# Harnack Inequality for Some Discontinuous Markov Processes with a Diffusion Part

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**Abstract**

In this paper we establish a Harnack inequality for nonnegative harmonic functions of some discontinuous Markov processes with a diffusion part.

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# 1 Introduction

Harnack inequality for nonnegative harmonic functions of diffusions in $\mathbb{R}^d$ has been a well-known fact for more than forty years. On the contrary, until recently very little was known about Harnack inequality for nonnegative harmonic functions of discontinuous Markov processes. The only exception was the rotationally invariant $\alpha$-stable process in $\mathbb{R}^d$, where Harnack inequality follows directly from the explicit form of the Poisson kernel for balls (i.e.,

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the exit distributions from balls). This situation changed several years ago with the paper [2] by Bass and Levin where they proved the Harnack inequality for the Markov process on $\mathbb{R}^d$ associated with the generator

$$\mathbf{L}f(x) = \int_{R^d \setminus \{0\}} [f(x+h) - f(x)] \frac{k(x,h)}{|h|^{d+\alpha}} dh$$

where $k(x, -h) = k(x, h)$ and $k$ is a positive function bounded between two positive numbers.

The paper [2] has stimulated and inspired the research in the subject, leading to several papers in recent years. Vondraček [10] adapted the arguments of [2] and proved that, when $X$ is a (not necessarily rotationally invariant) strictly $\alpha$-stable process, $\alpha \in (0, 2)$, with a Lévy measure comparable to the Lévy measure of the rotationally invariant $\alpha$-stable process, the Harnack inequality holds. In [4], the Harnack inequality was proved by using a different method for symmetric $\alpha$-stable processes under the assumptions that $\alpha \in (0, 1)$ and its Lévy measure is comparable to the Lévy measure of the rotationally invariant $\alpha$-stable process. In [3], Bass and Levin established upper and lower bounds on the transition densities of symmetric Markov chains on the integer lattice in $d$ dimensions, where the conductance between $x$ and $y$ is comparable to $|x-y|^{d+\alpha}$, $\alpha \in (0, 2)$. One of the key steps in proving the upper and lower bounds in [3] is the parabolic Harnack inequality. In [5], Chen and Kumagai showed that the parabolic Harnack inequality holds for symmetric stable-like processes in $d$-sets and established upper and lower bounds on the transition densities of these processes. All the processes mentioned above satisfy a certain scaling property which was used crucially in the proofs of the Harnack inequalities. In [8], Song and Vondraček extracted the essential ingredients of the Bass-Levin method by isolating three conditions that suffice to prove the Harnack inequality and showed that various classes of Markov processes, not necessarily having any scaling properties, satisfy the Harnack inequality. In the paper [1], Bass and Kassmann proved the Harnack inequality for a class of processes corresponding to non-local operators of variable order. Their method is also based on [2], but the arguments are more delicate.

In all the papers mentioned above, the corresponding Markov process did not have a continuous component. More precisely, the generator of the process was an integro-differential operator without a local part. A natural step forward was to study Harnack inequality for nonnegative harmonic functions of discontinuous processes with a diffusion component. This step was taken in a very recent paper [7], where the Harnack inequality was proved for a certain class of subordinate Brownian motions in $\mathbb{R}^d$, $d \geq 3$. By allowing the subordinator to have a drift, the class in question includes processes with both a diffusion and discontinuous component. A typical example of a process belonging to this class is an independent sum of a Brownian motion and rotationally invariant $\alpha$-stable process in $\mathbb{R}^d$, $d \geq 3$.

The purpose of this paper is to prove the Harnack inequality for nonnegative harmonic functions of another class of discontinuous Markov processes in $\mathbb{R}^d$, $d \geq 1$, with a diffusion part. Note that the processes dealt with in this paper are not Lévy processes in general, so the method of [7] does not apply. We describe now the processes that will be studied.

