Martin boundary of unbounded sets for purely discontinuous Feller processes

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Abstract

In this paper, we study the Martin kernels of general open sets associated with inaccessible points for a large class of purely discontinuous Feller processes in metric measure spaces.

1   Introduction and setup

This paper is a companion of [14] and here we continue our study of the Martin boundary of Greenian open sets with respect to purely discontinuous Feller processes in metric measure spaces. In [14], we have shown that (1) if D is a Greenian open set and $z_0 \in \partial D$ is accessible from D, then the Martin kernel of D associated with z_0 is a minimal harmonic function; (2) if D is an unbounded Greenian open set and ∞ is accessible from D, then the Martin kernel of D associated with ∞ is a minimal harmonic function. The goal of this paper is to study the Martin kernels of D associated with inaccessible boundary points of D, including ∞.

The background and recent progress on the Martin boundary is explained in the companion paper [14]. Martin kernels of bounded open sets D associated with both accessible and inaccessible boundary points of D have been studied in the recent preprint [5]. In this paper, we are mainly concerned with the Martin kernels of unbounded open sets associated with ∞ when ∞ is inaccessible from D. For completeness, we also spell out some of the details of the argument for dealing with the Martin kernels of unbounded open sets associated with inaccessible boundary points of D. To accomplish our task of studying the Martin kernels of general open sets, we follow the ideas of [1, 7] and first study the oscillation reduction

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of ratios of positive harmonic functions. In the case of isotropic α-stable processes, the oscillation reduction at infinity and Martin kernel associated with ∞ follow easily from the corresponding results at finite boundary points by using the sphere inversion and Kelvin transform. For the general processes dealt with in this paper, the Kelvin transform method does not apply.

Now we describe the setup of this paper which is the same as that of [14] and then give the main results of this paper.

Let $(\mathfrak{X}, d, m)$ be a metric measure space with a countable base such that all bounded closed sets are compact and the measure m has full support. For $x \in \mathfrak{X}$ and $r > 0$, let $B(x, r)$ denote the ball centered at x with radius r. Let $R_0 \in (0, \infty]$ be the localization radius such that $\mathfrak{X} \setminus B(x, 2r) \neq \emptyset$ for all $x \in \mathfrak{X}$ and all $r < R_0$.

Let $X = (X_t, \mathcal{F}_t, \mathbb{P}_x)$ be a Hunt process on $\mathfrak{X}$. We will assume the following

Assumption A: X is a Hunt process admitting a strong dual process $\widehat{X}$ with respect to the measure m and $\widehat{X}$ is also a Hunt process. The transition semigroups (P_t) and $(\widehat{P}_t)$ of X and $\widehat{X}$ are both Feller and strong Feller. Every semi-polar set of X is polar.

In the sequel, all objects related to the dual process $\widehat{X}$ will be denoted by a hat. We first recall that a set is polar (semi-polar, respectively) for X if and only if it is polar (semi-polar, respectively) for $\widehat{X}$.

If D is an open subset of $\mathfrak{X}$ and $\tau_D = \inf\{t > 0 : X_t \notin D\}$ the exit time from D, the killed process X^D is defined by $X_t^D = X_t$ if $t < \tau_D$ and $X_t^D = \partial$ where ∂ is an extra point added to $\mathfrak{X}$. Then, under assumption **A**, X^D admits a unique (possibly infinite) Green function (potential kernel) $G_D(x, y)$ such that for every non-negative Borel function f,

$$G_D f(x) := \mathbb{E}_x \int_0^{\tau_D} f(X_t) dt = \int_D G_D(x.y) \, m(dy) \,,$$

and $G_D(x, y) = \widehat{G}_D(y, x)$, $x, y \in D$, with $\widehat{G}_D(y, x)$ the Green function of $\widehat{X}^D$. It is assumed throughout the paper that $G_D(x, y) = 0$ for $(x, y) \in (D \times D)^c$. We also note that the killed process X^D is strongly Feller, see e.g. the first part of the proof of Theorem on [3, pp. 68–69].

Let ∂D denote the boundary of the open set D in the topology of $\mathfrak{X}$. Recall that $z \in \partial D$ is said to be regular for X if $\mathbb{P}_z(\tau_D = 0) = 1$ and irregular otherwise. We will denote the set of regular points of ∂D for X by D^{reg} (and the set of regular points of ∂D for $\widehat{X}$ by $\widehat{D}^{\mathrm{reg}}$). It is well known that the set of irregular points is semipolar, hence polar under **A**.

Suppose that D is Greenian, that is, the Green function $G_D(x, y)$ is finite away from the diagonal. Under this assumption, the killed process X^D is transient (and strongly Feller). In particular, for every bounded Borel function f on D, $G_D f$ is continuous.

The process X, being a Hunt process, admits a Lévy system (J, H) where $J(x, dy)$ is a kernel on $\mathfrak{X}$ (called the Lévy kernel of $\mathfrak{X}$), and $H = (H_t)_{t \geq 0}$ is a positive continuous additive functional of X. We assume that $H_t = t$ so that for every function $f : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$

2

vanishing on the diagonal and every stopping time T,

$$\mathbb{E}_x \sum_{0 < s \leq T} f(X_{s-}, X_s) = \mathbb{E}_x \int_0^T f(X_s, y) J(X_s, dy) ds .$$

Let $D \subset \mathfrak{X}$ be a Greenian open set. By replacing T with τ_D in the displayed formula above and taking $f(x,y) = \mathbf{1}_D(x) \mathbf{1}_A(y)$ with $A \subset \overline{D}^c$, we get that

$$\mathbb{P}_x(X_{\tau_D} \in A, \tau_D < \zeta) = \mathbb{E}_x \int_0^{\tau_D} J(X_s, A) ds = \int_D G_D(x, y) J(y, A) m(dy) \qquad (1.1)$$

where ζ is the life time of X. Similar formulae hold for $\widehat{X}$ and $\widehat{J}(x, dy) m(dx) = J(y, dx) m(dy)$.
Assumption C: The Lévy kernels of X and $\widehat{X}$ have the form $J(x, dy) = j(x, y) m(dy)$, $\widehat{J}(x, dy) = \widehat{j}(x, y) m(dy)$, where $j(x, y) = \widehat{j}(y, x) > 0$ for all $x, y \in \mathfrak{X}$, $x \neq y$.

We will always assume that Assumptions **A** and **C** hold true.

In the next assumption, z_0 is a point in $\mathfrak{X}$ and $R \leq R_0$.

Assumption C1(z_0, R): For all $0 < r_1 < r_2 < R$, there exists a constant $c = c(z_0, r_2/r_1) > 0$ such that for all $x \in B(z_0, r_1)$ and all $y \in \mathfrak{X} \setminus B(z_0, r_2)$,

$$c^{-1} j(z_0, y) \leq j(x, y) \leq c j(z_0, y), \qquad c^{-1} \widehat{j}(z_0, y) \leq \widehat{j}(x, y) \leq c \widehat{j}(z_0, y).$$

In the next assumption we require that the localization radius $R_0 = \infty$ and that D is unbounded. Again, z_0 is a point in $\mathfrak{X}$.

Assumption C2(z_0, R): For all $R \leq r_1 < r_2 < \infty$, there exists a constant $c = c(z_0, r_2/r_1) > 0$ such that for all $x \in B(z_0, r_1)$ and all $y \in \mathfrak{X} \setminus B(z_0, r_2)$,

$$c^{-1} j(z_0, y) \leq j(x, y) \leq c j(z_0, y), \qquad c^{-1} \widehat{j}(z_0, y) \leq \widehat{j}(x, y) \leq c \widehat{j}(z_0, y).$$

We *define* the Poisson kernel of X on an open set $D \in \mathfrak{X}$ by

$$P_D(x, z) = \int_D G_D(x, y) j(y, z) m(dy), \qquad x \in D, z \in D^c.$$

By (1.1), we see that $P_D(x, \cdot)$ is the density of the exit distribution of X from D restricted to $\overline{D}^c$:

$$\mathbb{P}_x(X_{\tau_D} \in A, \tau_D < \zeta) = \int_A P_D(x, z) m(dz), \qquad A \subset \overline{D}^c.$$

Recall that $f : \mathfrak{X} \to [0, \infty)$ is regular harmonic in D with respect to X if

$$f(x) = \mathbb{E}_x[f(X_{\tau_D}), \tau_D < \zeta], \quad \text{for all } x \in D ,$$

and it is harmonic in D with respect to X if for every relatively compact open $U \subset \overline{U} \subset D$,

$$f(x) = \mathbb{E}_x[f(X_{\tau_U}), \tau_D < \zeta], \quad \text{for all } x \in U .$$

3

Recall also that $f : D \to [0, \infty)$ is harmonic in D with respect to X^D if for every relatively compact open $U \subset \overline{U} \subset D$,

$$f(x) = \mathbb{E}_x[f(X^D_{\tau_U}), \tau_U < \zeta], \quad \text{for all } x \in U.$$

The next pair of assumptions is about an approximate factorization of positive harmonic functions. This approximate factorization plays a crucial role in proving the oscillation reduction. The first one is an approximate factorization of harmonic functions at a finite boundary point.

Assumption F1(z_0, R): Let $z_0 \in \mathfrak{X}$ and $R \leq R_0$. For any $\frac{1}{2} < a < 1$, there exists $C(a) = C(z_0, R, a) \geq 1$ such that for every $r \in (0, R)$, every open set $D \subset B(z_0, r)$, every nonnegative function f on $\mathfrak{X}$ which is regular harmonic in D with respect to X and vanishes in $B(z_0, r) \cap (\overline{D}^c \cup D^{\text{reg}})$, and all $x \in D \cap B(z_0, r/8)$,

$$C(a)^{-1} \mathbb{E}_x[\tau_D] \int_{\overline{B}(z_0, ar/2)^c} j(z_0, y) f(y) m(dy)$$

$$\leq f(x) \leq C(a) \mathbb{E}_x[\tau_D] \int_{\overline{B}(z_0, ar/2)^c} j(z_0, y) f(y) m(dy). \quad (1.2)$$

In the second assumption we require that the localization radius $R_0 = \infty$ and that D is unbounded.

