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Formal classification of parabolic Dulac maps

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Dulac or almost regular germs

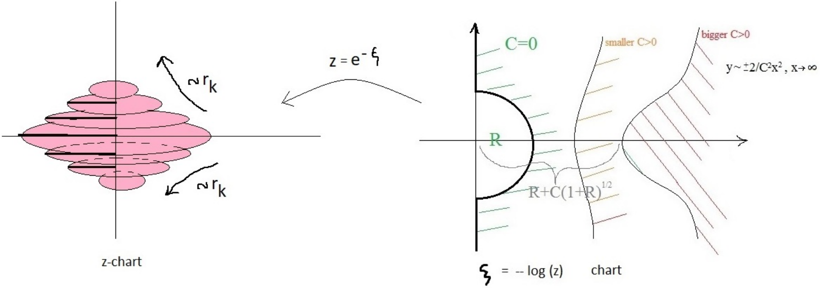
Definition [Ilyashenko].
Parabolic almost regular germ (*Dulac germ*):

- $f \in C^\infty(0,d)$
- extends to a holomorphic germ f to a *standard quadratic domain* Q:

$$Q = \Phi(\mathbb{C}_+ \setminus \overline{K(0,R)}),\ \ \Phi(\eta) = \eta + C(\eta+1)^{\frac{1}{2}},\ C,\ R > 0,$$

in the *logarithmic chart* $\xi = -\log z$.

Standard quadratic domain



$$r_k := r(\varphi_k) \sim e^{-C\sqrt{\frac{|k|\pi}{2}}},\ k \to \pm\infty,$$
$$\varphi_k \in \big((k-1)\pi,\ (k+1)\pi\big)$$

- f admits the *Dulac* asymptotic expansion:

$$f(z) \sim_{z\to 0} {\color{red}1 \cdot z} + \sum_{k=1}^{\infty} z^{\alpha_i} P_i(-\log z),$$

$$\text{i.e. } f(z) - z - \sum_{i=1}^{n} z^{\alpha_i} P_i(-\log z) = O(z^{\alpha_n}),\ n \in \mathbb{N},$$

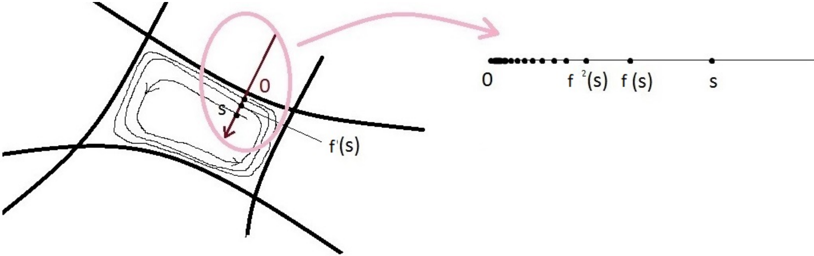
  - $\alpha_i > 1$, strictly increasing to $+\infty$,
  - α_i finitely generated [2],
  - P_i *polynomials*.
- $\mathbb{R}_+$ invariant under f (i.e. coefficients of $\widehat{f}$ real!)

[2] There exist β_k, $k = 1 \ldots n$, such that: $\alpha_i \in \mathbb{N}\beta_1 + \ldots + \mathbb{N}\beta_n$.

