1.) a)

$$(x-1)\underbrace{(x-2)T}_{=:U}=0$$

$$(x-1)U=0$$

$$(\tau_1 x)U=0$$

$$\tau_1(x(\tau_{-1}U))=0$$

$$\Rightarrow x(\tau_{-1}U)=0$$

$$\Rightarrow \tau_{-1}U=C\delta_0,\ C\in\mathbb{C}$$

$$\Rightarrow \boxed{U=C\delta_1,\ C\in\mathbb{C}}$$

$$(x-2)T=C\delta_1$$

Znamo da su rješenja $(x-2)T=0$ dana s

$$T_H=D\delta_2,\ D\in\mathbb{C}$$

Trebamo pogoditi još jedno partikularno rj.

Pretp. $T_P=E\delta_1,\ E\in\mathbb{C}.$

$$\langle(x-2)E\delta_1,\varphi\rangle=E\langle\delta_1,(x-2)\varphi\rangle$$

$$=E(-1)\overline{\varphi(1)}$$

$$=-E\langle\delta_1,\varphi\rangle$$

$$\Rightarrow -E=C\Rightarrow\boxed{E=-C}$$

Rj.: $T=-C\delta_1+D\delta_2,\ C,D\in\mathbb{C}$

$$\boxed{T=C_1\delta_1+C_2\delta_2,\ C_1,C_2\in\mathbb{C}}$$

b) $(x-i)(x-1)(x-2)T=\delta_1$

Nađimo najprije rješenje pripadne homogene j.

$(x-i)(x-1)(x-2)T_H=0$, $\frac{1}{x-i}$ je C^∞ f-ja

$$\Rightarrow (x-1)(x-2)T_H=0$$

$$\overset{a)}{\Rightarrow} T_H=C_1\delta_1+C_2\delta_2,\ C_1,C_2\in\mathbb{C}$$

$$\boxed{T=\frac{1-i}{2}\delta_1'+C_1\delta_1+C_2\delta_2,\ C_1,C_2\in\mathbb{C}}$$

Pretpostavimo rješenje u obliku $D\delta_1'$, $D\in\mathbb{C}$.

$$\langle(x-i)(x-1)(x-2)T_P,\varphi\rangle=D\langle\delta_1',(x-i)(x-1)(x-2)\varphi\rangle$$

$$=-D\langle\delta_1,(x-i)1(x-2)\varphi\rangle+D\underbrace{\langle\delta_1,(x-1)(\ldots)\rangle}_{=0}$$

$$=-D\overline{(1-i)}(-1)\overline{\varphi(1)}$$

$$=+D(1+i)\overline{\varphi(1)}=D(1+i)\langle\delta_1,\varphi\rangle\Rightarrow\boxed{D=\frac{1}{1+i}=\frac{1-i}{2}}$$

2.)

a) $\langle C_\lambda, \varphi\rangle = \lim\limits_{\varepsilon \searrow 0} \left(\int\limits_{-\infty}^{-\varepsilon} \frac{\cos(\lambda x)}{x} \overline{\varphi(x)}\,dx + \int\limits_{\varepsilon}^{\infty} \frac{\cos(\lambda x)}{x} \overline{\varphi(x)}\,dx \right)$

$= \lim\limits_{\varepsilon \searrow 0} \int\limits_{\varepsilon}^{\infty} \frac{\cos(\lambda x)}{x} \left(\overline{\varphi(x)} - \overline{\varphi(-x)}\right) dx$, $\operatorname{supp}\varphi \subseteq [-M, M]$

$|\langle C_\lambda, \varphi\rangle| \leq \lim\limits_{\varepsilon \searrow 0} \int\limits_{\varepsilon}^{M} \frac{|\cos(\lambda x)|}{x} \underbrace{|\overline{\varphi(x)} - \overline{\varphi(-x)}|}_{\leq 2x |\varphi'(c_x)|} dx$

$\leq 2\|\varphi'\|_{L^\infty} \lim\limits_{\varepsilon \searrow 0} \int\limits_{\varepsilon}^{M} dx = 2M\|\varphi'\|_{L^\infty}$

$\Rightarrow$ C_λ je distribucija reda 2.

