Ball Spaces and Kuhlmanns' Fixed Point Theory

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Fixed point methods play an important role in the construction of normal forms in local dynamical systems. In particular, suitable reformulations of conjugacy equations allow changes of coordinates to be obtained as fixed points of operators on spaces of formal series or analytic germs. Motivated by this approach, we discuss the theory of ball spaces developed by Katarzyna and Franz-Viktor Kuhlmann and its general fixed point principles.

A ball space consists of a set equipped with a distinguished family of nonempty subsets, called balls, without requiring an underlying metric or topology. Its central completeness notion is spherical completeness: every nonempty family of balls totally ordered by inclusion has nonempty intersection. We introduce this framework and explain how appropriate contraction conditions yield fixed point theorems encompassing both classical metric and ultrametric settings. We then relate this perspective to fixed point techniques arising in our work on the normalization of logarithmic transseries and Dulac germs. Particular attention is given to the role of completeness and the choice of a suitable space for solving conjugacy equations. Finally, motivated by ongoing work on hyperbolic dynamics in H-fields and surreal numbers, we discuss the prospects and challenges of extending these techniques to broader settings. The aim is to highlight how ball spaces can provide a common language for fixed point constructions and help identify the hypotheses needed for such extensions.

References

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