Suppose that $a$ is a continuous map from $\mathbb{R}^d$ into the space of symmetric $d \times d$ matrices, and suppose that there exist $0 < \lambda < \Lambda < \infty$ such that

$$\lambda|\xi|^2 \le \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad \forall x, \xi \in \mathbb{R}^d. \tag{1.1}$$

Suppose that $b$ is a bounded map from $\mathbb{R}^d$ into $\mathbb{R}^d$. Suppose further that $k(x, y)$ is a function on $\mathbb{R}^d \times \mathbb{R}^d$ which is bounded between two positive numbers $\kappa_1 < \kappa_2$. For $\alpha \in (1, 2)$ let

$$\begin{aligned} \mathbf{L}f(x) = \quad & \sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x)\frac{\partial}{\partial x_i} f(x) \\ & + \int_{\mathbb{R}^d} (f(x+y) - f(x) - y \cdot \nabla f(x) 1_{|y|<1}) \frac{k(x,y)}{|y|^{d+\alpha}} dy. \end{aligned} \tag{1.2}$$

It follows from [6] (see also [9]) that the martingale problem for $\mathbf{L}$ is well-posed. That is, there is a unique conservative Markov process $X = (X_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ on $(D([0,\infty), \mathbb{R}^d), \mathcal{B}(D([0,\infty), \mathbb{R}^d)))$ such that for any $f \in C_0^\infty(\mathbb{R}^d)$,

$$f(X_t) - f(X_0) - \int_0^t \mathbf{L}f(X_s)ds$$

is a $\mathbb{P}_x$-martingale for each $x \in \mathbb{R}^d$. Here $D([0,\infty), \mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued cadlag functions on $[0, \infty)$, and $\mathcal{B}(D([0,\infty), \mathbb{R}^d))$ is the Borel $\sigma$-field on $D([0,\infty), \mathbb{R}^d)$.

Recall that a nonnegative Borel function $h$ on $\mathbb{R}^d$ is said to be harmonic with respect to $X$ in a domain $D \subset \mathbb{R}^d$ if it is not identically infinite in $D$ and if for any bounded open subset $B \subset \overline{B} \subset D$,

$$h(x) = \mathbb{E}_x[h(X(\tau_B))1_{\tau_B<\infty}], \quad \forall x \in B,$$

where $\tau_B = \inf\{t > 0 : X_t \notin B\}$ is the first exit time of $B$.

We are going to prove the following Harnack inequality for nonnegative harmonic functions of $X$:

**Theorem 1.1** *For any domain $D$ of $\mathbb{R}^d$ and any compact subset $K$ of $D$, there exists a constant $C > 0$ such that for any function $h$ which is nonnegative in $\mathbb{R}^d$ and harmonic with respect to $X$ in $D$, we have*

$$h(x) \le Ch(y), \quad x, y \in K.$$

**Remark 1.2** *Note that we have assumed that $\alpha \in (1, 2)$. We do not think that this restriction is essential, but it comes from the method of proof we use.*

Proof of the theorem is based on the method developed in [2]. Necessary lemmas are stated in the next section. We only prove some of them, and refer the reader to proofs of similar lemmas in [8]. The last section contains the proof of the theorem following the idea from [1] which is a refinement of the one from [2].

We use capital letters $C_1, C_2, \dots$ for constants appearing in the statements of the results, and lowercase letters $c_1, c_2, \dots$ for constants appearing in proofs. The numbering of the latter constants starts afresh in every new proof.

# 2 Auxiliary lemmas

Recall that we assumed that $d \geq 1$ and $\alpha \in (1,2)$. Let

$$\mathbf{L}_1 f(x) = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x) \frac{\partial}{\partial x_i} f(x) ,$$

and let

$$\mathbf{L}_2 f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x) - y \cdot \nabla f(x) 1_{|y|<1}) \frac{k(x,y)}{|y|^{d+\alpha}} dy .$$

Then

$$\mathbf{L} f(x) = \mathbf{L}_1 f(x) + \mathbf{L}_2 f(x) .$$

Note that it follows from (1.1) that the functions $a_{ij}$ are uniformly bounded on $\mathbb{R}^d$.