Assumption F2(z_0, R): Let $z_0 \in \mathfrak{X}$ and $R > 0$. For any $1 < a < 2$, there exists $C(a) = C(z_0, R, a) \geq 1$ such that for every $r \geq R$, every open set $D \subset \overline{B}(z_0, r)^c$, every nonnegative function f on $\mathfrak{X}$ which is regular harmonic in D with respect to X and vanishes on $\overline{B}(z_0, r)^c \cap (\overline{D}^c \cup D^{\text{reg}})$, and all $x \in D \cap \overline{B}(z_0, 8r)^c$,

$$C(a)^{-1} P_D(x, z_0) \int_{B(z_0, 2ar)} f(z) m(dz)$$

$$\leq f(x) \leq C(a) P_D(x, z_0) \int_{B(z_0, 2ar)} f(z) m(dz). \quad (1.3)$$

Let $D \subset \mathfrak{X}$ be an open set. A point $z \in \partial D$ is said to be accessible from D with respect to X if

$$P_D(x, z) = \int_D G_D(x, w) j(w, z) m(dw) = \infty \quad \text{for all } x \in D, \quad (1.4)$$

and inaccessible otherwise.

In case D is unbounded we say that ∞ is accessible from D with respect to X if

$$\mathbb{E}_x \tau_D = \int_D G_D(x, w) m(dw) = \infty \quad \text{for all } x \in D \quad (1.5)$$

and inaccessible otherwise. The notion of accessible and inaccessible points was introduced in [2].

In [14], we have discussed the oscillation reduction and Martin boundary points at accessible points, and showed that the Martin kernel associated with an accessible point is a

minimal harmonic function. As in [14], the main tool in studying the Martin kernel associated with inaccessible points is the oscillation reduction at inaccessible points. To prove the oscillation reduction at inaccessible points, we need to assume one of the following additional conditions on the asymptotic behavior of the Lévy kernel:

Assumption E1(z_0, R): For every $r \in (0, R)$,

$$\lim_{d(z_0, y) \to 0} \sup_{z: d(z_0, z) > r} \frac{j(z, z_0)}{j(z, y)} = \lim_{d(y, z_0) \to 0} \inf_{z: d(z_0, z) > r} \frac{j(z, z_0)}{j(z, y)} = 1.$$

Assumption E2(z_0, R): For every $r > R$,

$$\lim_{d(z_0, z) \to \infty} \sup_{y: d(z_0, y) < r} \frac{j(z, z_0)}{j(z, y)} = \lim_{d(z_0, z) \to \infty} \inf_{y: d(z_0, y) < r} \frac{j(z, z_0)}{j(z, y)} = 1.$$

Combining Theorems 2.4 and 2.8 below for inaccessible points with the results in [14] for accessible ones, we have the following, which is the first main result of this paper.

Theorem 1.1 *Let $D \subset \mathfrak{X}$ be an open set. (a) Suppose that $z_0 \in \partial D$. Assume that there exists $R \leq R_0$ such that $\mathbf{C1}(z_0, R)$ and $\mathbf{E1}(z_0, R)$ hold, and that $\widehat{X}$ satisfies $\boldsymbol{F1}(z_0, R)$. Let $r \leq R$ and let f_1 and f_2 be nonnegative functions on $\mathfrak{X}$ which are regular harmonic in $D \cap B(z_0, r)$ with respect to $\widehat{X}$ and vanish on $B(z_0, r) \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$. Then the limit*

$$\lim_{D \ni x \to z_0} \frac{f_1(x)}{f_2(x)}$$

exists and is finite.
(b) Suppose that $R_0 = \infty$ and D is an unbounded subset of $\mathfrak{X}$. Assume that there is a point $z_0 \in \mathfrak{X}$ such that $\mathbf{C2}(z_0, R)$ and $\mathbf{E2}(z_0, R)$ hold, and that $\widehat{X}$ satisfies $\boldsymbol{F2}(z_0, R)$ for some $R > 0$. Let $r > R$ and let f_1 and f_2 be nonnegative functions on $\mathfrak{X}$ which are regular harmonic in $D \cap \overline{B}(z_0, r)^c$ with respect to $\widehat{X}$ and vanish on $\overline{B}(z_0, r)^c \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$. Then the limit

$$\lim_{D \ni x \to \infty} \frac{f_1(x)}{f_2(x)}$$

exists and is finite.

For $D \subset \mathfrak{X}$, let $\partial_M D$ denote the Martin boundary of D with respect to X^D in the sense of Kunita-Watanabe [15], see Section 3 for more details. A point $w \in \partial_M D$ is said to be minimal if the Martin kernel $M_D(\cdot, w)$ is a minimal harmonic function with respect to X^D. We will use $\partial_m D$ to denote the minimal Martin boundary of D with respect to X^D. A point $w \in \partial_M D$ is said to be a *finite Martin boundary point* if there exists a bounded (with respect to the metric d) sequence $(y_n)_{n \geq 1} \subset D$ converging to w in the Martin topology. A point $w \in \partial_M D$ is said to be an *infinite Martin boundary point* if there exists an unbounded (with respect to the metric d) sequence $(y_n)_{n \geq 1} \subset D$ converging to w in the Martin topology. We note that these two definitions do not rule out the possibility that a point $w \in \partial_M D$ is at

5

the same time finite and infinite Martin boundary point. We will show in Corollary 1.5(a) that under appropriate and natural assumptions this cannot happen. A point $w \in \partial_M D$ is said to be associated with $z_0 \in \partial D$ if there is a sequence $(y_n)_{n \geq 1} \subset D$ converging to w in the Martin topology and to z_0 in the topology of $\mathfrak{X}$. The set of Martin boundary points associated with z_0 is denoted by $\partial_M^{z_0} D$. A point $w \in \partial_M D$ is said to be associated with ∞ if w is an infinite Martin boundary point. The set of Martin boundary points associated with ∞ is denoted by $\partial_M^\infty D$. $\partial_M^f D$ and $\partial_m^f D$ will be used to denote the finite part of the Martin boundary and minimal boundary respectively. Note that $\partial_M^\infty D$ is the set of infinite Martin boundary points.

Recall that we denote the set of regular points of ∂D for X by D^{reg}. Here is our final assumption.

Assumption G: $\lim_{D \ni x \to z} G_D(x, y) = 0$ for every $z \in D^{\mathrm{reg}}$ and every $y \in D$.

From Theorem 1.1 and the results in [14], we have the following.

Theorem 1.2 *Let $D \subset \mathfrak{X}$ be an open set. (a) Suppose that $z_0 \in \partial D$. Assume that there exists $R \leq R_0$ such that $\mathbf{C1}(z_0, R)$ and $\mathbf{E1}(z_0, R)$ hold, and that $\widehat{X}$ satisfies $\mathbf{F1}(z_0, R)$. Then there is only one Martin boundary point associated with z_0.*
(b) Assume further that $\mathbf{G}$ holds, X satisfies $\mathbf{F1}(z_0, R)$, and that for all $r \in (0, R]$,

$$\sup_{x \in D \cap B(z_0, r/2)} \sup_{y \in \mathfrak{X} \setminus B(z_0, r)} \max(G_D(x, y), \widehat{G}_D(x, y)) =: c(r) < \infty, \tag{1.6}$$

and in case of unbounded D, for $r \in (0, r_0]$,

$$\lim_{x \to \infty} G_D(x, y) = 0 \quad \text{for all } y \in D \cap B(z_0, r).$$

Then the Martin boundary point associated with $z_0 \in \partial D$ is minimal if and only if z_0 is accessible from D with respect to X.

Corollary 1.3 *Suppose that the assumptions of Theorem 1.2(b) are satisfied for all $z_0 \in \partial D$ (with $c(r)$ in (1.6) independent of z_0). Suppose further that, for any inaccessible point $z_0 \in \partial D$, $\lim_{D \ni x \to z_0} j(x, z_0) = \infty$.*
(a) Then the finite part of the Martin boundary $\partial_M D$ can be identified with ∂D.
(b) If D is bounded, then ∂D and $\partial_M D$ are homeomorphic.

Theorem 1.4 *(a) Suppose that $R_0 = \infty$ and D is an unbounded open subset of $\mathfrak{X}$. If there is a point $z_0 \in \mathfrak{X}$ such that $\mathbf{C2}(z_0, R)$ and $\mathbf{E2}(z_0, R)$ hold, and $\widehat{X}$ satisfies $\mathbf{F2}(z_0, R)$, then there is only one Martin boundary point associated with ∞.*
(b) Assume further that $\mathbf{G}$ holds, X satisfies $\mathbf{F2}(z_0, R)$, and that for all $r \geq R$,

$$\sup_{x \in D \cap B(z_0, r/2)} \sup_{y \in \mathfrak{X} \setminus B(z_0, r)} \max(G_D(x, y), \widehat{G}_D(x, y)) =: c(r) < \infty \tag{1.7}$$

and

$$\lim_{x \to \infty} G_D(x, y) = 0 \quad \text{for all } y \in D. \tag{1.8}$$

Then the Martin boundary point associated with ∞ is minimal if and only if ∞ is accessible from D.

Corollary 1.5 *Let $R_0 = \infty$ and $D \subset \mathfrak{X}$ be unbounded. Suppose that the assumptions of Theorem 1.2(b) are satisfied for all $z_0 \in \partial D$ (with $c(r)$ in (1.6) independent of z_0) and that the assumptions of Theorem 1.4(a) and (b) are satisfied. Then*
(a) $\partial_M^f D \cap \partial_M^\infty D = \emptyset$.
(b) Suppose that, for any inaccessible point $z_0 \in \partial D$, $\lim_{D \ni x \to z_0} j(x, z_0) = \infty$. Then the Martin boundary $\partial_M D$ is homeomorphic with the one-point compactification of ∂D.

In case when X is an isotropic stable process, Theorems 1.2 and 1.4 were proved in [1].

In Section 2 we provide the proof of Theorem 1.1 for inaccessible points. Section 3 contains the proofs of Theorems 1.2 and 1.4. In Section 4 we discuss some Lévy processes in $\mathbb{R}^d$ satisfying our assumptions.

We will use the following conventions in this paper. $c, c_0, c_1, c_2, \cdots$ stand for constants whose values are unimportant and which may change from one appearance to another. All constants are positive finite numbers. The labeling of the constants $c_0, c_1, c_2, \cdots$ starts anew in the statement of each result. We will use ":=" to denote a definition, which is read as "is defined to be". We denote $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. Further, $f(t) \sim g(t)$, $t \to 0$ ($f(t) \sim g(t)$, $t \to \infty$, respectively) means $\lim_{t \to 0} f(t)/g(t) = 1$ ($\lim_{t \to \infty} f(t)/g(t) = 1$, respectively). Throughout the paper we will adopt the convention that $X_\zeta = \partial$ and $u(\partial) = 0$ for every function u.

