15) for $n=2$ we get the Diophantine quintuple $\{961, 3040, 26908, 43245, 276768\}$ with the property $D(36\cdot 31^4)$.

One question still unanswered is whether exists a Diophantine quintuple with the property $D(1)$. Therefore one may ask which is the least positive integer n_1, and which is the greatest negative integer n_2, for which there exists a Diophantine quintuple with the property $D(n_i)$, $i=1,2$. It holds: $n_1 \leq 256$ and $n_2 \geq -255$, since the sets $\{1, 33, 105, 320, 18240\}$ and $\{5, 21, 64, 285, 6720\}$ have the property $D(256)$, and the set $\{8, 32, 77, 203, 528\}$ has the property $D(-255)$.

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