**Lemma 2.1** *There exists a constant $C_1 > 0$ such that for any $x \in \mathbb{R}^d$ and any $r \in (0,1)$ we have*

$$\mathbb{P}_x(\sup_{s \leq t} |X_s - X_0| > r) \leq C_1 r^{-2} t.$$

**Proof.** Suppose that $x \in \mathbb{R}^d$ is fixed. Let $f$ be a $C^2$ function on $\mathbb{R}^d$ taking values in $[0,1]$ such that $f(y) = 0$ for $|y| \leq 1/2$ and $f(y) = 1$ for $|y| \geq 1$. Let $(f_n : n \geq 1)$ be a sequence of $C^2$ functions such that $0 \leq f_n \leq 1$,

$$f_n(y) = \begin{cases} f(y), & |y| \leq n+1 \\ 0, & |y| > n+2, \end{cases}$$

and that $(\partial^2/\partial x_i \partial x_j) f_n$ and $(\partial/\partial x_i) f_n$ are uniformly bounded. Then there exist positive constants $c_1$ and $c_2$ such that

$$|\nabla f_n(y)| \leq c_1, \quad y \in \mathbb{R}^d,$$

and

$$|f_n(y+z) - f_n(y) - z \cdot \nabla f_n(y)| \leq c_2 |z|^2, \quad y, z \in \mathbb{R}^d.$$

Put $f_r(y) = f((y-x)/r)$ and $f_{n,r}(y) = f_n((y-x)/r)$. For any $r \in (0,1)$, $y \in \mathbb{R}^d$, and $n \geq 1$, we have

$$\begin{aligned}
|\mathbf{L}_2 f_{n,r}(y)| \leq \quad & \Big|\int_{\mathbb{R}^d} (f_{n,r}(y+z) - f_{n,r}(y) - z \cdot \nabla f_{n,r}(y) 1_{|z|<r}) \frac{k(y,z)}{|z|^{d+\alpha}} dz\Big| \\
& + \int_{\mathbb{R}^d} |z \cdot \nabla f_{n,r}(y) 1_{r \leq |z| < 1}| \frac{k(y,z)}{|z|^{d+\alpha}} dz \\
\leq \quad & c_2 \kappa_2 r^{-2} \int_{|z|<r} |z|^{-(d+\alpha-2)} dz + 2 c_1 \kappa_2 r^{-1} \int_{|z| \geq r} |z|^{-(d+\alpha-1)} dz \\
\leq \quad & c_3 \kappa_2 r^{-\alpha},
\end{aligned}$$

where the positive constant $c_3$ depends on $\alpha, \kappa_1, \kappa_2$ and $d$.

By using the uniform bounds on functions $a_{ij}$, $b_i$, and the uniform bounds for partial derivatives $(\partial^2/\partial x_i \partial x_j) f_n$ and $(\partial/\partial x_i) f_n$, we obtain that for any $r \in (0,1)$, $y \in \mathbb{R}^d$, and $n \geq 1$,

$$|\mathbf{L}_1 f_{n,r}(y)| \leq c_4 r^{-2} + c_5 r^{-1} \leq c_6 r^{-2}.$$

Hence, there exists a positive constant $c_7$ such that for any $r \in (0,1)$, $y \in \mathbb{R}^d$, and $n \geq 1$,

$$|\mathbf{L} f_{n,r}(y)| \leq c_7 r^{-2}.$$

Therefore for any $r \in (0,1)$ and any $n \geq 1$,

$$\mathbb{E}_x f_{n,r}(X(\tau_{B(x,r)} \wedge t)) = \mathbb{E}_x \int_0^{\tau_{B(x,r)} \wedge t} \mathbf{L} f_{n,r}(X_s) ds \leq c_7 r^{-2} t.$$

Letting $n \uparrow \infty$, we get

$$\mathbb{E}_x f_r(X(\tau_{B(x,r)} \wedge t)) \leq c_7 r^{-2} t.$$

If $X$ exits $B(x,r)$ before time $t$, then $f_r(X(\tau_{B(x,r)} \wedge t)) = 1$, so the left hand side is greater than $\mathbb{P}_x(\tau_{B(x,r)} \leq t)$.