2   Oscillation reductions for inaccessible points

To handle the oscillation reductions at inaccessible points, in this section we will assume, in addition to the corresponding assumptions in [14], that $\mathbf{E1}(z_0, R)$ ($\mathbf{E2}(z_0, R)$ respectively) holds when we deal with finite boundary points (respectively infinity).

2.1   Infinity

Throughout this subsection we will assume that $R_0 = \infty$ and $D \subset \mathfrak{X}$ is an unbounded open set. We will deal with oscillation reduction at ∞ when ∞ is inaccessible from D with respect X. We further assume that there exists a point $z_0 \in \mathfrak{X}$ such that $\mathbf{E2}(z_0, R)$ and $\mathbf{C2}(z_0, R)$ hold, and that $\widehat{X}$ satisfies $\mathbf{F2}(z_0, R)$ for some $R > 0$. We will fix z_0 and R and use the notation $B_r = B(z_0, r)$. The next lemma is a direct consequence of assumption $\mathbf{E2}(z_0, R)$.

Lemma 2.1 *For any $q \geq 2$, $r \geq R$ and $\epsilon > 0$, there exists $p = p(\epsilon, q, r) > 16q$ such that for every $z \in \overline{B}^c_{pr/8}$ and every $y \in \overline{B}_{qr}$, it holds that*

$$(1 + \epsilon)^{-1} < \frac{j(z, y)}{j(z, z_0)} < 1 + \epsilon. \tag{2.1}$$

In the remainder of this subsection, we assume that $r \geq R$, and that D is an open set such that $D \subset \overline{B}^c_r$. For $p > q > 0$, let

$$D^p = D \cap \overline{B}^c_p, \qquad D^{p,q} = D^q \setminus D^p.$$

For $p > q > 1$ and a nonnegative function f on $\mathfrak{X}$ define

$$
\begin{aligned}
f^{pr,qr}(x) &= \mathbb{E}_x \left[f(\widehat{X}_{\widehat{\tau}_{D^{pr}}}) : \widehat{X}_{\widehat{\tau}_{D^{pr}}} \in D^{pr,qr} \right], \\
\widetilde{f}^{pr,qr}(x) &= \mathbb{E}_x \left[f(\widehat{X}_{\widehat{\tau}_{D^{pr}}}) : \widehat{X}_{\widehat{\tau}_{D^{pr}}} \in (D \setminus D^{qr}) \cup \overline{B}_r \right].
\end{aligned} \tag{2.2}
$$

Lemma 2.2 *Suppose that $r \geq R$, $D \subset \overline{B}^c_r$ is an open set and f is a nonnegative function on $\mathfrak{X}$ which is regular harmonic in D with respect to $\widehat{X}$ and vanishes on $\overline{B}^c_r \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$. Let $q \geq 2$, $\epsilon > 0$, and choose $p = p(\epsilon, q, r)$ as in Lemma 2.1. Then for every $x \in D^{pr/8}$,*

$$(1 + \epsilon)^{-1} \widehat{P}_{D^{pr/8}}(x, z_0) \int_{\overline{B}_{qr}} f(y) m(dy) \leq \widetilde{f}^{pr/8,qr}(x) \leq (1 + \epsilon) \widehat{P}_{D^{pr/8}}(x, z_0) \int_{\overline{B}_{qr}} f(y) m(dy). \tag{2.3}$$

Proof. Let $x \in D^{pr/8}$. Using Lemma 2.1 in the second inequality below, we get

$$
\begin{aligned}
\widetilde{f}^{pr/8,qr}(x) &= \int_{D \setminus D^{qr}} \widehat{P}_{D^{pr/8}}(x, y) f(y) m(dy) + \int_{\overline{B}_r} \widehat{P}_{D^{pr/8}}(x, y) f(y) m(dy) \\
&= \int_{D \setminus D^{qr}} \int_{D^{pr/8}} \widehat{G}_{D^{pr/8}}(x, z) \widehat{j}(z, y) m(dz) f(y) m(dy) \\
&\quad + \int_{\overline{B}_r} \int_{D^{pr/8}} \widehat{G}_{D^{pr/8}}(x, z) \widehat{j}(z, y) m(dz) f(y) m(dy) \\
&\leq (1 + \epsilon) \int_{D \setminus D^{qr}} \int_{D^{pr/8}} \widehat{G}_{D^{pr/8}}(x, z) \widehat{j}(z, z_0) m(dz) f(y) m(dy) \\
&\quad + (1 + \epsilon) \int_{\overline{B}_r} \int_{D^{pr/8}} \widehat{G}_{D^{pr/8}}(x, z) \widehat{j}(z, z_0) m(dz) f(y) m(dy) \\
&= (1 + \epsilon) \left(\int_{D \setminus D^{qr}} \widehat{P}_{D^{pr/8}}(x, z_0) f(y) m(dy) + \int_{\overline{B}_r} \widehat{P}_{D^{pr/8}}(x, z_0) f(y) m(dy) \right) \\
&= (1 + \epsilon) \widehat{P}_{D^{pr/8}}(x, z_0) \int_{\overline{B}_{qr}} f(y) m(dy).
\end{aligned}
$$

This proves the right-hand side inequality. The left-hand side inequality can be proved in the same way. $\square$

In the remainder of this subsection, we assume that $r \geq R$, $D \subset \overline{B}_r^c$ an open set and f_1 and f_2 are nonnegative functions on $\mathfrak{X}$ which are regular harmonic in D with respect $\widehat{X}$ and vanish on $\overline{B}_r^c \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$. Note that $f_i = f_i^{pr,qr} + \widetilde{f}_i^{pr,qr}$.

Lemma 2.3 *Let $r \geq R$, $q > 2$, $\epsilon > 0$, and choose $p = p(\epsilon, q, r)$ as in Lemma 2.1. If*

$$\int_{D^{3pr/8,qr}} f_i(y)m(dy) \leq \epsilon \int_{\overline{B}_{qr}} f_i(y)m(dy), \quad i = 1, 2, \tag{2.4}$$

then, for all $x \in D^{pr}$.

$$\frac{(1+\epsilon)^{-1} \int_{\overline{B}_{qr}} f_1(y)m(dy)}{(C\epsilon + 1 + \epsilon) \int_{\overline{B}_{qr}} f_2(y)m(dy)} \leq \frac{f_1(x)}{f_2(x)} \leq \frac{(C\epsilon + 1 + \epsilon) \int_{\overline{B}_{qr}} f_1(y)m(dy)}{(1+\epsilon)^{-1} \int_{\overline{B}_{qr}} f_2(y)m(dy)}. \tag{2.5}$$

Proof. Assume that $x \in D^{pr}$. Since $f_i^{pr/8,qr}$ is regular harmonic in $D^{pr/8}$ with respect to $\widehat{X}$ and vanishes on $\overline{B}_{pr/8}^c \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$, using $\mathbf{F2}(z_0, R)$ (with $a = 3/2$), we have

$$f_i^{pr/8,qr}(x) \leq C\widehat{P}_{D^{pr/8}}(x, z_0) \int_{B_{3pr/8}} f_i^{pr/8,qr}(y)m(dy).$$

Since $f_i^{pr/8,qr}(y) \leq f_i(y)$ and $f_i^{pr/8,qr}(y) = 0$ on $(D^{qr})^c$ except possibly at irregular points of D, by using that m does not charge polar sets and applying (2.4) we have

$$f_i^{pr/8,qr}(x) \leq C\widehat{P}_{D^{pr/8}}(x, z_0) \int_{D^{3pr/8,qr}} f_i(y)m(dy) \leq C\epsilon\widehat{P}_{D^{pr/8}}(x, z_0) \int_{\overline{B}_{qr}} f_i(y)m(dy).$$

By this and Lemma 2.2 we have

$$
\begin{aligned}
f_i(x) &= f_i^{pr/8,qr}(x) + \widetilde{f}_i^{pr/8,qr}(x) \\
&\leq C\epsilon\widehat{P}_{D^{pr/8}}(x, z_0) \int_{\overline{B}_{qr}} f_i(y)m(dy) + (1+\epsilon)\widehat{P}_{D^{pr/8}}(x, z_0) \int_{\overline{B}_{qr}} f_i(y)m(dy) \\
&= (C\epsilon + 1 + \epsilon)\widehat{P}_{D^{pr/8}}(x, z_0) \int_{\overline{B}_{qr}} f_i(y)m(dy)
\end{aligned}
$$

and

$$f_i(x) \geq \widetilde{f}_i^{pr/8,qr}(x) \geq (1+\epsilon)^{-1}\widehat{P}_{D^{pr/8}}(x, z_0) \int_{\overline{B}_{qr}} f_i(y)m(dy).$$

Therefore, (2.5) holds. $\qquad\square$

Suppose that ∞ is inaccessible from D with respect to X. Then there exists a point $x_0 \in D$ such that

$$\int_D G_D(x_0, y)m(dy) = \mathbb{E}_{x_0}\tau_D < \infty. \tag{2.6}$$

In the next result we fix this point x_0.

