Extensions of a Diophantine triple by adjoining smaller elements

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Abstract. In this paper, we prove that if $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples with $a_1 < a_2 < b < c < d$, then $a_2 > 24^3$, $a_2 > \max\{36a_1^3, 300a_1^2\}$, $b < a_2^{3/2}$, and $16a_1^2 b^3 < c < 16a_2 b^3$. The last inequalities imply that for a fixed Diophantine triple $\{b, c, d\}$ the number of Diophantine quadruples $\{a, b, c, d\}$ with $a < \min\{b, c, d\}$ is at most two. Moreover, we show that there are only finitely many quintuples $\{a_1, a_2, b, c, d\}$ as above.

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1. Introduction

Throughout this paper, m positive integers a_i $(1 \le i \le m)$ with the property that $a_i a_j + 1$ is a perfect square for all $1 \le i < j \le m$ are collectively referred to as *Diophantine m–tuple*. After the second author has proved in [12] that the definition above is satisfied for no m–tuple with $m \ge 6$, in [23] it is shown that it necessarily holds $m \le 4$. For the sake of convenience, when $m = 3$ or 4 we shall speak of Diophantine triple or quadruple, respectively.

A successful strategy in the study of Diophantine tuples is based on the idea of enlarging a given such set by finding an extra element with the required property. Most of the published works are devoted to extensions of Diophantine triples by adjoining a fourth element greater than the three already known. A possible explanation for this preference could be the existence of a neat formula giving a legitimate extension for any Diophantine triple $\{a, b, c\}$ (see [1] or [22]), namely

$$d = a + b + c + 2abc + 2rst,$$

where r, s, t are positive integers defined by relations $ab+1 = r^2$, $ac+1 = s^2$, $bc + 1 = t^2$. This integer d is the greatest root of the quadratic equation

$$(X + c - a - b)^2 = 4(ab + 1)(cX + 1), \tag{1.1}$$

whence the usual notation d_+ for it. Diophantine quadruples of the type $\{a, b, c, d_+\}$ are called *regular*. The currently open question of greatest interest in this area is the status of the next conjecture, which was posed implicitly in [1] or [22], and explicitly in [11].

Conjecture 1.1. *Any Diophantine triple* $\{a, b, c\}$ *has unique extension to a Diophantine quadruple* $\{a, b, c, d\}$ *by an element* $d > \max\{a, b, c\}$.

There are known many specific contexts in which this conjecture is valid, see, for instance, [3], [9], [10], [17], [18] for the most general results of the kind. In a different vein, in [10] it is shown that any Diophantine triple admits at most 8 extensions by an integer greater than the three given elements. A complete bibliography on Diophantine sets is maintained by the second author [13].

The other root of equation (1.1) is a non-negative integer d_-, smaller than $\max\{a, b, c\}$, for which all $ad_- + 1$, $bd_- + 1$, $cd_- + 1$ are perfect squares. Thus, when d_- is positive, one can produce a different Diophantine quadruple $\{a, b, c, d_-\}$ out of $\{a, b, c\}$. It is easy to verify that this happens precisely when $c > a + b + 2r$, with the convention that $a < b < c$. It is to be noted that quite recently, [20] studied the number of ways of extending a fixed Diophantine pair or triple to irregular Diophantine quadruples obtained by adjoining either smaller or larger elements than the given ones.

In the previous work, the present authors have initiated in [5] the study of extendibility of Diophantine triples by an integer smaller than all elements of the initial triple. They put forward a statement similar to Conjecture 1.1.

Conjecture 1.2. *Suppose that* $\{a_1, b, c, d\}$ *is a Diophantine quadruple with* $a_1 < b < c < d$. *Then,* $\{a_2, b, c, d\}$ *is not a Diophantine quadruple for any integer* a_2 *with* $a_1 \neq a_2 < b$.

Its validity is established for $c < 16b^3$ in [5]. In the same paper it is proved that there exists no Diophantine quadruple $\{a_1, b, c, d\}$ with $a_1 < b < c < d$ such that the quadruple $\{a_1+1, b, c, d\}$ is Diophantine as well. Moreover, as a consequence of Theorem 1.4 from [10], one sees that Conjecture 1.2 holds when $c \geq 200b^4$.

The aim of the present work is to point out further necessary conditions met by hypothetical counterexamples to our conjecture. The main findings can be summarized as follows.

Main Theorem. *Assume that* $\{a_1, b, c, d\}$ *and* $\{a_2, b, c, d\}$ *are Diophantine quadruples with* $a_1 < a_2 < b < c < d$. *Then, the following hold*:

(1) $a_2 > \max\{36a_1^3, 300a_1^2\}$.
(2) $b < a_2^{3/2}$ *for* $a_1 \geq 1$, *and* $b < a_2^{4/3}$ *for* $a_1 \geq 2$ *or* $a_1 = 1$ *and* $a_2 < 4 \cdot 10^5$.
(3) $a_2 > 24^3 = 13824$.
(4) $16a_1^2b^3 < c < 16a_2b^3$.

We will prove Main Theorem step by step in Sections 5 to 8. Indeed, Theorems 6.1, 6.2, 8.5 and Propositions 8.2, 8.3, 8.4, 8.6 and 8.7 together imply Main Theorem.

Let a, b, r with $a < r < b$ be positive integers such that $ab + 1 = r^2$. Following [24], we define an integer $c_\nu^\tau = c_\nu^\tau(a, b)$ by

$$c_\nu^\tau = \frac{1}{4ab}\left\{(\sqrt{b}+\tau\sqrt{a})^2(r+\sqrt{ab})^{2\nu}+(\sqrt{b}-\tau\sqrt{a})^2(r-\sqrt{ab})^{2\nu}-2(a+b)\right\} \tag{1.2}$$

with ν a positive integer and $\tau \in \{\pm\}$. Observe that if $b > a+2$, then $\{a, b, c_\nu^\tau\}$ is always a Diophantine triple for any ν and any τ.

The next corollary follows immediately from results obtained in the course of proving Main Theorem.

Corollary 1.3. *If $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples with $a_1 < a_2 < b < c < d$, then $c \neq c_\nu^\tau(a_2, b)$ for any positive integer ν and any $\tau \in \{\pm\}$. In particular, $b > 13a_2$.*

Main Theorem also has the following consequences.

Corollary 1.4. *Let $\{b, c, d\}$ be a Diophantine triple. Then, there exist at most two positive integers a with $a < \min\{b, c, d\}$ such that $\{a, b, c, d\}$ is a Diophantine quadruple.*

Corollary 1.5. *There are only finitely many quintuples $\{a_1, a_2, b, c, d\}$ with $a_1 < a_2 < b < c < d$ such that $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples.*

In fact, it can be shown that the largest element d satisfies $d < 10^{10^{26}}$.

Corollary 1.6. *Conjecture* 1.1 *implies Conjecture* 1.2.

Our proofs are based on three results from literature. A first one serves to bound from above the solutions of a relevant system of Pellian equations. The so-called method of hypergeometric functions provides several results of the kind, we shall prefer that established in [7] and recalled in Section 3 below. Another essential ingredient is a particular instance of the observation that in a Diophantine triple whose two smallest elements are very close to each other, the largest element has a standard form. Such a result facilitates explicit calculations we perform. The third result already available and which is extensively employed in our proofs gives an absolute lower bound for the second smallest entry in an irregular quadruple.