□

**Lemma 2.2** *Suppose $\varepsilon \in (0,1)$ is a constant. Then there exists $C_2 > 0$ such that for every $x \in \mathbb{R}^d$ and $r \in (0,1)$,*

$$\inf_{z \in B(x,(1-\varepsilon)r)} \mathbb{E}_z \tau_{B(x,r)} \geq C_2 \varepsilon^2 r^2.$$

**Proof.** The proof is an easy modification of Lemma 3.2 of [8]. □

**Lemma 2.3** *There exist $r_0 \in (0,1)$ and $C_3 > 0$ such that for any $x \in \mathbb{R}^d$ and any $r \in (0, r_0)$ we have*

$$\sup_{z \in B(x,r)} \mathbb{E}_z \tau_{B(x,r)} \leq C_3 r^2.$$

**Proof.** Let $g \in C_0^2(\mathbb{R}^d)$ be a function taking values in $[0,2]$ such that $g(y) = |y|^2$ for $|y| < 1$ and $g(y) = 2$ for $|y| \geq 2$. For $x \in \mathbb{R}^d$ any $r > 0$, put $f(y) = g((y-x)/r)$. Then for $y \in B(x,r)$,

$$\begin{aligned}
\mathbf{L}_1 f(y) &= \sum_{i=1}^d 2a_{ii}(y)r^{-2} + \sum_{i=1} 2b_i(y)(y_i - x_i)r^{-2} \\
&\geq \sum_{i=1}^d 2a_{ii}(y)r^{-2} - \sum_{i=1} 2|b_i(y)|\,|y_i - x_i|r^{-2} \\
&\geq \sum_{i=1}^d 2a_{ii}(y)r^{-2} - \sum_{i=1} 2|b_i(y)|r^{-1} \\
&\geq 2c_1 r^{-2} - 2c_2 r^{-1} \\
&\geq c_3 r^{-2}
\end{aligned}$$

provided $r$ is small enough. From the proof of Lemma 2.1 we know that, for any $r \in (0,1)$,

$$|\mathbf{L}_2 f(y)| \leq c_4 r^{-\alpha}, \quad y \in B(x,r).$$

Thus we know that there exist $r_0 \in (0,1)$ and $c_5 > 0$ such that for any $r \in (0, r_0)$,

$$\mathbf{L} f(y) \geq c_5 r^{-2}, \quad y \in B(x,r).$$

Therefore we have for $r \in (0, r_0)$,

$$\begin{aligned}
\mathbb{E}_x \tau_{B(x,r)} &= \lim_{t\uparrow\infty} \mathbb{E}_x[\tau_{B(x,r)} \wedge t] \\
&\leq c_5^{-1} r^2 \lim_{t\uparrow\infty} \mathbb{E}_x \int_0^{\tau_{B(x,r)}\wedge t} \mathbf{L} f(X_s)ds \\
&\leq c_5^{-1} r^2 \lim_{t\uparrow\infty} \mathbb{E}_x f(X(\tau_B(x,r) \wedge t)) \\
&\leq 2c_5^{-1} r^2.
\end{aligned}$$

□

**Lemma 2.4** *There exists $C_4 > 0$ such that for any $x \in \mathbb{R}^d$, any $r \in (0,1)$ and any closed subset $A$ of $B(x,r)$, we have*

$$\mathbb{P}_y(T_A < \tau_{B(x,2r)}) \geq C_4 r^{2-\alpha} \frac{|A|}{|B(x,r)|}, \forall y \in B(x,r).$$

**Proof.** The proof is similar to that of Lemma 3.4 of [8]

For $x, y \in \mathbb{R}^d$, let

$$j(x,y) = \frac{k(x,y)}{|x-y|^{d+\alpha}}.$$

Then $(j(x,y)dy, dt)$ is a Lévy system for $X$. Using the same argument as in the proof of Lemma 3.5 of [8], we can easily get the following result which does not depend on the continuous component of the process.

**Lemma 2.5** *There exist positive constants $C_5$ and $C_6$ such that if $x \in \mathbb{R}^d$, $r > 0$, $z \in B(x,r)$ and $H$ is a bounded nonnegative function with support in $B(x,2r)^c$, then*

$$\mathbb{E}_z H(X(\tau_{B(x,r)})) \le C_5(\mathbb{E}_z \tau_{B(x,r)}) \int H(y) j(x,y) dy$$

*and*

$$\mathbb{E}_z H(X(\tau_{B(x,r)})) \ge C_6(\mathbb{E}_z \tau_{B(x,r)}) \int H(y) j(x,y) dy.$$