9

Theorem 2.4 *Suppose that ∞ is inaccessible from D with respect to X. Let $r > 2d(z_0, x_0) \vee R$. For any two nonnegative functions f_1, f_2 on $\mathfrak{X}$ which are regular harmonic in D^r with respect to $\widehat{X}$ and vanish on $\overline{B}_r^c \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$ we have*

$$\lim_{D \ni x \to \infty} \frac{f_1(x)}{f_2(x)} = \frac{\int_{\mathfrak{X}} f_1(y)m(dy)}{\int_{\mathfrak{X}} f_2(y)m(dy)}. \tag{2.7}$$

Proof. First note that

$$\int_{B_{3r}} G_D(x_0, z)m(dz) \geq \mathbb{E}_{x_0}[D \cap B_{3r}] > 0.$$

By using **F2**(z_0, R) we see that $\int_{B_{8r}} f_i(y)m(dy) < \infty$. The function $v \mapsto G_D(x_0, v)$ is regular harmonic in D^r with respect to $\widehat{X}$ and vanishes on $\overline{B}_r^c \setminus D^r$ (so vanishes on $\overline{B}_r^c \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$). By using **F2**$(z_0, R)$ for $\widehat{X}$, we have for $i = 1, 2$,

$$\int_{D^{8r}} f_i(y)m(dy) \leq C \int_{B_{3r}} f_i(z)m(dz) \int_{D^{8r}} \widehat{P}_{D^r}(y, z_0)m(dy)$$

$$= C \int_{B_{3r}} G_D(x_0, z)m(dz) \int_{D^{8r}} \widehat{P}_{D^r}(y, z_0)m(dy) \frac{\int_{B_{3r}} f_i(z)m(dz)}{\int_{B_{3r}} G_D(x_0, z)m(dz)}$$

$$\leq C^2 \int_{D^{8r}} G_D(x_0, y)m(dy) \frac{\int_{B_{3r}} f_i(z)m(dz)}{\int_{B_{3r}} G_D(x_0, z)m(dz)}$$

$$\leq C^2 \int_D G_D(x_0, y)m(dy) \frac{\int_{B_{3r}} f_i(z)m(dz)}{\mathbb{E}_{x_0}[D \cap B_{3r}]} < \infty.$$

Hence $\int_{\mathfrak{X}} f_i(y)m(dy) < \infty$, $i = 1, 2$. Let $q_0 = 2$ and $\epsilon > 0$. For $j = 0, 1, \ldots$, inductively define the sequence $q_{j+1} = 3p(\epsilon, q_j, r)/8 > 6q_i$ using Lemma 2.1. Then for $i = 1, 2$,

$$\sum_{j=0}^{\infty} \int_{D^{q_{j+1}r, q_j r}} f_i(y)m(dy) = \int_{D^{q_0 r}} f_i(y)m(dy) < \infty.$$

If $\int_{D^{q_{j+1}r, q_j r}} f_i(y)m(dy) > \epsilon \int_{\overline{B}_{q_j r}} f_i(y)m(dy)$ for all $j \geq 0$, then

$$\sum_{j=0}^{\infty} \int_{D^{q_{j+1}r, q_j r}} f_i(y)m(dy) \geq \epsilon \sum_{j=0}^{\infty} \int_{\overline{B}_{q_j r}} f_i(y)m(dy) \geq \epsilon \sum_{j=0}^{\infty} \int_{\overline{B}_{q_0 r}} f_i(y)m(dy) = \infty.$$

Hence, there exists $k \geq 0$ such that $\int_{D^{q_{k+1}r, q_k r}} f_i(y)m(dy) \leq \epsilon \int_{\overline{B}_{q_k r}} f_i(y)m(dy)$. Moreover, since $\lim_{j \to \infty} \int_{D^{q_{j+1}r, q_j r}} f_i(y)m(dy) = 0$, there exists $j_0 \geq 0$ such that $\int_{D^{q_{j+1}r, q_j r}} f_i(y)m(dy) \leq \int_{D^{q_{k+1}r, q_k r}} f_i(y)m(dy)$ for all $j \geq j_0$. Hence for all $j \geq j_0 \vee k$ we have

$$\int_{D^{q_{j+1}r, q_j r}} f_i(y)m(dy) \leq \int_{D^{q_{k+1}r, q_k r}} f_i(y)m(dy) \leq \epsilon \int_{\overline{B}_{q_k r}} f_i(y)m(dy) \leq \epsilon \int_{\overline{B}_{q_j r}} f_i(y)m(dy).$$

Therefore, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0 \vee k$,

$$\int_{D^{q_{j+1}r, q_j r}} f_i(y) m(dy) \leq \epsilon \int_{\overline{B}_{q_j r}} f_i(y) m(dy) \qquad i = 1, 2,$$

and

$$(1 + \epsilon)^{-1} \int_{\mathfrak{X}} f_i(y) m(dy) < \int_{\overline{B}_{q_j r}} f_i(y) m(dy) < (1 + \epsilon) \int_{\mathfrak{X}} f_i(y) m(dy), \qquad i = 1, 2.$$

We see that the assumption of Lemma 2.3 are satisfied and conclude that (2.5) holds true: for $x \in D^{8q_{j+1}r/3}$,

$$\frac{(1 + \epsilon)^{-1} \int_{\overline{B}_{q_j r}} f_1(y) m(dy)}{(C\epsilon + 1 + \epsilon) \int_{\overline{B}_{q_j r}} f_2(y) m(dy)} \leq \frac{f_1(x)}{f_2(x)} \leq \frac{(C\epsilon + 1 + \epsilon) \int_{\overline{B}_{q_j r}} f_1(y) m(dy)}{(1 + \epsilon)^{-1} \int_{\overline{B}_{q_j r}} f_2(y) m(dy)}.$$

It follows that for $x \in D^{8q_{j+1}r/3}$,

$$\frac{(1 + \epsilon)^{-2} \int_{\mathfrak{X}} f_1(y) m(dy)}{(C\epsilon + 1 + \epsilon)(1 + \epsilon) \int_{\mathfrak{X}} f_2(y) m(dy)} \leq \frac{f_1(x)}{f_2(x)} \leq \frac{(C\epsilon + 1 + \epsilon)(1 + \epsilon) \int_{\mathfrak{X}} f_1(y) m(dy)}{(1 + \epsilon)^{-2} \int_{\mathfrak{X}} f_2(y) m(dy)}.$$

Since $\epsilon > 0$ was arbitrary, we conclude that (2.7) holds. $\qquad\square$

2.2 Finite boundary point

In this subsection, we deal with oscillation reduction at an inaccessible boundary point $z_0 \in \mathfrak{X}$ of an open set D. Throughout the subsection, we assume that there exists $R \leq R_0$ such that $\mathbf{E1}(z_0, R)$ and $\mathbf{C1}(z_0, R)$ hold, and that $\widehat{X}$ satisfies $\mathbf{F1}(z_0, R)$. We will fix this z_0. Again, for simplicity, we use notation $B_r = B(z_0, r)$, $r > 0$.

First, the next lemma is a direct consequence of assumption $\mathbf{E1}(z_0, R)$.

Lemma 2.5 *For any $q \in (0, 1/2]$, $r \in (0, R]$ and $\epsilon > 0$, there exists $p = p(\epsilon, q, r) < q/16$ such that for every $z \in B_{8pr}$ and every $y \in B_{qr}^c$,*

$$(1 + \epsilon)^{-1} < \frac{j(z, y)}{j(z_0, y)} < 1 + \epsilon. \tag{2.8}$$

Let $D \subset \mathfrak{X}$ be an open set. For $0 < p < q$, let $D_p = D \cap B_p$ and $D_{p,q} = D_q \setminus D_p$. For a function f on $\mathfrak{X}$, and $0 < p < q$, let

$$\widehat{\Lambda}_p(f) := \int_{\overline{B}_p^c} \widehat{j}(z_0, y) f(y) m(dy), \qquad \widehat{\Lambda}_{p,q}(f) =: \int_{D_{p,q}} \widehat{j}(z_0, y) f(y) m(dy). \tag{2.9}$$

For $0 < p < q < 1$ and $r \in (0, R]$, define

$$\begin{aligned}
f_{pr,qr}(x) &= \mathbb{E}_x \left[f(\widehat{X}_{\widehat{\tau}_{D_{pr}}}) : \widehat{X}_{\widehat{\tau}_{D_{pr}}} \in D_{pr,qr} \right], \\
\widetilde{f}_{pr,qr}(x) &= \mathbb{E}_x \left[f(\widehat{X}_{\widehat{\tau}_{D_{pr}}}) : \widehat{X}_{\widehat{\tau}_{D_{pr}}} \in (D \setminus D_{qr}) \cup B_r^c \right].
\end{aligned} \tag{2.10}$$

11

Lemma 2.6 *Let $q \in (0, 1/2]$, $R \in (0, r]$, $\epsilon > 0$, and choose $p = p(\epsilon, q, r)$ as in Lemma 2.5. Then for every $r \in (0, R]$, $D \subset B_r = B(z_0, r)$, nonnegative function f on $\mathfrak{X}$ which is regular harmonic in D with respect to $\widehat{X}$ and vanishes on $B_r \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$, and every $x \in D_{8pr}$,*

$$(1 + \epsilon)^{-1}(\mathbb{E}_x \widehat{\tau}_{D_{8pr}})\widehat{\Lambda}_{qr}(f) \leq \widetilde{f}_{8pr,qr}(x) \leq (1 + \epsilon)(\mathbb{E}_x \widehat{\tau}_{D_{8pr}})\widehat{\Lambda}_{qr}(f). \tag{2.11}$$

Proof. Let $x \in D_{8pr}$. Using Lemma 2.5 in the second inequality below, we get

$$
\begin{aligned}
\widetilde{f}_{8pr,qr}(x) &= \int_{D \setminus D_{qr}} \widehat{P}_{D_{8pr}}(x, y) f(y) m(dy) + \int_{B_R^c} \widehat{P}_{D_{8pr}}(x, y) f(y) m(dy) \\
&= \int_{D \setminus D_{qr}} \int_{D_{8pr}} \widehat{G}_{D_{8pr}}(x, z) \widehat{j}(z, y) m(dz) f(y) m(dy) \\
&\quad + \int_{B_R^c} \int_{D_{8pr}} \widehat{G}_{D_{8pr}}(x, z) \widehat{j}(z, y) m(dz) f(y) m(dy) \\
&\leq (1 + \epsilon)(\mathbb{E}_x \widehat{\tau}_{D_{8pr}}) \left(\int_{D \setminus D_{qr}} \widehat{j}(z_0, y) f(y) m(dy) + \int_{B_r^c} \widehat{j}(z_0, y) f(y) m(dy) \right) \\
&= (1 + \epsilon)(\mathbb{E}_x \widehat{\tau}_{D_{8pr}}) \int_{B_{qr}^c} \widehat{j}(z_0, y) f(y) m(dy) \\
&= (1 + \epsilon)(\mathbb{E}_x \widehat{\tau}_{D_{8pr}})\widehat{\Lambda}_{qr}(f).
\end{aligned}
$$

This proves the right-hand side inequality. The left-hand side inequality can be proved in the same way. $\qquad \square$

In the remainder of this subsection, we assume $r \in (0, R]$, $D \subset B_r$ is an open set and $z_0 \in \partial D$. We also assume that f_1 and f_2 are nonnegative functions on $\mathfrak{X}$ which are regular harmonic in D with respect to the process $\widehat{X}$, and vanish on $B_r \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$. Note that $f_i = (f_i)_{pr,qr} + (\widetilde{f}_i)_{pr,qr}$.