The paper is organized as follows. In the next two sections we fix notation employed throughout the paper and adapt several useful results from literature to our specific needs. Section 4 contains the main technical novelty. More precisely, Lemma 4.2 gives a much better lower bound for solutions of a relevant system of equations than the corresponding published results. Therefore, we think Lemma 4.2 might be of independent interest. After proving a lighter version of Main Theorem in Sections 5 and 6, we derive the corollaries stated above. The rest of the paper is devoted to completing the proof of Main Theorem as stated above.

2. Preliminaries

From now on we assume that $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples with $a_1 < a_2 < b < c < d$. Let $r_1, r_2, s_1, s_2, t, x_1, x_2, y, z$ be positive integers satisfying

$$a_1b + 1 = r_1^2,\ a_2b + 1 = r_2^2,\ a_1c + 1 = s_1^2,\ a_2c + 1 = s_2^2,\ bc + 1 = t^2,$$
$$a_1d + 1 = x_1^2,\ a_2d + 1 = x_2^2,\ bd + 1 = y^2,\ cd + 1 = z^2.$$

Considering x_1, x_2, y, z as unknowns, we obtain the following system of Pellian equations:

$$a_1z^2 - cx_1^2 = a_1 - c, \tag{2.1}$$
$$a_2z^2 - cx_2^2 = a_2 - c, \tag{2.2}$$
$$bz^2 - cy^2 = b - c. \tag{2.3}$$

By [21, Theorem 1.3] and [10, Lemma 2.3], any positive integer solution to (2.2) and (2.3) can be expressed as $z = v_m = w_n$ for some non-negative integers m, n, where $\{v_m\}$ and $\{w_n\}$ are recurrent sequences defined by

$$v_0 = z_{(0)},\ v_1 = s_2z_{(0)} + cx_{(0)},\ v_{m+2} = 2s_2v_{m+1} - v_m, \tag{2.4}$$
$$w_0 = z_{(1)},\ w_1 = tz_{(1)} + cy_{(1)},\ w_{n+2} = 2tw_{n+1} - w_n \tag{2.5}$$

satisfying either of the following:

(i) $m \equiv n \equiv 0 \pmod 2$ and $x_{(0)} = y_{(1)} = |z_{(0)}| = |z_{(1)}| = 1$ with $z_{(0)}z_{(1)} > 0$;
(ii) $m \equiv n \equiv 1 \pmod 2$ and $x_{(0)} = y_{(1)} = r_2$, $|z_{(0)}| = t$, $|z_{(1)}| = s_2$ with $z_{(0)}z_{(1)} > 0$.

In the next two sections we bound from above and from below the indices m and n in terms of entries of the Diophantine quadruples in question. To this end, we need an experimental result giving absolute lower bound for the second smallest element of an irregular Diophantine quadruple.

Lemma 2.1. ([7, Lemma 3.4]) *Let $\{a, b, c, d\}$ be an irregular Diophantine quadruple with $a < b < c < d$. Then:*

(1) *If $b \le 2a$, then $b > 21000$.*
(2) *If $2a < b \le 8a$, then $b > 130000$.*
(3) *If $b > 8a$, then $b > 4000$.*

Let a, b, r with $a < r < b$ be positive integers such that $ab + 1 = r^2$, and define an integer $c_\nu^\tau = c_\nu^\tau(a, b)$ with ν a positive integer and $\tau \in \{\pm\}$ by (1.2). Note that for $b = a + 2$ one has $c_1^- = 0$ and $c_{\nu+1}^- = c_\nu^+$. From the explicit formulas

$$c_1^\tau = a + b + 2\tau r,$$
$$c_2^\tau = 4ab(a + b + 2\tau r) + 4(a + b + \tau r),$$
$$c_3^\tau = 16a^2b^2(a + b + 2\tau r) + 8ab(3a + 3b + 4\tau r) + 3(3a + 3b + 2\tau r)$$

it is seen that for $b > a + 2$ it holds

$$c_2^- < 4ab^2 < c_2^+ < 16ab^2 < c_3^- < 16a^2b^3 < c_3^+ < c_4^- < \dots . \tag{2.6}$$

In [24, Theorem 8] it is shown that in any Diophantine triple $\{a, b, c\}$ with $a < b \le 4a$ one has $c = c_\nu^\tau$ for some ν and τ. The same conclusion holds on a wider range, as seen from [16, Lemma 4.1], where it was shown for $b \le 8a$, or the proof of Corollary 1.6 in [10], where the hypothesis is $b \le 13a$. Imposing a mild additional condition, the property is valid for a different range, including the interval $a^2 \le b \le 4a^2$ (see [8, Lemma 3.1]). In this paper we shall use a very recent result from the same family.

Lemma 2.2. ([25, Lemma 3.1]) *Let $\{a, b, c\}$ be a Diophantine triple and $a < b \le 24a$. Suppose that $\{1, 3, a, b\}$ is not a Diophantine quadruple. Then $c = c_\nu^\tau(a, b)$ for some ν and τ.*

Note that when $\{1, 3, a, b\}$ is a Diophantine quadruple, then $a = c_\nu^\tau(1, 3)$ by [24, Theorem 8] and $b = d_+(1, 3, a)$ by [14]. It can be seen that in this case a and b are consecutive terms of the sequence $(c_k)_{k\ge1}$ with $c_k = s_k^2 - 1$, where $s_0 = 1$, $s_1 = 3$, and $s_{k+2} = 4s_{k+1} - s_k$ for $k \ge 0$. Explicitly, the sequence $(c_k)_{k\ge1}$ starts with

$$8, 120, 1680, 23408, 326040, 4541160, 63250208, \dots . \tag{2.7}$$

3. Upper bounds for solutions

Theorem 3.1. *Let a_1, a_2 and N be integers with $0 < a_1 < a_2$, $a_2 \ge 5$ and $N \ge 3.804a_1'a_2^2(a_2 - a_1)^2$, where $a_1' = \max\{a_2 - a_1, a_1\}$. Assume that N is divisible by a_1a_2. Then, the numbers $\theta_1 = \sqrt{1 + a_2/N}$ and $\theta_2 = \sqrt{1 + a_1/N}$ satisfy*

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > \frac{a_1}{1.435 \cdot 10^{28} a_1' a_2 N} q^{-\lambda}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda = 1 + \frac{\log(10a_1^{-1}a_1'a_2N)}{\log(2.629a_1^{-1}a_2^{-1}(a_2 - a_1)^{-2}N^2)} < 2.$$

Proof. The validity of the assertion is obvious from the proof of [7, Theorem 2.1] with g replaced by 1. □

One can easily prove the following two lemmas in the same ways as [11, Lemma 12] and [19, Lemma 25] (cf. [11, Theorem 3]), respectively.