Motivation and history

- *first return maps* for polycycles with hyperbolic saddle singular points – n saddle vertices with hyperbolicity ratios $\beta_i > 0$ (Dulac)
- locally at the saddle

$$\begin{cases} \dot{x} = x + \text{h.o.t.} \\ \dot{y} = -\beta_i y + \text{h.o.t.} \end{cases}$$



Motivation and history

- *Dulac's problem*: accumulation of limit cycles on a hyperbolic polycycle possible?
- limit cycles = fixed points of the first return map
- Dulac: accumulation $\Rightarrow$ trivial power-log asymptotic expansion of the first return map $\Rightarrow$ trivial germ on $\mathbb{R}_+$ (Dulac's mistake)
- the problem: Dulac asymptotic expansion does not uniquely determine f on $\mathbb{R}_+$ (add **any exponentially small term w.r.t.** x!), e.g.

$$f(x) \sim x + x^2 - \log x,\ f(x) + e^{-1/x} \sim x + x^2 - \log x,\ x \to 0$$

- Ilyashenko's solution: first return maps extendable to a SQD
- SQD *sufficiently large complex domain*: by a variant of maximum modulus principle (*Phragmen-Lindelöf*), Dulac's expansion uniquely determines the germ on a SQD!

Questions

$\star$ goal: theory like the standard theory of Birkhoff, Ecalle, Voronin, Kimura, Leau etc. for parabolic analytic germs $\mathrm{Diff}(\mathbb{C}, 0)$

- **formal classification** of parabolic Dulac germs – by a **sequence** (!!! not necesarily convergent) of *formal power-logarithmic changes of variables*

$$\widehat{g} = \widehat{\varphi}^{-1} \circ f_0 \circ \widehat{\varphi},$$

$\widehat{f},\ f_0$ Dulac expansions, f_0 simple 3-monomial expression
$\widehat{\varphi}(z) = z + h.o.t.$ diffeo- with power-log asymptotic expansion

- simpler question: is a Dulac germ **formally embeddable as time-one map** in a flow of an analytic vector field $\xi(z)\frac{d}{dz}$ defined on a standard quadratic domain? ($=$ describe *trivial* analytic class)

$$g = \widehat{\varphi}^{-1} \circ \tilde{f}_0 \circ \widehat{\varphi},$$

$f,\ \tilde{f}_0$ Dulac germs,
$\tilde{f}_0$ time-one map of an analytic vector field on Q,

Why formal classification?

- motivated by **analytic classification** of parabolic Dulac germs

$$g = \varphi^{-1} \circ f \circ \varphi,$$

$f,\ g$ Dulac germs on Q, $\varphi(z) = z + o(z)$ analytic on Q

- φ admits $\widehat{\varphi}$ as its asymptotic expansion?
- domains of analytic 'summability' of $\widehat{\varphi}$

Historical results - germs of *parabolic analytic diffeomorphisms*

(Fatou $\sim$ end of 19^{th} century; Birkhoff~ 1950; Ecalle, Voronin$\sim 1980, \ldots$)

$f \in \mathrm{Diff}(\mathbb{C},0),\ f(z) = z + a_1 z^{k+1} + a_2 z^{k+2} + \ldots,\quad k \in \mathbb{N}$

- Formal embedding

= formal reduction to a **time-one map of a vector field**:

$$f_0(z) = \mathrm{Exp}\left(\frac{z^{k+1}}{1+\rho z^k}\frac{d}{dx}\right).\mathrm{id} = z + z^{k+1} + \left(\rho + \frac{k+1}{2}\right)z^{2k+1} + \ldots$$

Step-by-step elimination of monomials from f:

$$\varphi_\ell(z) = \begin{cases} az,\ a \neq 1, \\ z + cz^\ell,\ \ell \in \mathbb{N} \end{cases} \quad \leftrightarrow \widehat{\varphi}(z) = az + \sum_{k=2}^{\infty} c_k z^k \in \mathbb{C}[[z]]$$

(formal changes of variables)

$\Rightarrow\ (k,\rho),\ k \in \mathbb{N},\ \rho \in \mathbb{C} \ldots$ ($\rho = \mathrm{Res}_0\left(\frac{1}{z - f(z)}\right)$) formal invariants for f.

Example

$f(z) = z + z^2 + z^3 + \ldots = \frac{z}{1-z}$ time-one map of $z^2 \frac{d}{dy}$.