Lemma 2.7 *Let $R \in (0, 1]$, $q < 1/2$, $\epsilon > 0$, and let $p = p(\epsilon, q, r)$ be as in Lemma 2.5. If*

$$\widehat{\Lambda}_{8pr/3,qr}(f_i) \leq \epsilon \widehat{\Lambda}_{qr}(f_i), \quad i = 1, 2, \tag{2.12}$$

then for $x \in D_{pr}$

$$\frac{(1 + \epsilon)^{-1}\widehat{\Lambda}_{qr}(f_1)}{(C\epsilon + 1 + \epsilon)\widehat{\Lambda}_{qr}(f_2)} \leq \frac{f_1(x)}{f_2(x)} \leq \frac{(C\epsilon + 1 + \epsilon)\widehat{\Lambda}_{qr}(f_1)}{(1 + \epsilon)^{-1}\widehat{\Lambda}_{qr}(f_2)}. \tag{2.13}$$

Proof. Assume that $x \in D_{pr}$. Since $(f_i)_{8pr,qr}$ is regular harmonic in D_{8pr} with respect to $\widehat{X}$ and vanish on $B_{8pr} \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$, using **F1**$(z_0, R)$ (with $a = 2/3$), we have

$$(f_i)_{8pr,qr}(x) \leq C(\mathbb{E}_x \tau_{D_{8pr}})\widehat{\Lambda}_{8pr/3}((f_i)_{8pr,qr}).$$

Since $(f_i)_{8pr,qr}(y) \le f_i(y)$ and $(f_i)_{8pr,qr}(y) = 0$ on D_{qr}^c except possibly at irregular points of D, applying (2.12) we have

$$(f_i)_{8pr,qr}(x) \le C(\mathbb{E}_x \tau_{D_{8pr}}) \widehat{\Lambda}_{8pr/3,qr}(f_i) \le C\epsilon (\mathbb{E}_x \tau_{D_{8pr}}) \widehat{\Lambda}_{qr}(f_i).$$

By this and Lemma 2.6, we have that

$$\begin{aligned}
f_i(x) &= (f_i)_{8pr,qr}(x) + (\widetilde{f_i})_{8pr,qr}(x) \\
&\le C\epsilon (\mathbb{E}_x \tau_{D_{8pr}}) \widehat{\Lambda}_{qr}(f_i) + (1+\epsilon)(\mathbb{E}_x \tau_{D_{8pr}}) \widehat{\Lambda}_{qr}(f_i) \\
&= (C\epsilon + 1 + \epsilon)(\mathbb{E}_x \tau_{D_{8pr}}) \widehat{\Lambda}_{qr}(f_i)
\end{aligned}$$

and

$$f_i(x) \ge (\widetilde{f_i})_{8pr,qr}(x) \ge (1+\epsilon)^{-1}(\mathbb{E}_x \tau_{D_{8pr}}) \widehat{\Lambda}_{qr}(f_i).$$

Therefore, (2.13) holds. $\qquad\square$

Assume that z_0 is inaccessible from D with respect to X. Then there exist a point x_0 in D such that

$$P_D(x_0, z_0) = \int_D G_D(x_0, v) j(v, z_0) m(dv) < \infty.$$

In the next result we fix this point x_0

Theorem 2.8 *Suppose that z_0 is inaccessible from D with respect to X. Let $r < 2d(z_0, x_0) \wedge R$. For any two nonnegative functions f_1, f_2 on $\mathfrak{X}$ which are regular harmonic in D_r with respect to $\widehat{X}$ and vanish on $B_r \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$, we have*

$$\lim_{D \ni x \to z_0} \frac{f_1(x)}{f_2(x)} = \frac{\int_{\mathfrak{X}} \widehat{j}(z_0, y) f_1(y) m(dy)}{\int_{\mathfrak{X}} \widehat{j}(z_0, y) f_2(y) m(dy)}. \tag{2.14}$$

Proof. First note that

$$\int_{\overline{B}_{r/3}^c} \widehat{j}(z_0, z) G_D(x_0, z) m(dz) \ge \int_{D \cap \overline{B}_{r/3}^c} j(z, z_0) G_{D \cap \overline{B}_{r/3}^c}(x_0, z) m(dz) = P_{D \cap \overline{B}_{r/3}^c}(x_0, z_0) > 0.$$

Since $\widehat{X}$ satisfies **F1**(z_0, R), we have $\widehat{\Lambda}_{r/8}(f_i) < \infty$. The function $v \mapsto G_D(x_0, v)$ is regular harmonic in D_r with respect to $\widehat{X}$ and vanishes on $B_r \setminus D_r$ (so vanishes on $B_r \cap (\overline{D}^c \cup \widehat{D}^{\mathrm{reg}})$). By using **F1**$(z_0, R)$ for $\widehat{X}$ we have

$$\int_{B_{r/8}} \widehat{j}(z_0, y) f_i(y) m(dy) \le C \int_{\overline{B}_{r/3}^c} \widehat{j}(z_0, z) f_i(z) m(dz) \int_{B_{r/8}} \widehat{j}(z_0, y) \mathbb{E}_y[\widehat{\tau}_{D_r}] m(dy)$$

$$= C \int_{\overline{B}_{r/3}^c} \widehat{j}(z_0, z) G_D(x_0, z) m(dz) \int_{B_{r/8}} \widehat{j}(z_0, y) \mathbb{E}_y[\widehat{\tau}_{D_r}] m(dy) \frac{\int_{\overline{B}_{r/3}^c} \widehat{j}(z_0, z) f_i(z) m(dz)}{\int_{\overline{B}_{r/3}^c} \widehat{j}(z_0, z) G_D(x_0, z) m(dz)}$$

$$\le C^2 \int_{B_{r/8}} \widehat{j}(z_0, y) G_D(x_0, y) m(dz) \frac{\int_{\overline{B}_{r/3}^c} \widehat{j}(z_0, z) f_i(z) m(dz)}{\int_{\overline{B}_{r/3}^c} \widehat{j}(z_0, z) G_D(x_0, z) m(dz)}$$

13

$$\leq C^2 P_D(x_0, z_0) \frac{\widehat{\Lambda}_{r/3}(f_i)}{P_{D \cap \overline{B}_{r/3}^c}(x_0, z_0)} < \infty.$$

Therefore

$$\widehat{\Lambda}(f_i) := \int_{\mathfrak{X}} \widehat{j}(z_0, y) f_i(y) m(dy) = \int_{B_{r/8}} \widehat{j}(z_0, y) f_i(y) m(dy) + \widehat{\Lambda}_{r/8}(f_i) < \infty.$$

Let $q_0 = 1/2$ and $\epsilon > 0$. For $j = 0, 1, \ldots$, inductively define the sequence $q_{j+1} = p(\epsilon, q_j, r)$ as in Lemma 2.5. Then

$$\sum_{j=0}^{\infty} \widehat{\Lambda}_{q_{j+1}r, q_j r}(f_i) = \int_{D_{r/2}} \widehat{j}(z_0, y) f_i(y) m(dy) \leq \int_{\mathfrak{X}} \widehat{j}(z_0, y) f_i(y) m(dy) < \infty.$$

If $\widehat{\Lambda}_{q_{j+1}r, q_j r}(f_i) > \epsilon \widehat{\Lambda}_{q_j r}(f_i)$ for all $j \geq 0$, then

$$\sum_{j=0}^{\infty} \widehat{\Lambda}_{q_{j+1}r, q_j r}(f_i) \geq \epsilon \sum_{j=0}^{\infty} \widehat{\Lambda}_{q_j r}(f_i) \geq \epsilon \sum_{j=0}^{\infty} \widehat{\Lambda}_{q_0 r}(f_i) = \infty.$$

Hence, there exists an integer $k \geq 0$ such that $\widehat{\Lambda}_{q_{k+1}r, q_k r}(f_i) \leq \epsilon \widehat{\Lambda}_{q_k r}(f_i)$. Moreover, since $\lim_{j \to \infty} \widehat{\Lambda}_{q_{j+1}r, q_j r}(f_i) = 0$, there exists $j_0 \geq 0$ such that $\widehat{\Lambda}_{q_{j+1}r, q_j r}(f_i) \leq \widehat{\Lambda}_{q_{k+1}r, q_k r}(f_i)$ for all $j \geq j_0$. Hence for all $j \geq j_0 \vee k$ we have

$$\widehat{\Lambda}_{q_{j+1}r, q_j r}(f_i) \leq \widehat{\Lambda}_{q_{k+1}r, q_k r}(f_i) \leq \epsilon \widehat{\Lambda}_{q_k r}(f_i) \leq \epsilon \widehat{\Lambda}_{q_j r}(f_i).$$

Therefore for all $j \geq j_0$,

$$\widehat{\Lambda}_{q_{j+1}r, q_j r}(f_i) \leq \epsilon \widehat{\Lambda}_{q_j r}(f_i), \qquad i = 1, 2,$$

and

$$(1 + \epsilon)^{-1} \widehat{\Lambda}(f_i) < \widehat{\Lambda}_{q_j r}(f_i) < (1 + \epsilon) \widehat{\Lambda}(f_i), \qquad i = 1, 2.$$

Hence the assumption of Lemma 2.7 are satisfied and consequently (2.13) holds: for $x \in D_{q_{j+1}r}$,

$$\frac{(1 + \epsilon)^{-1} \widehat{\Lambda}_{q_j r}(f_1)}{(C\epsilon + 1 + \epsilon) \widehat{\Lambda}_{q_j r}(f_2)} \leq \frac{f_1(x)}{f_2(x)} \leq \frac{(C\epsilon + 1 + \epsilon) \widehat{\Lambda}_{q_j r}(f_1)}{(1 + \epsilon)^{-1} \widehat{\Lambda}_{q_j r}(f_2)}.$$

It follows that $x \in D_{q_{j+1}r}$,

$$\frac{(1 + \epsilon)^{-2} \widehat{\Lambda}(f_1)}{(C\epsilon + 1 + \epsilon)(1 + \epsilon) \widehat{\Lambda}(f_2)} \leq \frac{f_1(x)}{f_2(x)} \leq \frac{(C\epsilon + 1 + \epsilon)(1 + \epsilon) \widehat{\Lambda}(f_1)}{(1 + \epsilon)^{-2} \widehat{\Lambda}(f_2)}.$$

Since $\epsilon > 0$ was arbitrary, we conclude that (2.14) holds. $\qquad \square$

14

3 Proof of Theorems 1.2 and 1.4

Let D be a Greenian open subset of $\mathfrak{X}$. Fix $x_0 \in D$ and define

$$M_D(x, y) := \frac{G_D(x, y)}{G_D(x_0, y)}, \qquad x, y \in D, \ y \neq x_0.$$

Combining [14, Lemmas 3.2 and 3.4] and our Theorems 2.4 and 2.8 we have the following.