Lemma 3.2. *Let $N = a_1a_2c$ and let θ_1, θ_2 be as in Theorem 3.1. Then, all positive solutions to the system of Pellian equations (2.1) and (2.2) satisfy*

$$\max\left\{\left|\theta_1 - \frac{s_1a_2x_1}{a_1a_2z}\right|, \left|\theta_2 - \frac{s_2a_1x_2}{a_1a_2z}\right|\right\} < \frac{c}{2a_1} z^{-2}.$$

Lemma 3.3. *Assume that $c \geq 4a_2b^3$. If $z = w_n$ with $n \geq 4$, then*

$$\log z > \frac{n}{2}\log(4bc).$$

Lemma 3.4. *Assume that $c \geq 4a_2b^3$. If $z = v_m = w_n$ has a solution for some integers m and n with $n \geq 4$, then*

$$n < \frac{8\log(8.471\cdot 10^{13}a_1^{1/2}(a_1')^{1/2}a_2^2c)\log(1.622a_1^{1/2}a_2^{1/2}(a_2-a_1)^{-1}c)}{\log(4bc)\log(0.2629a_1(a_1')^{-1}a_2^{-1}(a_2-a_1)^{-2}c)}.$$

Proof. Note that it holds $a_2 - a_1 \geq 3$. Indeed, [5, Theorem 1.3] assures that $a_2 - a_1 \neq 1$ and for $a_2 - a_1 = 2$, $\{a_1, a_2, b, c, d\}$ would be a Diophantine quintuple, which contradicts [18, Corollary 2] or [23, Theorem 1]. In particular, we may assume $a_2 \geq 5$. Actually, in the only possible case not covered by the claim just proved one has $a_1 = 1$ and $a_2 = 4$. However, then, $b + 1 = u^2$ and $4b + 1 = v^2$ together yield $v^2 - 4u^2 = -3$, which has no solution in positive integers other than $(v, u) = (1, 1)$.

Since $a_1a_2c \geq 4a_1a_2^2b^3 > 3.804a_2^5$, we may apply Theorem 3.1 with

$$q = a_1a_2z,\ p_1 = s_1a_2x_1,\ p_2 = s_2a_1x_2 \text{ and } N = a_1a_2c.$$

Combining Theorem 3.1 with Lemma 3.2, one gets

$$z^{2-\lambda} < 0.7175\cdot 10^{28}a_1a_1'a_2^4c^2 < \left(8.471\cdot 10^{13}a_1^{1/2}(a_1')^{1/2}a_2^2c\right)^2$$

and

$$\frac{1}{2-\lambda} = \frac{\log(2.629a_1a_2(a_2-a_1)^{-2}c^2)}{\log\left(\frac{2.629a_1(a_2-a_1)^{-2}c}{10a_1'a_2}\right)} < \frac{2\log(1.622a_1^{1/2}a_2^{1/2}(a_2-a_1)^{-1}c)}{\log(0.2629a_1(a_1')^{-1}a_2^{-1}(a_2-a_1)^{-2}c)}.$$

The assertion now follows from comparing the above inequalities with those in Lemma 3.3. □

4. Lower bounds for solutions

Lemma 4.1. *Assume that $z = v_m = w_n$ has a solution for some integers m and n. If $c > b^3$, then*

$$m \leq \frac{4}{3}n + \frac{2}{3} \quad \textit{and} \quad m \leq 1.4n.$$

Proof. The first inequality can be easily shown in a similar fashion to [12, Lemma 4]. The second one is a direct consequence of the first if $n \geq 10$ and follows by simple computations in the few remaining cases. □

Lemma 4.2. *Assume that $c > \max\{b^3, a_2^2b^2\}$ and $b > 4000$. If $z = v_m = w_n$ has a solution for some integers m and n with $n \geq 4$, then $m \equiv n \pmod 2$ and $n > b^{-1/2}c^{1/4}$ for odd n and $n > \frac{5}{7}b^{-1/2}c^{1/2}$ for even n.*

Proof. In the case (i) $m \equiv n \equiv 0 \pmod 2$ and $|z_{(0)}| = |z_{(1)}| = 1$ with $z_{(0)}z_{(1)} > 0$, [4, Lemma 2.4] and its proof show that $m > b^{-1/2}c^{1/2}$. It follows from $c > b^3$ and Lemma 4.1 with $n \geq 4$ that

$$n \geq \frac{5}{7}m > \frac{5}{7}b^{-1/2}c^{1/2}.$$

In the case (ii) $m \equiv n \equiv 1 \pmod 2$ and $|z_{(0)}| = t$, $|z_{(1)}| = s$ with $z_{(0)}z_{(1)} > 0$, as in the proof of [21, Lemma 3.1], we have

$$\pm t\{a_2(m^2-1) - b(n^2-1)\} \equiv 2r_2s_2(n-m) \pmod{8c}, \tag{4.1}$$
$$\pm s_2\{a_2(m^2-1) - b(n^2-1)\} \equiv 2r_2t(n-m) \pmod{8c}. \tag{4.2}$$

Since $\gcd(s_2t, c) = 1$, these congruences imply

$$\bigl(a_2(m^2-1) - b(n^2-1)\bigr)^2 \equiv 4r_2^2(n-m)^2 \pmod{c}. \tag{4.3}$$

We show that $n \leq b^{-1/2}c^{1/4}$ entails $|b(n^2-1)-a_2(m^2-1)|+2r_2(m-n) < c^{1/2}$.

First we examine the situation when $b(n^2-1) \geq a_2(m^2-1)$. Since $m \geq n$ by [10, Lemma 2.9] and $a_2 \geq 5$, Lemma 4.1 with $n \geq 5$ implies that

$$\begin{aligned} b(n^2-1) - a_2(m^2-1) + 2r_2(m-n) &\leq bn^2 - 2\sqrt{a_2b(m^2-1)} + 2r_2(m-n) \\ &< bn^2 \leq c^{1/2}. \end{aligned}$$

In the opposite situation $b(n^2-1) < a_2(m^2-1)$, by hypotheses $c > \max\{b^3, a_2^2b^2\}$ and $b > 4000$ it holds

$$\begin{aligned} a_2(m^2-1) - b(n^2-1) + 2r_2(m-n) &< (1.96a_2 - b)n^2 + b + 0.8n\sqrt{a_2b+1} \\ &< 0.96c^{1/2} + c^{1/3} + a_2^{1/2}c^{1/4} \\ &< 0.96c^{1/2} + 2b^{-1/2}c^{1/2} \\ &\leq \bigl(0.96 + 2\cdot 4001^{-1/2}\bigr)c^{1/2} < c^{1/2}. \end{aligned}$$

Hence, (4.3) boils down to

$$\bigl|b(n^2-1) - a_2(m^2-1)\bigr| = 2r_2(m-n), \tag{4.4}$$

so that (4.1) and (4.2) become

$$r_2(m-n)(\pm t + s_2) \equiv 0 \pmod{4c}.$$

In view of Lemma 4.1 and $c > \max\{b^3, a_2^2b^2\}$, one has

$$r_2(m-n)| \pm t + s_2| < 0.9a_2^{1/2}bc^{1/2}n \leq 0.9a_2^{1/2}b^{1/2}c^{3/4} < c.$$

Therefore, it necessarily holds $(m-n)(t-s_2) = 0$, which implies either $m = n$ or $a_2 = b$. In the former case, with (4.4) and $n \geq 4$ one readily sees that one also has $a_2 = b$. This contradiction is due to the assumption $n \leq b^{-1/2}c^{1/4}$. □

5. The first step of the proof

The goal of this section is to prove the following.

Theorem 5.1. *Assume that* $\{a_1, b, c, d\}$ *and* $\{a_2, b, c, d\}$ *are Diophantine quadruples with* $a_1 < a_2 < b < c < d$*. Then, the following hold*:

(1) $a_2 > 2a_1$.
(2) $16a_1^2b^3 < c < 16a_2^2b^3$.

We begin by ameliorating [5, Theorem 1.4].

Theorem 5.2. *If* $\{a_1, b, c\}$ *and* $\{a_2, b, c\}$ *are Diophantine triples with* $a_1 < a_2 < b < c \leq 16a_1^2b^3$*, then* $\{a_1, a_2, b, c\}$ *is a Diophantine quadruple.*

Proof. This is nothing but [5, Theorem 1.4] with the assumption $c < 16b^3$ replaced by $c \leq 16a_1^2b^3$. The proof proceeds along exactly the same lines as that of [5, Theorem 1.4]. □

On account of [23, Theorem 1], Theorem 5.2 has the following corollary.