Example

$g(z) = e^z - 1 = z + z^2 + z^3 + \ldots$ not a time-one map of a vector field, *formally embeddable* in $z^2 \frac{d}{dy}$

Historical results - germs of *analytic diffeomorphisms*

- Is g analytically embeddable, or just formally?

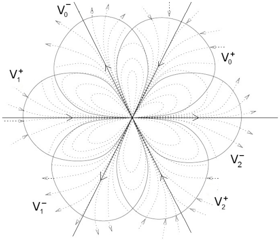
$\leftrightarrow$ Does $\widehat{\varphi}$ converge to an analytic function at 0?

Leau-Fatou flower theorem (1987):

$\star$ $2k$ analytic conjugacies φ_i of f to f_0, all expanding in $\widehat{\varphi}$

$\star$ defined on $2k$ *petals* invariant under local discrete dynamics

$\star$ k attracting directions: $(-a_1)^{-\frac{1}{k}}$; k repelling directions: $a_1^{-\frac{1}{k}}$



$k = 3 \rightarrow$ 6 petals, $f(z) = z + z^4 + \ldots$

$\rightarrow$ in general, analytic embedding in a flow **only on open sectors**

$\rightarrow$ the analytic class of f in direct relation with this question

Formal embedding into flows for Dulac germs (non-analytic at 0)

• elimination **term-by-term** by an *adapted* 'sequence' of non-analytic *elementary changes of variables*:

$$\varphi(z)=az;\ \ \varphi_{\alpha,m}(z)=z+cz^{\alpha}\boldsymbol{\ell}^m,\ m\in\mathbb{Z},\ \alpha>0,\ (\alpha,m)\succ(1,0).$$

Example (MRRZ, 2016)

0. $f(z)=z-z^2\boldsymbol{\ell}^{-1}+z^2+z^3$,

1. $\varphi_1(z)=z+c_1z\boldsymbol{\ell},\ c_1\in\mathbb{C}$,

   $f_1(z)=\varphi_1^{-1}\circ f\circ\varphi_1(z)=z-z^2\boldsymbol{\ell}^{-1}+a_1z^2\boldsymbol{\ell}+h.o.t,$

2. $\varphi_2(z)=z+c_2z\boldsymbol{\ell}^2,\ c_2\in\mathbb{R}$,

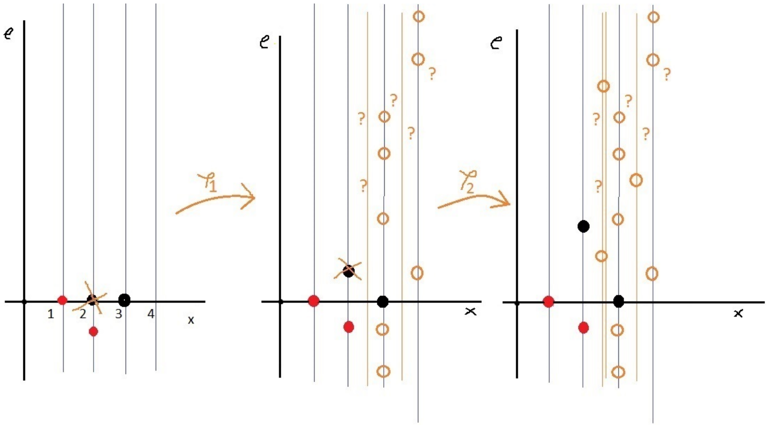
   $f_2(z)=\varphi_2^{-1}\circ f\circ\varphi_2(z)=z+z^2\boldsymbol{\ell}^{-1}\qquad+a_2z^2\boldsymbol{\ell}^2+h.o.t,$

3. $\varphi_3(z)=z+c_3z\boldsymbol{\ell}^3,\ c_3\in\mathbb{R}$,

   $f_3(z)=\varphi_3^{-1}\circ f\circ\varphi_3(z)=z+z^2\boldsymbol{\ell}^{-1}\qquad\qquad+a_2z^2\boldsymbol{\ell}^3+h.o.t,$

   $\vdots$

The visualisation of the reduction procedure



the control of the support!