Theorem 3.1 *(a) Suppose that* $\mathbf{E1}(z_0, R)$ *holds and that* $\widehat{X}$ *satisfies* $\mathbf{F1}(z_0, R)$. *Then*

$$M_D(x, z_0) := \lim_{D \ni v \to z_0} \frac{G_D(x, v)}{G_D(x_0, v)} \tag{3.1}$$

exists and is finite. In particular, if z_0 *is inaccessible from* D *with respect to* X, *then*

$$M_D(x, z_0) = \frac{\int_{\mathfrak{X}} \widehat{j}(z_0, y) G_D(x, y) m(dy)}{\int_{\mathfrak{X}} \widehat{j}(z_0, y) G_D(x_0, y) m(dy)} = \frac{P_D(x, z_0)}{P_D(x_0, z_0)}. \tag{3.2}$$

(b) Suppose that $\mathbf{E2}(z_0, R)$ *holds and that* $\widehat{X}$ *satisfies* $\mathbf{F2}(z_0, R)$. *Then for every* $x \in D$ *the limit*

$$M_D(x, \infty) := \lim_{D \ni v \to \infty} \frac{G_D(x, v)}{G_D(x_0, v)} \tag{3.3}$$

exists and is finite. In particular, if ∞ *is inaccessible from* D *with respect to* X, *then*

$$M_D(x, \infty) = \frac{\mathbb{E}_x \tau_D}{\mathbb{E}_{x_0} \tau_D}. \tag{3.4}$$

Since both X^D and $\widehat{X}^D$ are strongly Feller, the process X^D satisfies Hypothesis (B) in [15]. See [14, Section 4] for details. Therefore D has a Martin boundary $\partial_M D$ with respect to X^D satisfying the following properties:

(M1) $D \cup \partial_M D$ is a compact metric space (with the metric denoted by d_M);

(M2) D is open and dense in $D \cup \partial_M D$, and its relative topology coincides with its original topology;

(M3) $M_D(x, \cdot)$ can be uniquely extended to $\partial_M D$ in such a way that

    (a) $M_D(x, y)$ converges to $M_D(x, w)$ as $y \to w \in \partial_M D$ in the Martin topology;

    (b) for each $w \in D \cup \partial_M D$ the function $x \to M_D(x, w)$ is excessive with respect to X^D;

    (c) the function $(x, w) \to M_D(x, w)$ is jointly continuous on $D \times ((D \setminus \{x_0\}) \cup \partial_M D)$ in the Martin topology and

    (d) $M_D(\cdot, w_1) \neq M_D(\cdot, w_2)$ if $w_1 \neq w_2$ and $w_1, w_2 \in \partial_M D$.

Proof of Theorem 1.2: (a) Using Theorem 3.1(a), by the same argument as in the proof of [14, Theorem 1.1(a)], we have that $\partial_M^{z_0} D$ consists of a single point.

(b) If z_0 is accessible from D with respect to X, then by [14, Theorem 1.1 (b)] the Martin kernel $M_D(\cdot, z_0)$ is minimal harmonic for X^D.

Assume that z_0 is inaccessible from D with respect to X. Since $x \mapsto P_D(x, z_0)$ is *not* harmonic with respect to X^D, we conclude from (3.3) that the Martin kernel $M_D(\cdot, z_0)$ is not harmonic, and in particular, that z_0 is *not* a minimal Martin boundary point. $\square$

Proof of Corollary 1.3: (a) Let $\Xi : \partial D \to \partial_M^f D$ so that $\Xi(z)$ is the unique Martin boundary point associated with $z \in \partial D$. Since every finite Martin boundary point is associated with some $z \in \partial D$, we see that Ξ is onto. We show now that Ξ is 1-1. If not, there are $z, z' \in \partial D$, $z \neq z'$, such that $\Xi(z) = \Xi(z') = w$. Then $M_D(\cdot, z) = M_D(\cdot, w) = M_D(\cdot, z')$. It follows from the proof of [14, Corollary 1.2(a)] that z and z' can not be both accessible. If one of them, say z, is accessible and the other, z', is inaccessible, then we can not have $M_D(\cdot, z) = M_D(\cdot, z')$ since $M_D(\cdot, z)$ is harmonic while $M_D(\cdot, z')$ is not. Now let's assume that both z and z' are inaccessible. Then $M_D(\cdot, z) = \frac{P_D(\cdot, z)}{P_D(x_0, z)}$ and $M_D(\cdot, z') = \frac{P_D(\cdot, z')}{P_D(x_0, z')}$. From $M_D(\cdot, z) = M_D(\cdot, z')$ we deduce that

$$P_D(x, z) P_D(x_0, z') = P_D(x, z') P_D(x_0, z), \qquad \text{for all } x \in D.$$

By treating $P_D(x_0, z')$ and $P_D(x_0, z)$ as constants, the above equality can be written as

$$\int_D G_D(x, y) j(y, z) m(dy) = c \int_D G_D(x, y) j(y, z') m(dy), \qquad \text{for all } x \in D.$$

By the uniqueness principle for potentials, this implies that the measures $j(y, z) m(dy)$ and $c j(y, z') m(dy)$ are equal. Hence $j(y, z) = c j(y, z')$ for m-a.e. $y \in D$. But this is impossible (for example, let $y \to z$; then $j(y, z) \to \infty$, while $c j(y, z')$ stays bounded because of **C1**(z, R)). We conclude that $z = z'$.

(b) The proof of this part is exactly the same as that of [14, Corollary 1.2(b)]. $\square$

Proof of Theorem 1.4:

(a) Using Theorem 3.1(b), by the same argument as in the proof of [14, Theorem 1.3(a)], we have that $\partial_M^\infty D$ is a single point which we will denote by ∞.

(b) If ∞ is inaccessible from D with respect to X, then by [14, Theorem 1.2 (b)] the Martin kernel $M_D(\cdot, \infty)$ is minimal harmonic for X^D.

Assume that ∞ is inaccessible from D with respect to X. Since the function $x \mapsto \mathbb{E}_x \tau_D = \int_D G_D(x, y) m(dy)$ is *not* harmonic with respect to X^D, by (3.3) we conclude that the Martin kernel $M_D(\cdot, \infty)$ is not harmonic, and in particular, ∞ is *not* a minimal Martin boundary point. $\square$

Proof of Corollary 1.5: (a) In the same way as in the proof of [14, Corollary 1.4(a)] it suffices to show that it cannot happen that $M_D(\cdot, \infty) = M_D(\cdot, z)$ for any $z \in \partial D$. If both

16

∞ and z are accessible, this was shown in the proof of [14, Corollary 1.4(a)]. If one of the two points is accessible and the other inaccessible, then clearly the two Martin kernels are different since one is harmonic while the other is not. Assume that both ∞ and z are inaccessible. Then $M_D(\cdot, \infty) = \frac{\mathbb{E}.\tau_D}{\mathbb{E}_{x_0}\tau_D}$ and $M_D(\cdot, z) = \frac{P_D(\cdot, z)}{P_D(x_0, z)}$. Therefore,

$$P_D(x_0, z)\mathbb{E}_x \tau_D = P_D(x, z)\mathbb{E}_{x_0}\tau_D, \qquad \text{for all } x \in D.$$

By treating $\mathbb{E}_{x_0}\tau_D$ and $P_D(x_0, z)$ as constants, the above equality can be written as

$$\int_D G_D(x, y)m(dy) = c \int_D G_D(x, y)j(y, z)m(dy), \qquad \text{for all } x \in D.$$

By the uniqueness principle for potentials, this implies that the measures $m(dy)$ and $cj(y, z)m(dy)$ are equal. Hence $1 = cj(y, z)$ for m-a.e. $y \in D$ which clearly contradicts $\mathbf{C1}(z, R)$.

(b) The proof of this part is exactly the same as that of [14, Corollary 1.4(b)].

4   Examples

In this section we discuss several classes of Lévy processes in $\mathbb{R}^d$ satisfying our assumptions.

4.1   Subordinate Brownian motions

In this subsection we discuss subordinate Brownian motions in $\mathbb{R}^d$ satisfying our assumptions.

We will list conditions on subordinate Brownian motions one by one under which our assumptions hold true.

Let $W = (W_t, \mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$, $S = (S_t)$ an independent driftless subordinator with Laplace exponent ϕ and define the subordinate Brownian motion $Y = (Y_t, \mathbb{P}_x)$ by $Y_t = W_{S_t}$. Let j_Y denote the Lévy density of Y.

The Laplace exponent ϕ is a Bernstein function with $\phi(0+) = 0$. Since ϕ has no drift part, ϕ can be written in the form

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\,\mu(dt)\,.$$

Here μ is a σ-finite measure on $(0, \infty)$ satisfying $\int_0^\infty (t \wedge 1)\,\mu(dt) < \infty$. μ is called the Lévy measure of the subordinator S. ϕ is called a complete Bernstein function if the Lévy measure μ of S_t has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \geq 0$ for every non-negative integer n. We will assume that ϕ is a complete Bernstein function.

When ϕ is unbounded and Y is transient, the mean occupation time measure of Y admits a density $G(x, y) = g(|x - y|)$ which is called the Green function of Y, and is given by the formula

$$g(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} u(t)\, dt\,. \tag{4.1}$$

Here u is the potential density of the subordinator S.

We first discuss conditions that ensure $\mathbf{E1}(z_0, R)$.