Corollary 5.3. *If* $\{a_1, b, c, d\}$ *and* $\{a_2, b, c, d\}$ *are Diophantine quadruples with* $a_1 < a_2 < b < c < d$*, then* $c > 16a_1^2b^3$ *and* $d > 16a_2^2c^3$.

Note that the former inequality in Corollary 5.3 proves the first inequality of assertion (2) in Theorem 5.1, and that since $d_+(a_i, b, c) < 4a_ibc + 4c$ $(i \in \{1, 2\})$, the latter inequality in Corollary 5.3 means that $\{a_i, b, c, d\}$ $(i \in \{1, 2\})$ are irregular Diophantine quadruples.

After all these preparations, we are ready to prove Theorem 5.1. We argue by reduction to absurd.

Proof of Theorem 5.1. First of all, we claim that $n \geq 4$. From [10, Lemma 2.5] we know that $m \geq 3$. In the case (i) $m \equiv n \equiv 0 \pmod 2$ and $|z_{(0)}| = |z_{(1)}| = 1$ with $z_{(0)}z_{(1)} > 0$, since $w_2 = 2ct \pm (2bc + 1) < 4c^2$ and

$$\begin{aligned} v_4 &= 4(2a_2c + 1)cs_2 \pm (8a_2^2c^2 + 8a_2c + 1) \\ &\geq 8a_2c^2(s_2 - a_2) + 4c(s_2 - 2a_2) - 1 > 8a_2^2c^2, \end{aligned}$$

one has $n \neq 2$. In the case (ii) $m \equiv n \equiv 1 \pmod 2$ and $|z_{(0)}| = t$, $|z_{(1)}| = s_2$ with $z_{(0)}z_{(1)} > 0$, [24, Theorem 8] implies that $c > a_2 + b + 2r_2$, that is, $c > 4a_2b + a_2 + b$ (by [24, Lemma 4]), which together with [10, Lemma 2.6] implies that $n \neq 3$. We therefore obtain $n \geq 4$.

Since $a_2 \leq 2a_1$ and $c > 16a_1^2b^3$ (by Corollary 5.3) together imply $c > 4a_2^2b^3$, under either of the assumptions $a_2 \leq 2a_1$ and $c \geq 16a_2^2b^3$ one can apply Lemmas 3.4 and 4.2 to get

$$n < 8\varphi, \quad \text{with} \quad n > b^{-1/2}c^{1/4}, \tag{5.1}$$

where

$$\varphi = \frac{\log(8.471 \cdot 10^{13} a_1^{1/2}(a_1')^{1/2}a_2^2c)\log(1.622a_1^{1/2}a_2^{1/2}(a_2 - a_1)^{-1}c)}{\log(4bc)\log(0.2629a_1(a_1')^{-1}a_2^{-1}(a_2 - a_1)^{-2}c)}.$$

(1) Suppose that $a_2 \leq 2a_1$. Then, from $a_2 - a_1 \geq 3$ (see the proof of Lemma 3.4) one gets

$$a_1^{1/2}(a_2-a_1)^{-1} < \frac{a_2^{1/2}}{3}, \tag{5.2}$$

$$a_2(a_2-a_1)^2 \leq \frac{a_2^3}{4}. \tag{5.3}$$

If $b \leq 4\sqrt{5}\,a_2$, then $c > 16a_1^2b^3$ and $b > 4000$ together show that

$$c > 16\left(\frac{a_2}{2}\right)^2 b^3 \geq \frac{4}{80}b^5 > 0.05 \cdot 4000b^4 = 200b^4.$$

However, from [10, Theorem 1.4] one has $d_+(a_1,b,c) = d = d_+(a_2,b,c)$, which contradicts $a_1 < a_2$. Thus, we may assume that $b > 4\sqrt{5}\,a_2$. It follows from (5.2) and (5.3) that

$$\varphi < \frac{\log(8.471 \cdot 10^{13}a_2^3c)\log(0.5407a_2c)}{\log(16\sqrt{5}\,a_2c)\log(1.0516a_2^{-3}c)}. \tag{5.4}$$

Note that the right-hand side of (5.4) is a decreasing function of c provided $1.0516a_2^{-3}c > 1$. Since $c \geq 4a_2^2b^3 > 4 \cdot \left(4\sqrt{5}\right)^3 a_2^5$, one gets

$$n < \frac{8\log(24.2455 \cdot 10^{16}a_2^8)\log\left(1547.5738a_2^6\right)}{\log(102400a_2^6)\log(3009.8548a_2^2)}.$$

From (5.1) and $c/b^2 \geq 4a_2^2b \geq 16004a_2^2$ it follows that

$$16004^{1/4}a_2^{1/2} < n < \frac{32\log(148.9632a_2)\log(3.4011a_2)}{\log(6.8399a_2)\log(54.8621a_2)},$$

which yields $a_2 \leq 7$.

We combine the upper bound for a_2 just obtained with the lower bound on b given by Lemma 2.1 to get $b > \lambda a_2$ for some λ much bigger than the value $4\sqrt{5}$ employed previously. This way we obtain a smaller bound on a_2, which entails a bigger λ. After a few rounds, if needed, the game ends when $a_2 < 5$. However, as noted in the proof of Lemma 3.4, one has $a_2 \geq 5$, a contradiction.

From $a_2 \leq 7$ and $b \geq 4001$ we infer that $b \geq \lambda a_2$ with $\lambda = 4001/7$. The reasoning detailed above leads to the inequalities

$$16004^{1/4}a_2^{1/2} < n < \frac{8\log\left(8.471 \cdot 10^{13} \cdot 4\lambda^3a_2^8\right)\log\left(0.5407 \cdot 4\lambda^3a_2^6\right)}{\log\left(16\lambda^4a_2^6\right)\log\left(1.0516 \cdot 4\lambda^3a_2^2\right)}$$

which hold only for $a_2 \leq 1$. This contradiction shows that one cannot have $a_2 \leq 2a_1$.

(2) Now we assume, for the sake of contradiction, that we have $c \geq 16a_2^2b^3$. By a similar argument to the above, we can see that $b > 8\sqrt{5}\,a_2$. Moreover, in view of what we just did, we can additionally assume $a_2 > 2a_1$.

Then,

$$a_1^{1/2}(a_2-a_1)^{1/2} \le \frac{a_2}{2}, \tag{5.5}$$

$$a_1^{1/2}a_2^{1/2}(a_2-a_1)^{-1} \le \sqrt{2}. \tag{5.6}$$

It follows from (5.1), (5.5) and (5.6) that one has

$$2\cdot 4001^{1/4}a_2^{1/2} < \frac{8\log(3.879\cdot 10^{18}a_2^8)\log(2.1008\cdot 10^5 a_2^5)}{\log(6.5528\cdot 10^6 a_2^6)\log(24076.6a_2)},$$

from which one derives $a_2 \le 3$. This contradiction completes the proof of Theorem 5.1. □

6. The second step of the proof

In this section, utilizing the results and the methods in the previous section, we improve the lower bound for a_2 and the upper bound for c given in Theorem 5.1. We first show the following.