# 3 Proof of Harnack inequality

**Theorem 3.1** *Let $r_0$ be as in Lemma 2.3 and $r \in (0, r_0)$. There exists a constant $C_7 > 0$ such that for any $z_0 \in \mathbb{R}^d$ and any nonnegative bounded function in $\mathbb{R}^d$ which is harmonic with respect to $X$ in $B(z_0, r)$ we have*

$$u(x) \le C_7 u(y), \quad x, y \in B(z_0, r/2).$$

**Proof.** Suppose that $u$ is nonnegative and bounded in $\mathbb{R}^d$ and harmonic with respect to $X$ in $B(z_0, r)$. By looking at $u + \epsilon$ and letting $\epsilon \downarrow 0$, we may suppose that $u$ is bounded from below by a positive constant. By looking at $au$ for a suitable $a > 0$, we may suppose that $\inf_{B(z_0, r/2)} u = 1/2$. We want to bound $u$ from above in $B(z_0, r/2)$ by a constant depending only on $r$, $d$ and $\alpha$. Choose $z_1 \in B(z_0, r/2)$ such that $u(z_1) \le 1$. Let $\gamma = 2 - \alpha$. Choose $\rho \in (1, \gamma^{-1})$.

For $i \ge 1$ let

$$r_i = \frac{c_1 r}{i^\rho},$$

where $c_1$ is a constant to be determined later. We require first of all that $c_1$ is small enough so that

$$\sum_{i=0}^{\infty} r_i \le \frac{r}{8}. \tag{3.1}$$

Recall that by Lemma 2.4, there exists $c_2 > 0$ such that for any $\bar{z} \in \mathbb{R}^d, \bar{r} \in (0,1), A \subset B(\bar{z}, \bar{r}/2)$ and $\bar{x} \in B(\bar{z}, \bar{r}/2)$,

$$\mathbb{P}_{\bar{x}}(T_A < \tau_{B(\bar{z},\bar{r})}) \ge c_2 \bar{r}^\gamma \frac{|A|}{|B(\bar{z}, \bar{r}/2)|}. \tag{3.2}$$

Let $c_3$ be a constant such that

$$c_3 \le c_2 2^{-4+\rho\gamma}\,.$$

Once $c_1$ and $c_3$ have been chosen, choose $K_1$ sufficiently large so that

$$\frac{1}{4} c_2 K_1 \exp(r^\gamma c_1 c_3 i^{1-\rho\gamma}) c_1^{3\gamma+d} r^{3\gamma} \ge 2 i^{3\rho\gamma+\rho d} \tag{3.3}$$

for all $i \ge 1$. Such a choice is possible since $\rho\gamma < 1$. Note that $K_1$ will depend only on $r, d$ and $\alpha$ as well as constants $c_1, c_2, c_3$. Suppose now that there exists $x_1 \in B(z_0, r/2)$ with $u(x_1) \ge K_1$. We will show that in this case there exists a sequence $\{(x_j, K_j) : j \ge 1\}$ with $x_{j+1} \in B(x_j, 2r_j) \subset B(z_0, 3r/4)$, $K_j = u(x_j)$, and

$$K_j \ge K_1 \exp(r^\gamma c_1 c_3 j^{1-\rho\gamma}). \tag{3.4}$$

Since $1-\rho\gamma > 0$, we have $K_j \to \infty$, a contradiction to the assumption that $u$ is bounded. We can then conclude that $u$ must be bounded by $K_1$ on $B(z_0, r/2)$, and hence $u(x) \le 2K_1 u(y)$ if $x, y \in B(z_0, r/2)$.

Suppose that $x_1, x_2, \ldots, x_i$ have been selected and that (3.4) holds for $j = 1, \ldots, i$. We will show that there exists $x_{i+1} \in B(x_i, 2r_i)$ such that if $K_{i+1} = u(x_{i+1})$, then (3.4) holds for $j = i+1$; we then use induction to conclude that (3.4) holds for all $j$.