The description of the formal change of variables

• more than just a *formal series composition* of changes of variables: a transfinite composition, $\rightarrow$ produces a **transseries** $\widehat{\varphi}$:

$\star$ in the process, prove that *every change has its successor change*

$\star$ prove the *formal convergence* of composition of changes of variables: by **transfinite induction**[1] in the *formal topology*[2]

---

[1] a generalization of the mathematical induction from $\mathbb{N}$ to ordinal numbers: existence of a *successor element* and a *limit element*,
[2] i.e. in each step of composition the support remains well-ordered; the coefficient of each monomial in the support stabilizes in the course of composition.

A broader class closed to embeddings: the class of power-log transseries $\widehat{\mathcal{L}}$

...contains both the Dulac germ expansions $f \mapsto \widehat{f}$ and the *formal changes of variables*

$$\widehat{\mathcal{L}} \ldots \ \widehat{f}(z) = \sum_{\alpha \in S} \sum_{k=N_\alpha}^{\infty} a_{\alpha,k} z^{\alpha} \boldsymbol{\ell}^k, \ a_{\alpha,k} \in \mathbb{R}, \ N_\alpha \in \mathbb{Z},$$

$S \subseteq (0, \infty)$ well-ordered (here: finitely gen.)

Similarly we define $\widehat{\mathcal{L}}_2, \ \widehat{\mathcal{L}}_3$, etc. and

$$\widehat{\mathfrak{L}} := \cup_{k \in \mathbb{N}} \widehat{\mathcal{L}}_k.$$

(iterated logarithms admitted!)

(L. van den Dries, A. Macintyre, D. Marker, *Logarithmic-exponential series*. Ann. Pure Appl. Logic 111 (2001))

$\boldsymbol{\ell} := -\frac{1}{\log x}$

Theorem (Formal embedding theorem for Dulac germs, MRRZ 2016)

$\widehat{f}(z) = z - az^{\alpha}\boldsymbol{\ell}^{m} + h.o.t.$ *parabolic Dulac,* $a > 0$, $\alpha > 1$, $m \in \mathbb{N}_{-}$.
$\Rightarrow$ *formally in* $\widehat{\mathcal{L}}$ *conjugated to:*

$$f_0(z) = \exp\Big(\frac{-z^{\alpha}\boldsymbol{\ell}^{m}}{1 - \frac{\alpha}{2}z^{\alpha-1}\boldsymbol{\ell}^{k} + \big(\frac{k}{2} - \rho\big)z^{\alpha-1}\boldsymbol{\ell}^{k+1}}\frac{d}{dz}\Big).\mathrm{id} =$$

$$= z - z^{\alpha}\boldsymbol{\ell}^{m} + \rho z^{2\alpha-1}\boldsymbol{\ell}^{2m+1} + h.o.t.$$

$\star$ (α, m, ρ), $\rho \in \mathbb{R}\ldots$ formal invariants ($\rho = \big[\frac{\boldsymbol{\ell}}{z}\big]\frac{1}{z-f(z)}$) for Dulac germ

$\star$ $f_0(z)$ a time-one map of an analytic vector field on SQD $(\mathbb{Q}_+)$

Example continued

Example (continued)

$$f_0(z) = \exp\Big(- \frac{z^2 \boldsymbol{\ell}^{-1}}{1 - z\boldsymbol{\ell}^{-1} + \big(b - \frac{1}{2}\big) z} \Big).\mathrm{id} =$$

$$= z - z^2 \boldsymbol{\ell}^{-1} + b z^3 \boldsymbol{\ell}^{-1} + h.o.t.,$$

$f_0 = \widehat{\varphi}^{-1} \circ \widehat{f} \circ \widehat{\varphi}, \quad \widehat{\varphi} \in \widehat{\mathcal{L}}$ – a *transfinite* change of variables