Theorem 6.1. *Assume that* $\{a_1, b, c, d\}$ *and* $\{a_2, b, c, d\}$ *are Diophantine quadruples with* $a_1 < a_2 < b < c < d$. *Then, the following hold*:

(1) $a_2 > 4a_1$.
(2) $16a_1^2b^3 < c < 4a_2^2b^3$.

Proof. The reasoning has much in common with that employed to establish Theorem 5.1, therefore we shall point out only the differences.

To prove part (1), we suppose that we have $2a_1 < a_2 \le 4a_1$. Then it obviously holds $c > 16a_1^2b^3 \ge a_2^2b^3$ and one can readily show that $b > 2\sqrt{5}a_2$. Thus, from Lemma 4.2 we conclude that $n > (2\sqrt{5}a_2^3)^{1/4}$. In addition, from Lemma 4.2 and $b \ge 4001$ we see that $n > 4001^{1/4}a_2^{1/2}$. Instead of (5.3) we use the inequality

$$a_2(a_2-a_1)^3 \le \frac{27}{16}a_1a_2^3$$

to obtain

$$\varphi < \frac{\log(4.2355\cdot 10^{13}a_2^3c)\log(1.622\sqrt{2}\,c)}{\log(8\sqrt{5}\,a_2c)\log\left(0.2629\cdot 16\cdot 27^{-1}a_2^{-3}c\right)}.$$

As $0.2629\cdot 16\cdot 27^{-1}a_2^{-3}c > 0.1557a_2^{-1}b^3 > b$, the right-hand side is decreasing with c. Comparison of the lower bound for n obtained above with the upper bound given by Lemma 3.4 results in the relation

$$4001^{1/4}a_2^{1/2} < \frac{80\log(88.5740a_2)\log(2.9002a_2)}{3\log(3.4199a_2)\log(3.7328a_2)},$$

which implies $a_2 \le 29$.

We resume the reasoning, using $b \ge \lambda a_2$ with $\lambda = 4001/29$ instead of $\lambda = 2\sqrt{5}$. The outcome of computations is $a_2 \le 8$. One more iteration decreases the bound on a_2 to 5. Since no further improvement is obtained this way, we concentrate on c instead of a_2.

Note that from $2a_1 < a_2 = 5 \le 4a_1$ it follows that $a_1 = 2$, so that with Lemma 3.4 we get

$$\varphi < \frac{\log(8.471 \cdot 10^{13} \cdot \sqrt{6} \cdot 25c) \log(1.622\sqrt{10} \cdot 3^{-1}c)}{\log(4bc) \log\bigl(0.2629 \cdot 0.4 \cdot 27^{-1}c\bigr)}.$$

By using $c < 16a_2^2b^3 = 400b^3$, we obtain

$$\varphi < \frac{3\log(5.1875 \cdot 10^{15}c) \log(1.7098c)}{4\log(0.6324c) \log\bigl(256.752^{-1}c\bigr)}.$$

As $c > 16a_1^2b^3 = 64b^3$ implies $c/b^2 > 16c^{1/3}$, from

$$2c^{1/12} < n < \frac{6\log(5.1875 \cdot 10^{15}c) \log(1.7098c)}{\log(0.6324c) \log\bigl(256.752^{-1}c\bigr)}$$

it results $c < 5 \cdot 10^{11}$, which is not compatible with $c > 64 \cdot 4001^3$.

Now suppose that $c > 4a_2^2b^3$. Then, as seen in the proof of Theorem 5.1, one has $\lambda = 4\sqrt{5}$. Using part (1), one obtains

$$a_1^{1/2}(a_2 - a_1)^{1/2} < \frac{\sqrt{3}}{4}a_2, \tag{6.1}$$

$$a_1^{1/2}a_2^{1/2}(a_2 - a_1)^{-1} < \frac{2}{3}, \tag{6.2}$$

which entail

$$16004^{1/4}a_2^{1/2} < \frac{160\log(134.1657a_2) \log(4.9904a_2)}{3\log(6.8399a_2) \log(752.4637a_2)}.$$

Hence, $a_2 \le 12$. With $\lambda = 4001/12$ one gets $a_2 < 5$. This contradiction shows that the assertion in part (2) is true. □

Theorem 6.2. *Assume that $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples with $a_1 < a_2 < b < c < d$. Then, the following hold:*

(1) $a_2 > a_1^2$.
(2) $b < a_2^2$.

Proof. (1) Assuming that $a_2 \le a_1^2$, from $a_2 > 4a_1$ one obtains $a_1 \ge 5$, which implies $a_2 \ge 21$ and $c > 16a_2b^3 > 16a_2^4$.

According to inequalities established in the proof of Theorem 6.1, one has

$$\varphi < \frac{\log\bigl(8.471 \cdot 10^{13} \cdot \frac{\sqrt{3}}{4}a_2^3c\bigr) \log\bigl(1.622 \cdot \frac{2}{3}c\bigr)}{\log(4bc) \log\bigl(0.2629a_2^{-7/2}c\bigr)}.$$

A slightly larger upper bound is obtained by using the inequalities $b > 200^{-1/4}c^{1/4}$, $8a_2^3 < c^{3/4}$ and $a_2^{-7/2}c > 2^{7/2}c^{1/8}$. Since $c/b^2 > 20^{4/3}c^{1/3}$, with Lemma 3.4 one gets

$$\varphi < \frac{56\log\bigl(1.7182 \cdot 10^7c\bigr) \log\bigl(1.0814c\bigr)}{5\log\bigl(1.0506c\bigr) \log\bigl(6125.8247c\bigr)}. \tag{6.3}$$

We discuss separately the two possible outcomes of comparison of b to a_1^2.

Case 1: $b \geq a_1^2$. Then it holds $c > 16a_1^8$ and $c/b^2 > 16a_1^4$. Lemma 4.2 yields

$$2a_1 < n < \frac{448 \log\big(11.3475a_1\big) \log\big(1.4282a_1\big)}{5 \log\big(1.4229a_1\big) \log\big(4.2063a_1\big)},$$

whence $a_1 \leq 53$. Introducing the quantity $\mu = 4001/53^2$ into argument, one has $b \geq \mu a_1^2$ and one gets $a_1 \leq 37$. Then one updates $\mu = 4001/37^2$, which leads to $a_1 \leq 18$. After two more iterations one arrives at $a_1 \leq 4$, a contradiction that shows that Case 1 is impossible.

Case 2: $b < a_1^2$. Note that this hypothesis implies $a_1 \geq 64$ and $a_2 \geq 257$. Working with inequality (6.3), one readily obtains

$$256^{1/3} c^{1/12} < n < \frac{448 \log\big(1.7182 \cdot 10^7 c\big) \log\big(1.0814c\big)}{5 \log\big(1.0506c\big) \log\big(6125.8247c\big)},$$

which is true only for $c < 10^{15}$. As this inequality contradicts $c > 16 \cdot 64^2 \cdot 4001^3 > 4 \cdot 10^{15}$, we conclude that Case 2 is not possible.

(2) Once again, we reason by reduction to absurd. So, assume that one has $a_2^2 \leq b$. In view of Theorem 6.1 and (2), it is natural to distinguish the next two cases.