Let

$$A_i = \{y \in B(x_i, r_i/4) : u(y) \ge K_i r^{2\gamma}\}$$

First we prove that

$$\frac{|A_i|}{|B(x_i, r_i/4)|} \le \frac{1}{4}. \tag{3.5}$$

To prove this claim, we suppose to the contrary that $|A_i|/|B(x_i, r_i/4)| > 1/4$. Let $F$ be a compact subset of $A_i$ with $|F|/|B(x_i, r_i/4)| > 1/4$. Recall that $r \ge 8r_i$. By the definition of harmonicity, (3.2), (3.3) and (3.4),

$$\begin{aligned}
1 &\ge u(z_1) \ge \mathbb{E}_{z_1}[u(X_{T_F \wedge \tau_{B(z_0,r)}}); T_F < \tau_{B(z_0,r)}] \\
&\ge K_i r_i^{2\gamma} \mathbb{P}_{z_1}(T_F < \tau_{B(z_0,r)}) \\
&\ge c_2 K_i r_i^{2\gamma} r^\gamma \frac{|F|}{|B(z_0, r/4)|} \\
&\ge 2^{-2} c_2 K_i r_i^{3\gamma} (r_i/r)^d \\
&\ge 2^{-2} c_2 K_1 \exp(r^\gamma c_1 c_3 i^{1-\rho\gamma}) r_i^{3\gamma} (r_i/r)^d \\
&\ge 2^{-2} c_2 K_1 \exp(r^\gamma c_1 c_3 i^{1-\rho\gamma}) c_1^{3\gamma+d} r^{3\gamma}\, i^{-3\rho\gamma} i^{-\rho d} \\
&\ge 2 i^{3\rho\gamma+\rho d} i^{-3\rho\gamma-\rho d} = 2\,,
\end{aligned}$$

where the last line follows by (3.3). This is a contradiction, and therefore (3.5) is valid.

Write $\tau_i$ for $\tau_{B(x_i,r_i/2)}$. Set $M_i = \sup_{B(x_i,r_i)} u$. Let $E_i$ be a compact subset of $B(x_i, r_i/4)\setminus A_i$ such that $|E_i|/|B(x_i, r_i/4)| \geq 1/2$. In view of (3.5) such a choice is possible. Let $p_i = \mathbb{P}_{x_i}(T_{E_i} < \tau_i)$. We have

$$\begin{aligned} K_i = u(x_i) &= \mathbb{E}_{x_i}[u(X_{T_{E_i}\wedge\tau_i}); T_{E_i} < \tau_i] \\ &+ \mathbb{E}_{x_i}[u(X_{T_{E_i}\wedge\tau_i}); T_{E_i} \geq \tau_i, X_{\tau_i} \in B(x_i, r_i)] \\ &+ \mathbb{E}_{x_i}[u(X_{T_{E_i}\wedge\tau_i}); T_{E_i} \geq \tau_i, X_{\tau_i} \notin B(x_i, r_i)]. \end{aligned} \tag{3.6}$$

Since $E_i$ is compact, we have

$$\mathbb{E}_{x_i}[u(X_{T_{E_i}\wedge\tau_i}); T_{E_i} < \tau_i] \leq K_i r_i^{2\gamma}\mathbb{P}_{x_i}(T_{E_i} < \tau_i) \leq K_i r_i^{2\gamma}.$$

We also have

$$\mathbb{E}_{x_i}[u(X_{T_{E_i}\wedge\tau_i}); T_{E_i} \geq \tau_i, X_{\tau_i} \in B(x_i, r_i)] \leq M_i(1 - p_i).$$

Inequality (3.5) implies in particular that there exists $y_i \in B(x_i, r_i/4)$ with $u(y_i) \leq K_i r_i^{2\gamma}$. We then have, by Lemmas 2.2, 2.3 and 2.5,

$$\begin{aligned} K_i r_i^{2\gamma} &\geq u(y_i) \geq \mathbb{E}_{y_i}[u(X_{\tau_i}) : X_{\tau_i} \notin B(x_i, r_i)] \\ &\geq c_4\mathbb{E}_{x_i}[u(X_{\tau_i}) : X_{\tau_i} \notin B(x_i, r_i)]\,. \end{aligned} \tag{3.7}$$