Parallel construction: the (formal) Fatou coordinate and Abel equation " = " (formal) embedding in a vector field

'Equivalent' problems:

1. (formal) conjugation of f to f_0 (time-one map of an analytic vector field)
2. (formal) Fatou coordinate for f

$\Psi(f(z)) - \Psi(z) = 1$ (Abel equation)
$\widehat{\Psi}(\widehat{f}(z)) - \widehat{\Psi}(z) = 1$ (formal Abel equation)

$\Psi = \Psi_0 \circ \varphi$, $\widehat{\Psi} = \Psi_0 \circ \widehat{\varphi}$

$*$ the Fatou coordinate represents the *time*:

$$\widehat{\Psi}(\widehat{f}^t(x_0)) - \widehat{\Psi}(x_0) = t.$$

Non-uniqueness of asymptotic expansion of a germ in $\widehat{\mathcal{L}}$

When do we say that $\widehat{\Psi}$ is the transserial asymptotic expansion of Ψ?

Caution! *Transserial* asymptotic expansion is not well-defined (unique), if we do not prescribe a canonical summation method on limit ordinal steps (dictated here by Abel equation)!

$\rightarrow$ ambiguity: choice of the sum in $\boldsymbol{\ell}$ at limit ordinal steps

Example

$f(z) = z + z^2 \frac{\boldsymbol{\ell}}{1-\boldsymbol{\ell}} + z^5$

Some possible asymptotic expansions:

$$\widehat{f_1}(z) = z + z^2(\boldsymbol{\ell} + \boldsymbol{\ell}^2 + \boldsymbol{\ell}^3 + \dots) + z^5$$
$$\widehat{f_2}(z) = z + z^2(\boldsymbol{\ell} + \boldsymbol{\ell}^2 + \boldsymbol{\ell}^3 + \dots) - z^3 + z^5, \text{ etc.}$$

- $\widehat{f_1}$: canonical (convergent sum) at the first limit ordinal step:

$$\boldsymbol{\ell} + \boldsymbol{\ell}^2 + \boldsymbol{\ell}^3 + \dots \mapsto \frac{\boldsymbol{\ell}}{1-\boldsymbol{\ell}}$$

- $\widehat{f_2}$: $\boldsymbol{\ell} + \boldsymbol{\ell}^2 + \boldsymbol{\ell}^3 + \dots \mapsto \frac{\boldsymbol{\ell}}{1-\boldsymbol{\ell}} + e^{-\frac{3}{\boldsymbol{\ell}}} \quad \left(z = e^{-1/\boldsymbol{\ell}}\right)$

Moreover: (?) canonical choice if series in ℓ was **divergent** (Fatou coordinate)

Sketch of the proof / method of summation

$f(z) \sim \widehat{f}(z) = z + z^{\alpha_1} P_1(-\log z) + z^{\alpha_2} P_2(-\log z) + \dots$

- solve (formal) Abel equation by *blocks*

$$\widehat{\Psi}(z + z^{\alpha_1} P_1(\boldsymbol{\ell}^{-1}) + \dots) - \widehat{\Psi}(z) = 1$$

- $\widehat{\Psi}(z) := \sum z^{\beta_i} \widehat{T}_i(\boldsymbol{\ell})$
- In each step, $\widehat{T}_i$ obtained solving one differential equation:

$$\frac{d}{dz}\Big(z^{\beta_i} \widehat{T}_i(\boldsymbol{\ell})\Big) := z^{\beta_i - 1} R(\boldsymbol{\ell}),$$

$$(*)\ \widehat{T}_i(\boldsymbol{\ell}) = z^{-\beta_i} \int z^{\beta_i - 1} R(\boldsymbol{\ell}) dz,$$

  β_i a finite combination of α_i; R a rational function in $\boldsymbol{\ell}$.