Case A: $a_1 \leq 4$. Besides the inequalities (6.1) and (6.2), we employ those specific to the case at hand, namely $c > 16a_1^2 a_2^6$, $c/b^2 > 16a_1^2 a_2^2$, and

$$\frac{a_1}{a_2(a_2 - a_1)^3} > \frac{a_1}{a_2^4}.$$

Lemma 4.2 yields

$$2a_1^{1/2} a_2^{1/2} < \frac{27 \log\big(43.7473a_2\big) \log\big(1.6083a_2\big)}{\log\big(1.6817a_2\big) \log\big(2.0509a_2\big)}.$$

This gives bounds of the type $a_2 \leq UBA(a_1)$, specifically,

$$\begin{aligned} &a_2 \leq 382 \quad \text{for} \quad a_1 = 1, \quad a_2 \leq 203 \quad \text{for} \quad a_1 = 2, \\ &a_2 \leq 141 \quad \text{for} \quad a_1 = 3, \quad a_2 \leq 109 \quad \text{for} \quad a_1 = 4. \end{aligned}$$

Now we work with c and use $UBA(a_1)$ to bound from above φ. The outcome of routine calculations using inequalities established in previous proofs is

$$\begin{aligned} (4a_1)^{1/3} c^{1/12} &< \frac{8 \log\left(8.471 \cdot 10^{13} \cdot 2^{-5/3} a_1^{-1/3} c^{17/12}\right) \log\left(1.622 \cdot \frac{2}{3} c\right)}{\log\left(4^{2/3} a_2^{-2/3} c^{4/3}\right) \log\left(0.2629 \cdot 16^{2/3} a_1^{7/3} c^{1/3}\right)} \\ &< \frac{51 \log\left(3.0015 \cdot 10^9 a_1^{-4/17} c\right) \log\left(1.0814c\right)}{2 \log\left(2UBA(a_1)^{-1/2} c\right) \log\left(4.6517 a_1^7 c\right)}. \end{aligned}$$

If $a_1 = 1$, then $c < 8.168 \cdot 10^{16}$ and $b \leq 172186$; if $a_1 = 2$, then $c < 1.637 \cdot 10^{15}$ and $b \leq 29463$; if $a_1 = 3$, then $c < 1.621 \cdot 10^{14}$ and $b \leq 10402$; if $a_1 = 4$, then $c < 3.099 \cdot 10^{13}$ and $b \leq 4946$.

Having tight bounds for a_1, a_2, and b, a short computation gives all triples $\{a_1, a_2, b\}$ presumably extendible to Diophantine quadruples $\{a_i, b, c, d\}$. We fix a value for a_1, then we search for values $4001 \leq b \leq 172186$ such that

$\{a_1, b\}$ is a Diophantine pair. For each such b we look for $a_2 \le UBA(a_1)$ with the property that $\{a_2, b\}$ is a Diophantine pair too. Next we use Lemmas 3.4 and 4.2 to bound from above c, say, by $UBC(a_1)$. More precisely, we find 164 triples with $a_1 = 1$ but only 73 of them satisfy the necessary condition $UBC(a_1) > 16a_1^2b^3$. For $a_1 = 2$ there are 51 triples (a_1, a_2, b), 16 out of which pass the test on the corresponding upper bound for c. The only survivors when $a_1 = 3$ are $(a_2, b) = (64, 4641)$ and $(60, 5208)$. The test eliminates all candidate triples with $a_1 = 4$.

For each triple (a_1, a_2, b) thus obtained we perform the following algorithm. Consider the equations

$$a_1c + 1 = s_1^2, \quad bc + 1 = t^2, \quad a_2c + 1 = s_2^2.$$

Elimination of c between the first two relations results in the quadratic equation

$$a_1t^2 - bs_1^2 = a_1 - b \tag{6.4}$$

whose solutions are given by finitely many formulas of the type $s_1 = \rho_1u+\theta_1v$, $t = \rho_2u + \theta_2v$, where (u, v) solves the associated Pell equation. Here the constants ρ_i, θ_i are obtained by well-known procedures for determining the fundamental solutions to quadratic equations (for an implementation see, for instance, [26]).

We retain only those units (u, v) for which the corresponding t satisfies

$$4a_1b^2 < (bc)^{1/2} < t < 2a_2b^2 \tag{6.5}$$

(see Theorem 6.1). Finally we check for each survivor whether $a_2(t^2-1)/b+1$ is square.

This procedure implemented in Pari [27] finds that none of the 91 triples (a_1, a_2, b) satisfies all the required conditions.

Case B: $a_1 > 4$. Now the inequality $a_2 > a_1^2$ is stronger than $a_2 > 4a_1$, so we shall employ it together with Lemma 4.2.

Note that under the current hypothesis one has $a_2 \ge 26$, as well as

$$\begin{aligned}\big(a_1(a_2 - a_1)\big)^{1/2} &< a_2^{3/4},\\ \frac{\sqrt{a_1a_2}}{a_2 - a_1} &< \frac{1.244}{a_2^{1/4}},\\ \frac{a_1}{a_2(a_2 - a_1)^3} &> \frac{5}{a_2^4},\end{aligned}$$

which imply

$$\varphi < \frac{\log\big(8.471 \cdot 10^{13}a_2^{11/4}c\big)\log\big(1.622 \cdot 1.244a_2^{-1/4}c\big)}{\log(4bc)\log\big(0.2629 \cdot 5a_2^{-4}c\big)}.$$

Together with $c > 400a_2^6$ and $n > 2a_1^{1/2}b^{1/4} > 20^{1/2} \cdot 4000^{1/4}$, this yields

$$20^{1/2} \cdot 4000^{1/4} < \frac{805\log\big(77.4716a_2\big)\log\big(3.203a_2\big)}{32\log\big(2.5148a_2\big)\log\big(22.9303a_2\big)}.$$

Hence, $a_2 \le 3$, which is a contradiction. □

Corollary 6.3. *Under the hypothesis of Theorem* 6.2 *it holds* $b > 4a_1^2$.

Proof. Assume the contrary. Then one has $a_1^2 < b < 4a_1^2$, so we can apply Theorem 1.1 from [8], which gives that $\{a_1, b, c, d\}$ is a regular Diophantine quadruple. Therefore, $d = d_+(a_1, b, c) < d_+(a_2, b, c)$, in contradiction with the assumption that $\{a_2, b, c, d\}$ is a Diophantine quadruple. □

7. Proof of Corollaries

Proof of Corollary 1.3. If $c = c_\nu^\tau(a_2, b)$ for some ν and τ, then inequalities (2.6) and Theorem 6.1 together yield $c = c_3^-(a_2, b)$, which, in view of the inequality $b < a_2^2$ obtained in Theorem 6.2, contradicts [10, Proposition 1.5 (4)]. As for the second assertion, if $b \le 13a_2$, then the proof of [10, Corollary 1.6] implies that $c = c_\nu^\tau$ for some ν and τ, which contradicts the first assertion. □

Proof of Corollary 1.4. Assume that there exist three positive integers a_1, a_2, a_3 with $a_1 < a_2 < a_3 < \min\{b, c, d\}$ such that $\{a_i, b, c, d\}$ $(i \in \{1, 2, 3\})$ are Diophantine quadruples. Applying Theorem 6.1 to $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ one has $c < 4a_2^2b^3$, while applying Theorem 6.1 to $\{a_2, b, c, d\}$ and $\{a_3, b, c, d\}$ yields $16a_2^2b^3 < c$, which is a contradiction. □

Proof of Corollary 1.5. Assume that $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples with $a_1 < a_2 < b < c < d$. By the discussion after Corollary 5.3, we know that $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are irregular. In terminology of [12], $\{a_1, b, c\}$ is a standard triple of the second kind, since $b > a_2 > 4a_1$ and $c > b^3$ by Theorem 6.1. It follows from Proposition 4 in [12] that $c < 10^{2171}$. Therefore, as in Section 9 of [12], we can get that $d < 10^{10^{26}}$. □

Proof of Corollary 1.6. As seen in the previous proof, any counterexample to Conjecture 1.2 gives rise to two irregular Diophantine quadruples, whose existence would falsify Conjecture 1.1. □

8. The final step of the proof

In this section, the goal is to improve the results established above. We attack problems from a different angle by employing Lemmas 2.1 and 2.2.