By [8, Lemma A.1], for all $t > 0$, we have

$$\mu(t) \leq (1 - 2e^{-1})^{-1} t^{-2} \phi'(t^{-1}) \leq (1 - 2e^{-1})^{-1} t^{-1} \phi(t^{-1}). \tag{4.2}$$

Thus

$$\mu(t) \leq (1 - 2e^{-1})^{-1} \phi'(M^{-1}) t^{-2}, \qquad t \in (0, M]. \tag{4.3}$$

In [10], we have shown that there exists $c \in (0, 1)$ such that

$$\mu(t + 1) \geq c\mu(t), \qquad t \geq 1. \tag{4.4}$$

As a consequence of this, one can easily show that there exist $c_1, c_2 > 0$ such that

$$\mu(t) \geq c_1 e^{-c_2 t}, \qquad t \geq 1. \tag{4.5}$$

In fact, it follows from (4.4) that for any $n \geq 1$, $\mu(n + 1) \geq c^n \mu(1)$. Thus, for any $t \geq 1$,

$$\begin{aligned}
\mu(t) &\geq \mu([t] + 1) \geq c^{[t]} \mu(1) = \mu(1) e^{[t] \log c} \\
&= \mu(1) e^{([t] - t) \log c} e^{t \log c} \geq c^{-1} \mu(1) e^{t \log c}.
\end{aligned}$$

The following is a refinement of (4.4) and [7, Lemma 3.1].

Lemma 4.1 *Suppose that the Laplace exponent ϕ of S is a complete Bernstein function. Then, for any $t_0 > 0$,*

$$\limsup_{\delta \to 0} \sup_{t > t_0} \frac{\mu(t)}{\mu(t + \delta)} = 1 \,.$$

Proof. This is proof is similar to the proof of [7, Lemma 3.1], which in turn is a refinement of the proof of [10, Lemma 13.2.1]. Let $\eta > 0$ be given. Since μ is a complete monotone function, there exists a measure m on $[0, \infty)$ such that

$$\mu(t) = \int_{[0,\infty)} e^{-tx} m(dx), \qquad t > 0.$$

Choose $r = r(\eta, t_0) > 0$ such that

$$\eta \int_{[0,r]} e^{-t_0 x} m(dx) \geq \int_{(r,\infty)} e^{-t_0 x} m(dx).$$

Then for any $t > t_0$, we have

$$\eta \int_{[0,r]} e^{-tx} m(dx) = \eta \int_{[0,r]} e^{-(t-t_0)x} e^{-t_0 x} m(dx) \geq \eta e^{-(t-t_0)r} \int_{[0,r]} e^{-t_0 x} m(dx)$$

$$\geq e^{-(t-t_0)r} \int_{(r,\infty)} e^{-t_0 x} m(dx) = \int_{(r,\infty)} e^{-(t-t_0)r} e^{-t_0 x} m(dx) \geq \int_{(r,\infty)} e^{-tx} m(dx).$$

18

Thus for any $t > t_0$ and $\delta > 0$,

$$
\begin{aligned}
\mu(t + \delta) &\geq \int_{[0,r]} e^{-(t+\delta)x} m(dx) \geq e^{-r\delta} \int_{[0,r]} e^{-tx} m(dx) \\
&= e^{-r\delta}(1 + \eta)^{-1} \left(\int_{[0,r]} e^{-tx} m(dx) + \eta \int_{[0,r]} e^{-tx} m(dx) \right) \\
&\geq e^{-r\delta}(1 + \eta)^{-1} \left(\int_{[0,r]} e^{-tx} m(dx) + \int_{(r,\infty)} e^{-tx} m(dx) \right) \\
&= e^{-r\delta}(1 + \eta)^{-1} \int_{[0,\infty)} e^{-tx} m(dx) = e^{-r\delta}(1 + \eta)^{-1}\mu(t).
\end{aligned}
$$

Therefore

$$
\limsup_{\delta \to 0} \sup_{t > t_0} \frac{\mu(t)}{\mu(t + \delta)} \leq 1 + \eta.
$$

Since η is arbitrary and μ is decreasing, the assertion of the lemma is valid. $\qquad\square$

The Lévy measure of Y has a density with respect to the Lebesgue measure given by $j_Y(x) = j(|x|)$ with

$$
j(r) = \int_0^\infty g(t, r)\mu(t) dt, \qquad r \neq 0,
$$

where

$$
g(t, r) = (4\pi t)^{-d/2} \exp(-\frac{r^2}{4t}).
$$

As a consequence of (4.4), one can easily get that there exists $c \in (0, 1)$ such that

$$
j(r + 1) \geq cj(r), \qquad r \geq 1. \tag{4.6}
$$

Using this, we can show that there exist $c_1, c_2 > 0$ such that

$$
j(r) \geq c_1 e^{-c_2 r}, \qquad r \geq 1. \tag{4.7}
$$

Lemma 4.2 *Suppose that the Laplace exponent ϕ of S is a complete Bernstein function. For any $r_0 \in (0, 1)$,*

$$
\limsup_{\eta \to 0} \sup_{r > r_0} \frac{\int_0^\eta g(t, r)\mu(t) dt}{j(r)} = 0.
$$

Proof. For any $\eta \in (0, 1)$ and $r \in (r_0, 2]$, we have

$$
\frac{\int_0^\eta g(t, r)\mu(t) dt}{j(r)} \leq \frac{\int_0^\eta g(t, r_0)\mu(t) dt}{j(2)}.
$$

Thus

$$
\lim_{\eta \to 0} \sup_{r \in (r_0, 2]} \frac{\int_0^\eta g(t, r)\mu(t) dt}{j(r)} = 0.
$$

19

Thus we only need to show that

$$\limsup_{\substack{\eta \to 0 \\ r > 2}} \frac{\int_0^\eta g(t,r)\mu(t)dt}{j(r)} = 0.$$

It follows from (4.3) that

$$\int_0^\eta (4\pi t)^{-d/2} \exp(-\frac{r^2}{4t})\mu(t)dt \le c_1 \int_0^\eta t^{-(\frac{d}{2}+2)} \exp(-\frac{r^2}{4t})dt \le c_3 \int_0^\eta \exp(-\frac{r^2}{8t})dt$$

$$= c_3 \int_{r^2/(8\eta)}^\infty e^{-s} \frac{r^2}{8s^2}ds \le c_4 r^2 \int_{r^2/(8\eta)}^\infty e^{-s/2}ds = c_5 r^2 \exp(-\frac{r^2}{16\eta}).$$

Now combining this with (4.7) we immediately arrive at the desired conclusion. $\qquad \square$

Lemma 4.3 *Suppose that the Laplace exponent ϕ of S is a complete Bernstein function. For any $r_0 \in (0,1)$,*

$$\limsup_{\substack{\delta \to 0 \\ r > r_0}} \frac{j(r)}{j(r+\delta)} = 1. \tag{4.8}$$

Proof. For any $\epsilon \in (0,1)$, choose $\eta \in (0,1)$ such that

$$\sup_{r > r_0} \frac{\int_0^\eta g(t,r)\mu(t)dt}{j(r)} \le \epsilon.$$

Then for any $r > r_0$, $\int_\eta^\infty g(t,r)\mu(t)dt \ge (1-\epsilon)j(r)$. Fix this η. It follows from Lemma 4.1 that there exists $\delta_0 \in (0, \eta/2)$ such that

$$\frac{\mu(t)}{\mu(t+\delta)} \le 1+\epsilon, \qquad t \ge \eta, \delta \in (0, \delta_0].$$

For $t > \eta$, $0 \le (r+\delta-t)^2 = (r+\delta)^2 - 2tr + t(t-\delta) - \delta t$ and so $t(t-\delta) \ge 2tr + \delta t - (r+\delta)^2$. Thus

$$\frac{(r+\delta)^2}{4t} - \frac{r^2}{4(t-\delta)} = \frac{(r+\delta)^2(t-\delta) - r^2 t}{4t(t-\delta)} = \frac{\delta(2tr + \delta t - (r+\delta)^2)}{4t(t-\delta)} \le \frac{\delta}{4}.$$

Consequently, for $r > r_0$ and $\delta \in (0, \delta_0)$,

$$j(r+\delta) \ge \int_\eta^\infty (4\pi t)^{-d/2} \exp(-\frac{(r+\delta)^2}{4t})\mu(t)dt$$

$$\ge e^{-\delta/4} \int_\eta^\infty (4\pi t)^{-d/2} \exp(-\frac{r^2}{4(t-\delta)})\mu(t)dt$$

$$\ge e^{-\delta/4} \int_{\eta-\delta}^\infty (4\pi(t+\delta))^{-d/2} \exp(-\frac{r^2}{4t})\mu(t+\delta)dt$$

$$\geq \quad e^{-\delta/4} \left(\frac{\eta}{\eta + \delta} \right)^{d/2} (1 + \epsilon)^{-1} \int_{\eta}^{\infty} g(t, r) \mu(t) dt$$

$$\geq \quad e^{-\delta/4} \left(\frac{\eta}{\eta + \delta} \right)^{d/2} (1 + \epsilon)^{-1} (1 - \epsilon) j(r).$$

Now choose $\delta^* \in (0, \delta_0)$ such that

$$e^{-\delta/4} \left(\frac{\eta}{\eta + \delta} \right)^{d/2} \geq (1 + \epsilon)^{-1}, \qquad \delta \in (0, \delta^*].$$

Then for all $r > r_0$ and $\delta \in (0, \delta^*]$,

$$j(r + \delta) \geq (1 + \epsilon)^{-2} (1 - \epsilon) j(r),$$

which is equivalent to

$$\frac{j(r)}{j(r + \delta)} \leq \frac{(1 + \epsilon)^2}{(1 - \epsilon)},$$

which implies (4.8). $\qquad\square$

Lemma 4.4 *If the Laplace exponent ϕ of S is a complete Bernstein function, then $\mathbf{E1}(z_0, R)$ holds for Y.*

Proof. Fix $r_0, \varepsilon > 0$ and use the notation $B_r = B(0, r)$. By Lemma (4.3) there exists $\eta = \eta(\epsilon, r_0) > 0$ such that for all $\eta \leq \eta(\epsilon, r_0)$,

$$\sup_{r > r_0} \frac{j(r)}{j(r + \eta)} < 1 + \epsilon.$$

Let $\delta := \frac{2\eta}{r_0} \wedge 1$. For $y \in B_{\delta r_0/2}$ and $z \in B_{2r_0}^c$ we have

$$r_0 < \frac{|z|}{2} = |z| - \frac{|z|}{2} \leq |z| - |y| \leq |z - y| \leq |z| + |y| \leq |z| + \frac{\delta r_0}{2} \leq |z| + \eta,$$
$$r_0 < |z| \leq |z - y| + |y| \leq |z - y| + \eta.$$