- $(*)$ solvable analytically (T_i analytic on Q) as well as formally ($\widehat{T}_i \in \mathbb{C}[[z]]$) by partial integration
  $\rightarrow$ principle of summation at limit ordinal steps: $\widehat{T}_i \mapsto T_i$
  (integral sum)

- $\widehat{\Psi} := \Psi_\infty + \widehat{R}$, where Ψ_∞ contains *only finitely many* infinite blocks

- analytic Fatou coordinate on small sectors around $\mathbb{R}_+$: *iterative summation* of the Abel equation along the orbit of f/f^{-1}, after subtracting sufficiently many blocks:

$$R(f(z)) - R(z) = \delta(z),$$

$\delta(z)$ of arbitrarily small order.

$$\Rightarrow \; R(z) := -\sum_{k=0}^{\infty} \delta(f^{\circ(\pm)k}(z)), \; j \in \mathbb{Z}.$$

Converges *locally uniformly on small sectors around* $\mathbb{R}_+$.

$Q.E.D.$

Example of blocks computation in the Fatou coordinate of a Dulac germ

Example

$f(z) = z + z^2\boldsymbol{\ell}^{-1} + z^3 \;\Rightarrow\; \Psi(z + z^2\boldsymbol{\ell}^{-1} + z^3) - \Psi(z) = 1.\ (*)$

Computation of the first block of Ψ by formal T. expansion of $(*)$:

$$\Psi_0'(z)z^2\boldsymbol{\ell}^{-1} = 1 \;\Rightarrow\; \Psi_0(z) = \int z^{-2}\boldsymbol{\ell}\,dz$$

- Integration by parts: $\widehat{\Psi}_0(z) = z^{-1}\sum_{n\in\mathbb{N}} n!\boldsymbol{\ell}^n$ (divergent series in $\boldsymbol{\ell}$ in the first block!)
- Analytic integration on SQD: $\Psi_0(z) = \int_*^z y^{-2}\boldsymbol{\ell}(y)\,dy$

? appropriate sum of divergent series above ? integral sum

$$\sum_n n!\boldsymbol{\ell}^n \mapsto \frac{\int_*^z y^{-2}\boldsymbol{\ell}(y)\,dy}{z^{-1}}.$$

A Fatou coordinate $\leftrightarrow$ embedding in a flow

Theorem (MRRZ2)

There exists a unique (up to an additive constant) formal Fatou coordinate $\widehat{\Psi}$ *for the Dulac expansion* $\widehat{f}$ *in* $\widehat{\mathfrak{L}}$*. Moreover, it is in* $\widehat{\mathcal{L}}_2^\infty$*.*

Theorem (MRRZ2)

There exists an analytic Fatou coordinate $\Psi \in C^\infty(0,d)$ *(that is, an analytic embedding* $\{f_t\}_t$*,* $f_t \in C^\infty(0,d)$*) which admits the formal Fatou coordinate* $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ *as its "integral asymptotic expansion".*

Note: the analytic construction extendable to complex sectors coresponding to attracting/repelling petals for the local dynamics of Dulac f

The solution: the notion of *sectional* asymptotic expansions [MRRZ2]

⋆ the notion of a *sectional* asymptotic expansion-a section is a linear operator attributing a particular germ to partial expansions on intermediate limit ordinal levels

⋆ the *integral section*: a canonical choice dictated by the solution of the Abel equation!

References

MRRZ Mardešić, P., Resman, M., Rolin, J.P., Županovic, V., Normal forms and embeddings for power-log transseries, Advances in Mathematics 303 (2016), 888-953

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MRRZ3 Mardesic, P., Resman, M., Rolin, J.P., Zupanovic, V.: Length of epsilon-neighborhoods of orbits of Dulac maps (preprint, 2018), https://arxiv.org/pdf/1606.02581v3.pdf