Lemma 8.1. *Assume that* $\{a_1, b, c, d\}$ *and* $\{a_2, b, c, d\}$ *are Diophantine quadruples with* $a_1 < a_2 < b < c < d$. *If* $b \le 24a_2$, *then* $\{1, 3, a_2, b\}$ *is a Diophantine quadruple. In particular,* $a_1 \notin \{1, 3\}$, $a_2 \ge 1680$ *and* $b \ge 23408$.

Proof. The first assertion follows from Lemma 2.2 and Corollary 1.3. This in conjunction with Lemma 2.1 and (2.7) gives the lower bounds on a_2 and b. If $a_1 = 1$ or 3, then $\{a_1, a_2, b, c, d\}$ is a Diophantine quintuple, whose existence is prohibited by [23, Theorem 1]. □

Proposition 8.2. *If* $\{a_1, b, c, d\}$ *and* $\{a_2, b, c, d\}$ *are Diophantine quadruples with* $a_1 < a_2 < b < c < d$, *then* $4a_1^2 < a_2$.

Proof. Assume that $a_2 \le 4a_1^2$. From $a_2^2 > b > 4000$ it follows that $a_2 \ge 64$, whence $a_1 \ge 4$. Note that $a_1 = 4$ enforces $a_2 = 64$, a situation not compatible with the hypothesis that both $\{a_i, b, c, d\}$ are Diophantine quadruples (because a_2/a_1 is a perfect square). Thus, it holds $a_1 \ge 5$.

By part (1) of Theorem 6.2 it is clear that one has

$$\sqrt{a_2 - a_1} < \sqrt{a_2} \le 2a_1, \ \frac{\sqrt{a_2 a_1}}{a_2 - a_1} < \frac{\sqrt{a_1}}{a_1 - 1} \le \frac{5}{4\sqrt{a_1}}.$$

If $b > \rho a_1^2$ for some positive ρ, then $c > 16a_1^2 b^3 > 16\rho^3 a_1^8$ and $c/b^2 > 16\rho a_1^4$. Thus, Lemmas 3.4 and 4.2 give

$$\rho^{1/4} a_1 < \frac{4\log\left(4337.152 \cdot 10^{13} \rho^3 a_1^{27/2}\right) \log\left(32.44\rho^3 a_1^{15/2}\right)}{\log\left(64\rho^4 a_1^{10}\right) \log\left(0.2629\rho^3 a_1/16\right)}.$$

From Corollary 1.3 it is seen that we can take $\rho = 13$ in the previous relation, which leads to $a_1 \le 20$ and consequently $a_2 \le 1600$. This inequality together with Lemma 8.1 imply that $b > 24a_2$, so that we can resume the reasoning from the previous paragraph with $\rho = 24$. The new bound thus found is $a_1 \le 14$.

From now on we apply Lemmas 3.4 and 4.2 with focus on a_2. Our current knowledge allows us to use the following inequalities:

$$\sqrt{a_1(a_2 - a_1)} < a_2^{3/4},$$

$$\frac{\sqrt{a_1 a_2}}{a_2 - a_1} \le \frac{\sqrt{14a_2}}{a_2 - 14} \le \frac{32\sqrt{14}}{25\sqrt{a_2}},$$

$$c > 16a_1^2 b^3 \ge 4a_2 b^3 > 4 \cdot 24^3 a_2^4.$$

We find

$$96^{1/4} a_2^{1/2} < \frac{378\log(583.4707a_2)\log(40.685a_2)}{5\log(22.1305a_2)\log(52833406a_2)},$$

whence $a_2 \le 169$. In order to examine values of a_2 close to this upper bound, we consider first $a_2 \ge 145$, which in turn implies $7 \le a_1 \le 12$.

We treat the remaining cases with the help of the multiplicative analogue of the idea introduced in the proof of Theorem 5.1. Specifically, instead of inequalities $b > \lambda a_2$ we use $b > a_2^{\varepsilon}$. Initially we have

$$c > 16 \cdot 49 \cdot 24^3 a_2^3 > 169^{3.15767} a_2^3 \ge a_2^{6.15767},$$

$$\frac{c}{b^2} > a_2^{2.91864}, \quad \frac{\sqrt{a_1 a_2}}{a_2 - a_1} \le \frac{145\sqrt{12}}{133\sqrt{a_2}},$$

$$4b > 96a_2 > a_2^{1.88975}, \quad 0.2629a_1 \ge 1.8403 > a_2^{0.11889}.$$

Hence,

$$a_2^{0.72966} < \frac{22.0068\log(36.6091a_2)\log(1.3777a_2)}{(\log a_2)^2},$$

whence $a_2 \le 157$. We resume the reasoning with this value instead of 169. Note that we can obtain a further gain by replacing $b > 24a_2$ by $b \ge (4001/157)a_2$. The outcome of calculations is the improved bound $a_2 \le 147$.

The next upper bound for a_2 is computed as 134. At this moment the working hypotheses are modified as follows: $a_2 \le 144$ and $a_1 \le 11$.

One continues on these lines until a bound $a_2 < 64$ is obtained, when one concludes that the assumption $a_2 \le 4a_1^2$ is refuted. □

Proposition 8.3. *If $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples with $a_1 < a_2 < b < c < d$, then $c < 16a_2b^3$.*

Proof. As before, we reason by contradiction. So assume it holds $16a_2b^3 \le c$.

We examine separately the situation for $a_1 = 1$ by applying Lemmas 3.4 and 4.2 for $\{a_2, b, c, d\}$ with $b > \lambda a_2$. It follows from Lemma 8.1 that $b > 24a_2$, where we get $a_2 \le 21$, a bound incompatible with $a_2 \ge 64$.

When $a_1 \ge 2$, our calculations initiated with $\lambda = 13$ give $a_2 \le 52$, so that we can resume the computations with $\lambda = 4001/52$. The resulting upper bound $a_2 \le 19$ is smaller than the lower bound $a_2 \ge 64$. □

Proposition 8.4. *If $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples with $a_1 < a_2 < b < c < d$, then $b < a_2^{3/2}$.*

Proof. Suppose, for the sake of a contradiction, that $a_2^{3/2} \le b$. Note that $b > 24a_2$, since otherwise it would follow $a_2 \le 576$, in contradiction with Lemma 8.1.

We first show that one necessarily has $a_1 \le 7$. Indeed, in view of Proposition 8.2, for $a_1 \ge 8$ one obtains

$$\sqrt{a_1(a_2 - a_1)} < 2^{-1/4}a_2^{3/4}, \quad c > (4\sqrt{2})^4 a_2^{9/2},$$

$$\frac{c}{b^2} > (4\sqrt{2})^4 a_2^{3/2}, \quad \frac{\sqrt{a_1a_2}}{a_2 - a_1} < \frac{16\sqrt{2}}{31a_2^{1/4}},$$

whence

$$4\sqrt{2}\, a_2^{3/8} < \frac{8\log\left(7.2942 \cdot 10^{16} a_2^{29/4}\right)\log\left(1212.3391 a_2^{17/4}\right)}{\log\left(4096a_2^6\right)\log\left(2153.6768 a_2^{1/2}\right)}.$$

This inequality implies $a_2 \le 245$, which is incompatible with $a_2 > 4a_1^2 \ge 4 \cdot 8^2 = 256$, see Proposition 8.2.