Hence,

$$\frac{j(|z - y|)}{j(|z|)} \leq \frac{j(|z - y|)}{j(|z - y| + \eta)} \leq \sup_{r > r_0} \frac{j(r)}{j(r + \eta)} < 1 + \epsilon$$

and

$$\frac{j(|z|)}{j(|z - y|)} \leq \frac{j(|z|)}{j(|z| + \eta)} \leq \sup_{r > r_0} \frac{j(r)}{j(r + \eta)} < 1 + \epsilon.$$

This finishes the proof of the lemma. $\qquad\square$

We now briefly discuss (1.8), $\mathbf{C1}(z_0, R)$, (1.6), $\mathbf{F1}(z_0, R)$, and $\mathbf{G}$. First note that, if Y is transient then (1.8) holds (see [12, Lemma 2.10]). For the remainder of this section, we

21

will always assume that ϕ is a complete Bernstein function and the Lévy density μ of ϕ is infinite, i.e. $\mu(0, \infty) = \infty$. We consider the following further assumptions on ϕ:

H: *there exist constants $\sigma > 0$, $\lambda_0 > 0$ and $\delta \in (0, 1]$ such that*

$$\frac{\phi'(\lambda t)}{\phi'(\lambda)} \leq \sigma \, t^{-\delta} \quad \text{for all} \ \ t \geq 1 \ \ \text{and} \ \ \lambda \geq \lambda_0 \, . \tag{4.9}$$

When $d \leq 2$, we assume that $d + 2\delta - 2 > 0$ where δ is the constant in (4.9), and there are $\sigma' > 0$ and

$$\delta' \in \left(1 - \tfrac{d}{2}, (1 + \tfrac{d}{2}) \wedge (2\delta + \tfrac{d-2}{2}) \right) \tag{4.10}$$

such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \geq \sigma' x^{-\delta'} \quad \text{for all} \ \ x \geq 1 \ \ \text{and} \ \ \lambda \geq \lambda_0 \, . \tag{4.11}$$

Assumption **H** was introduced and used in [8] and [9]. It is easy to check that if ϕ is a complete Bernstein function satisfying satisfying a weak lower scaling condition at infinity

$$a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t) \, , \qquad \lambda \geq 1, t \geq 1 \, , \tag{4.12}$$

for some $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0, 1)$, then **H** is automatically satisfied. One of the reasons for adopting the more general setup above is to cover the case of geometric stable and iterated geometric stable subordinators. Suppose that $\alpha \in (0, 2)$ for $d \geq 2$ and that $\alpha \in (0, 2]$ for $d \geq 3$. A geometric $(\alpha/2)$-stable subordinator is a subordinator with Laplace exponent $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$. Let $\phi_1(\lambda) := \log(1 + \lambda^{\alpha/2})$, and for $n \geq 2$, $\phi_n(\lambda) := \phi_1(\phi_{n-1}(\lambda))$. A subordinator with Laplace exponent ϕ_n is called an iterated geometric subordinator. It is easy to check that the functions ϕ and ϕ_n satisfy **H** but they do not satisfy (4.12).

It follows from [9, Lemma 5.4] and [14, Section 4.2] that if Y is transient and **H** is true, then there exists $R > 0$ such that the assumptions **G**, **C1**(z_0, R), (1.6) and **F1**(z_0, R) hold for all $z_0 \in \mathbb{R}^d$. Thus using these facts and Lemma 4.4, we have the following as a special case of Theorems 1.1(a) and 1.2(b).

Corollary 4.5 *Suppose that $Y = (Y_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is a transient subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$. Suppose ϕ is a complete Bernstein function with the infinite Lévy measure μ and assume that **H** holds. Let $r \leq 1$ and let f_1 and f_2 be nonnegative functions on $\mathbb{R}^d$ which are regular harmonic in $D \cap B(z_0, r)$ with respect to the process Y, and vanish on $B(z_0, r) \cap (\overline{D}^c \cup D^{\mathrm{reg}})$. Then the limit*

$$\lim_{D \ni x \to z_0} \frac{f_1(x)}{f_2(x)}$$

exists and is finite. Moreover, the Martin boundary point associated with $z \in \partial D$ is minimal if and only if z is accessible from D.

4.2 Unimodal Lévy process

Let Y be an isotropic unimodal Lévy process whose characteristic exponent is $\Psi_0(|\xi|)$, that is,

$$\Psi_0(|\xi|) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) j_0(|y|) dy \qquad (4.13)$$

where the function $x \mapsto j_0(|x|)$ is the Lévy density of Y. If Y is transient, let $x \mapsto g_0(|x|)$ denote the Green function of Y.

Let $0 < \alpha < 2$. Suppose that $\Psi_0(\lambda) \sim \lambda^\alpha \ell(\lambda)$, $\lambda \to 0$, and ℓ is a slowly varying function at 0. Then by [4, Theorems 5 and 6] we have the following asymptotics of j_0 and g_0.

Lemma 4.6 *Suppose that $\Psi_0(\lambda) \sim \lambda^\alpha \ell(\lambda)$, $\lambda \to 0$, and ℓ is a slowly varying function at 0 ,*

(a) It holds that

$$j_0(r) \sim r^{-d} \Psi_0(r^{-1}), \qquad r \to \infty. \qquad (4.14)$$

(b) If $d \geq 3$, then Y is transient and

$$g_0(r) \sim r^{-d} \Psi_0(r^{-1})^{-1}, \qquad r \to \infty. \qquad (4.15)$$

We further assume that the Lévy measure of X is infinite. Then by [16, Lemma 2.5] the density function $x \to p_t(|x|)$ of X is continuous and, by the strong Markov property, so is the density function of X^D. Using the upper bound of $p_t(|x|)$ in [6, Theorem 2.2] (which works for all $t > 0$) and the monotonicity of $r \to p_t(r)$, we see that the Green function of X^D is continuous for all open set D. From this and (4.15), we have that if $d \geq 3$, the Lévy measure is infinite, and $\Psi_0(\lambda) \sim \lambda^\alpha \ell(\lambda)$, then **G** and (1.7) hold (see [13, Proposition 6.2]). It is proved in [13] under some assumptions much weaker than the above that $\mathbf{F2}(z_0, R)$ holds for all $z_0 \in \mathbb{R}^d$. From (4.14) we have that $\mathbf{E2}(z_0, R)$ and $\mathbf{C2}(z_0, R)$ hold for all $z_0 \in \mathbb{R}^d$. Using the above facts, we have the following as a special case of Theorems 1.1(b) and 1.4(b).

Corollary 4.7 *Suppose that $d \geq 3$ and that $Y = (Y_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is an isotropic unimodal Lévy process whose characteristic exponent is given by $\Psi_0(|\xi|)$. Suppose that $0 < \alpha < 2$, that the Lévy measure of X is infinite, and that $\Psi_0(\lambda) \sim \lambda^\alpha \ell(\lambda)$, $\lambda \to 0$, and ℓ is a slowly varying function at 0. Let $r > 1$, D be an unbounded open set and let f_1 and f_2 be nonnegative functions on $\mathbb{R}^d$ which are regular harmonic in $D \cap \overline{B}(z_0, r)^c$ with respect to the process Y, and vanish on $\overline{B}(z_0, r)^c \cap (\overline{D}^c \cup D^{\mathrm{reg}})$. Then the limit*

$$\lim_{D \ni x \to \infty} \frac{f_1(x)}{f_2(x)}$$

exists and is finite. Moreover, the Martin boundary point associated with ∞ is minimal if and only if ∞ is accessible from D.

Remark 4.8 Using [11, Lemma 3.3] instead of [4, Theorem 6], one can see that Corollary 4.7 holds for $d > 2\alpha$ when Y is a subordinate Brownian motion whose Laplace exponent ϕ is a complete Bernstein function and that $\phi(\lambda) \sim \lambda^{\alpha/2}\ell(\lambda)$ where $0 < \alpha < 2$ and ℓ is a slowly varying function at 0,

Remark 4.9 If Y is a Lévy process satisfying $\mathbf{E1}(z_0, R)$, ($\mathbf{E2}(z_0, R)$, respectively) then the Lévy process Z with Levy density $j_Z(x) := k(x/|x|)j_Y(x)$ also satisfies $\mathbf{E1}(z_0, R)$ ($\mathbf{E2}(z_0, R)$, respectively) when k is a continuous function on the unit sphere and bounded between two positive constant. In fact, since

$$\left| \frac{z-y}{|z-y|} - \frac{z}{|z|} \right| \leq \left| \frac{z-y}{|z-y|} - \frac{z-y}{|z|} \right| + \left| \frac{z-y}{|z|} - \frac{z}{|z|} \right| \leq \frac{||z| - |z-y||}{|z|} + \frac{|y|}{|z|} \leq \frac{2|y|}{|z|},$$

we have

$$\left| \frac{z-y}{|z-y|} - \frac{z}{|z|} \right| \leq \frac{2|y|}{r} \quad \text{for all } |z| > r \quad \text{and} \quad \left| \frac{z-y}{|z-y|} - \frac{z}{|z|} \right| \leq \frac{2r}{|z|} \quad \text{for all } |y| < r.$$

Moreover, since k is bounded below by a positive constant,

$$\left| \frac{k(z/|z|)}{k((z-y)/|z-y|)} - 1 \right| \leq c|k(z/|z|) - k((z-y)/|z-y|)|.$$

Thus by uniform continuity of k on the unit sphere, we see that for all $r > 0$

$$\lim_{|y| \to 0} \sup_{z:|z|>r} \frac{k(z/|z|)}{k((z-y)/|z-y|)} = \lim_{|y| \to 0} \sup_{z:|z|>r} \frac{k(z/|z|)}{k((z-y)/|z-y|)} = 1,$$

and

$$\lim_{|z| \to \infty} \sup_{y:|y|<r} \frac{k(z/|z|)}{k((z-y)/|z-y|)} = \lim_{|z| \to \infty} \inf_{y:|y|<r} \frac{k(z/|z|)}{k((z-y)/|z-y|)} = 1.$$

When Y is a symmetric stable process, this includes not necessarily symmetric strictly stable processes with Levy density $ck(x/|x|)|x|^{-d-\alpha}$ where k is a continuous function on the unit sphere bounded between two positive constant.

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