Therefore

$$\mathbb{E}_{x_i}[u(X_{T_{E_i}\wedge\tau_i}); T_{E_i} \geq \tau_i, X_{\tau_i} \notin B(x_i, r_i)] \leq c_5 K_i r_i^{2\gamma}$$

for a positive constant $c_5$. Consequently we have

$$K_i \leq (1 + c_5)K_i r_i^{2\gamma} + M_i(1 - p_i). \tag{3.8}$$

Rearranging, we get

$$M_i \geq K_i \left(\frac{1 - (1 + c_5)r_i^{2\gamma}}{1 - p_i}\right). \tag{3.9}$$

By (3.2) and by the fact that $|E_i|/|B(x_i, r_i/4)| \geq 1/2$,

$$p_i \geq \frac{1}{2}c_2 r_i^{\gamma}. \tag{3.10}$$

By choosing

$$c_1 \leq \left(\frac{1}{4}\frac{c_2}{1 + c_5}\right)^{\gamma}\frac{1}{r}\,,$$

and by using the fact that the $r_i$'s are decreasing, we get

$$(1 + c_5)r_i^{2\gamma} \leq \frac{1}{4}c_2 r_i^{\gamma} \tag{3.11}$$

for all $i$. Therefore

$$M_i \geq K_i \left( \frac{1 - \frac{1}{2} p_i}{1 - p_i} \right) > (1 + \frac{p_i}{2}) K_i.$$

Using the definition of $M_i$ and (3.10), there exists a point $x_{i+1} \in \overline{B(x_i, r_i)} \subset B(x_i, 2r_i)$ such that

$$K_{i+1} = u(x_{i+1}) \geq K_i (1 + c_2 r_i^\gamma / 4).$$

Taking logarithms and writing

$$\log K_{i+1} = \log K_i + \sum_{j=1}^{i} [\log K_{j+1} - \log K_j],$$

we have

$$\begin{aligned}
\log K_{i+1} &\geq \log K_1 + \sum_{j=1}^{i} \log(1 + c_2 r_j^\gamma / 4) \\
&\geq \log K_1 + \frac{c_2}{8} \sum_{j=1}^{i} r_j^\gamma \\
&= \log K_1 + \frac{c_2}{8} r^\gamma c_1^\gamma \sum_{j=1}^{i} j^{-\rho\gamma} \\
&\geq \log K_1 + \frac{c_2}{8} r^\gamma c_1 i^{1-\rho\gamma} \\
&\geq \log K_1 + r^\gamma c_1 c_3 (i+1)^{1-\rho\gamma},
\end{aligned}$$

where the last line follows by choice of $c_3$.

Hence (3.4) holds for $i + 1$ provided we choose $c_1$ small enough so that (3.1) and (3.11) holds. The proof is now finished. □

By using standard chain argument, we can easily get the following consequence of the theorem above.

**Corollary 3.2** *For any domain $D$ of $\mathbb{R}^d$ and any compact subset $K$ of $D$, there exists a constant $C_8 > 0$ such that for any function $h$ which is nonnegative bounded in $\mathbb{R}^d$ and harmonic with respect to $X$ in $D$, we have*

$$h(x) \leq C_8 h(y), \quad x, y \in K.$$

**Proof of Theorem 1.1.** It remains to remove the boundedness assumption in the corollary above. This is done in the same way as in the proof of Theorem 2.4 in [8]. We include the proof for reader's convenience.

Choose a bounded domain $U$ such that $K \subset U \subset \overline{U} \subset D$. If $h$ is harmonic with respect to $X$ in $D$, then

$$h(x) = \mathbb{E}_x[h(X(\tau_U))1_{\{\tau_U<\infty\}}], \quad x \in U.$$

For any $n \geq 1$, define

$$h_n(x) = \mathbb{E}_x[(h \wedge n)(X(\tau_U))1_{\{\tau_U<\infty\}}], \quad x \in \mathbb{R}^d.$$

Then $h_n$ is a bounded nonnegative function on $\mathbb{R}^d$, harmonic with respect to $X$ in $U$, and

$$\lim_{n\uparrow\infty} h_n(x) = h(x), \quad x \in \mathbb{R}^d.$$

It follows from Corollary 3.2 that there exists a constant $c = c(U, K) > 0$ such that

$$h_n(x) \leq c h_n(y), \quad x, y \in K, n \geq 1.$$

Letting $n \uparrow \infty$, we get that

$$h(x) \leq c h(y), \quad x, y \in K.$$

□

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