From now on the reasoning parallels that employed in Case A of Theorem 6.2. With the help of Lemmas 3.4 and 4.2 one finds $a_2 \le UBA(a_1)$. Using this bound and the same lemmas, one obtains $c < UBC(a_1)$, whence $b \le UBB(a_1)$ by Theorem 5.1. Then one can explicitly enumerate the triples (a_1, a_2, b), in number of $NT(a_1)$. For each triple thus obtained one considers the corresponding equation (6.4). According to Proposition 8.3, instead of condition (6.5) one requires

$$4a_1b^2 < (bc)^{1/2} < t < 4a_2^{1/2}b^2. \tag{8.1}$$

For $a_1 = 7$ (6), we find $UBA(a_1) = 322$ (446), $UBC(a_1) = 1.7 \cdot 10^{14}$ ($5.9 \cdot 10^{14}$), $UBB(a_1) = 6007$ (10080), and $NT(a_1) = 0$. Table 1 summarizes the data thus obtained for $a_1 \le 5$.

a_1	1	2	3	4	5
UBA	23961	4870	1982	1100	659
UBB	4641588	375923	99065	38840	18171
UBC	$1.6 \cdot 10^{21}$	$3.4 \cdot 10^{18}$	$1.4 \cdot 10^{17}$	$1.5 \cdot 10^{16}$	$2.4 \cdot 10^{15}$
NT	282	181	117	55	10

TABLE 1. Experimental data for Proposition 8.4

Since none of the 645 equations of type (6.4) does have solutions satisfying condition (8.1), we conclude that the assumption $a_2^{3/2} \le b$ is refuted. $\square$

Now we are ready to prove part (2) of Main Theorem.

Theorem 8.5. *If $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples with $a_1 < a_2 < b < c < d$, then $b < a_2^{4/3}$ for $a_1 \ge 2$ or $a_1 = 1$ and $a_2 < 400000$.*

Proof. The argument closely follows the reasoning employed to obtain Proposition 8.4. We shall therefore point out the salient differences.

Start by assuming that there exists a Diophantine quadruple with $a_2^{4/3} \le b$. We claim that one necessarily has $b > 24a_2$. Indeed, in the opposite case one obtains $a_2 \le 24^3$. This in conjunction with Lemma 8.1 and (2.7) leads to the conclusion that $(a_2, b) = (1680, 23408)$. It is a matter of easy computation to find that the only possibilities for $23408a_1 + 1$ to be a perfect square when $4a_1^2 < 1680$ are $a_1 = 1$ or 3. These are rejected because their existence contradicts the non-extendibility of Diophantine quadruples.

Now, that we know that $b > 24a_2$, with the help of Proposition 8.4 we find that $a_2 \ge 577$. From now we proceed as in the previous proof. With the notation introduced there, we get $UBA(a_1) \le 542$ for $a_1 \ge 8$. The other upper bounds on a_2 are much higher, ranging from $UBA(1) < 1.855 \cdot 10^7$ to $UBA(2) = 260664$ to $UBA(7) = 985$. Finally it is found that $NT(2) = 5$, $NT(3) = NT(4) = 7$, and $NT(a_1) = 0$ for $a_1 \ge 5$. Moreover, for $a_1 = 1$ and $a_2 \le 400000$ our computations exhibit only 2 admissible values for b. The proof is concluded by checking that none of the 21 equations of type (6.4) does have solutions fulfilling all the required conditions. $\square$

From the proof just concluded it is clear that removing the hypothesis $a_2 \le 400000$ is just a matter of extensive computation. It is highly unlikely to find any extra equation (6.4) for the unexplored values of a_2.

Part (3) of the Main Theorem is by now almost obvious.

Proposition 8.6. *If $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples with $a_1 < a_2 < b < c < d$, then $a_2 > 24^3$.*

Proof. Supposing $a_2 \le 24^3$, from the previous theorem we get $b < 24a_2$. The argument employed in the second paragraph of the previous proof gives the desired conclusion. $\square$

In order to complete the proof of Main Theorem, it remains to show the following.

Proposition 8.7. *If $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ are Diophantine quadruples with $a_1 < a_2 < b < c < d$, then $a_2 > \max\{36a_1^3, 300a_1^2\}$.*

Proof. Once again, we argue by contradiction. Assume first that there exist two Diophantine quadruples with $a_2 \le 36a_1^3$.

With the help of Propositions 8.2 and 8.6 we obtain

$$\frac{\sqrt{a_1 a_2}}{a_2 - a_1} < \frac{\sqrt{2}a_2^{1/4}}{2a_2^{1/2} - 1} < 0.7102 a_2^{-1/4}.$$

By Corollary 1.3 and Theorem 5.1 one has $c > 35152 a_1^2 a_2^3$ as well as $c/b^2 > 208a_1^2 a_2 \ge 13(4/3)^{4/3} a_2^{5/3}$, so that

$$\varphi < \frac{\log\bigl(2.9778 \cdot 10^{18} a_1^{5/2} a_2^{11/2}\bigr) \log\bigl(40493.15 a_1^2 a_2^{11/4}\bigr)}{\log\bigl(1827904 a_1^2 a_2^4\bigr) \log\bigl(256.7072\bigr)}.$$

The expression in the right side is increasing with a_1, therefore

$$13^{1/4} \cdot \left(\frac{4}{3}\right)^{1/3} a_2^{5/12} < \frac{8\log\bigl(5.2641 \cdot 10^{17} a_2^{27/4}\bigr) \log\bigl(10123.2875 a_2^{15/4}\bigr)}{\log\bigl(456976 a_2^5\bigr) \log\bigl(256.7072\bigr)},$$

that is

$$a_2^{5/12} < \frac{3.49295 \log\bigl(422.065 a_2\bigr) \log\bigl(11.6974 a_2\bigr)}{\log\bigl(13.5512 a_2\bigr)}.$$

This inequality is false for $a_2 > 14301$. Hence, it remains to search for Diophantine 2-tuples (a_i, b) $(i = 1, 2)$ satisfying the conditions $4a_1^2 < a_2 \le 36a_1^3$, $24^3 < a_2 \le 14301$, and $13a_2 < b < a_2^{4/3}$.

Our current knowledge allows us to restrict the search to values $179712 = 13 \cdot 24^3 < b \le 347127 = \lfloor 14301^{4/3} \rfloor$. A short computation finds only three triples (a_1, a_2, b) for which the associated equation (6.4) needs to be considered. As none of them has solutions in the required range, the assumption $36a_1^3 \ge a_2$ is false.

To obtain the other part of the conclusion, we suppose that $a_2 \le 300a_1^2$. Then $c > 4 \cdot 75^{-1} \cdot 13^3 a_2^4$, $c/b^2 > (52/75)a_2^2$, and $a_1 \ge \bigl(24^3/300\bigr)^{1/2} > 6$, whence

$$a_2^{1/2} < \frac{8.2596 \log\bigl(222.6345 a_2\bigr) \log\bigl(3.6989 a_2\bigr)}{\log\bigl(5.7143 a_2\bigr)}$$

and hence $a_2 \le 14116$. The range where the hypothetical values for a_2 sit has been explored as described in the previous paragraph and no suitable values have been found. Therefore, the inequality $a_2 \le 300a_1^2$ is refuted. $\square$

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