b) $\langle x C_\lambda, \varphi\rangle = \langle C_\lambda, \underset{x\in\mathbb{R}}{\overline{x}}\varphi\rangle = \lim\limits_{\varepsilon \searrow 0} \left(\int\limits_{-\infty}^{-\varepsilon} \frac{\cos(\lambda x)}{\cancel{x}} \cancel{x} \overline{\varphi(x)}\,dx + \int\limits_{\varepsilon}^{\infty} \frac{\cos(\lambda x)}{\cancel{x}} \cancel{x} \overline{\varphi(x)}\,dx \right)$

$= \int\limits_{-\infty}^{+\infty} \cos(\lambda x) \overline{\varphi(x)}\,dx$

$= \langle \cos(\lambda x), \varphi\rangle$

$\Rightarrow \boxed{x C_\lambda = \cos\lambda x}$

$xT = \cos\lambda x \Rightarrow T = C\delta_0 + C_\lambda$, $C \in \mathbb{C}$

c) $C_\lambda \xrightarrow[\lambda\to 0]{*} \operatorname{pv}\left(\frac{1}{x}\right)$ u $\mathcal{D}'(\mathbb{R})$

$\left|\langle C_\lambda, \varphi\rangle - \langle \operatorname{pv}(\tfrac{1}{x}), \varphi\rangle\right| \leq \lim\limits_{\varepsilon \searrow 0} \int\limits_{\varepsilon}^{M} \frac{|\cos(\lambda x) - 1|}{x} |\overline{\varphi(x)} - \overline{\varphi(-x)}|\,dx$

$\leq 2\|\varphi'\|_{L^\infty} \int\limits_{0}^{M} \underbrace{|\cos(\lambda x) - \overset{\cos(0)}{\overset{\|}{1}}|}\,dx$

$\leq 2M\|\varphi'\|_{L^\infty}\, \eta$

$\cos(\lambda\cdot)$ je jednoliko nepr. na segmentu $[0, M]$ pa $\exists\, \delta > 0$

$\underbrace{|\lambda x| < \delta}_{\lambda \leq \frac{\delta}{M}} \Rightarrow |\cos(\lambda x) - 1| < \eta$

2.) c) $C_\lambda \xrightarrow[\lambda\to\infty]{*} 0$ u $\mathcal{D}'(\mathbb{R})$

$$\cancel{|\langle C_\lambda,\varphi\rangle| = \lim_{\varepsilon\to 0}\left|\int_{\lambda\varepsilon}^{\infty}\frac{\cos(y)}{y}\left(\overline{\varphi}\left(\tfrac{y}{\lambda}\right)-\overline{\varphi}\left(-\tfrac{y}{\lambda}\right)\right)dy\right| \leq}$$

$\cancel{y=\lambda x,\quad dy=\lambda dx,\quad x=\varepsilon \to y=\lambda\varepsilon}$

$$\Phi(x) := \frac{\varphi(x)-\varphi(-x)}{x}$$

$$\lim_{x\to 0}\Phi(x) = \frac{2x\,\varphi'(c_x)}{x} = 2\varphi'(0) \quad\Rightarrow\quad \Phi\in C([0,+\infty\rangle)$$

$\&\ \Phi\equiv 0$ na $\langle M,+\infty\rangle$

$$\Phi'(x) = \frac{x(\varphi'(x)+\varphi'(-x)) - \varphi(x)+\varphi(-x)}{x^2}$$

$$= \frac{x(\varphi'(x)+\varphi'(-x)) - 2x\,\varphi'(c_x)}{x^2}$$

$$= \frac{(\varphi'(x)-\varphi'(c_x)) + (\varphi'(-x)-\varphi'(c_x))}{x}, \quad c_x\in\langle -x,x\rangle$$

$$= \frac{(x-c_x)\,\varphi''(d_x) + (-x-c_x)\,\varphi''(e_x)}{x}, \quad \begin{array}{l} d_x\in\langle c_x,x\rangle \\ e_x\in\langle -x,-c_x\rangle\end{array}$$

$$= \underset{\substack{\downarrow\\ 1}}{\frac{x-c_x}{x}}\ \underset{\substack{\downarrow\\ \varphi''(0)}}{\varphi''(d_x)} - \underset{\substack{\downarrow\\ 1}}{\frac{x+c_x}{x}}\ \underset{\substack{\downarrow\\ \varphi''(0)}}{\varphi''(e_x)}$$

$$\longrightarrow 0$$

$$\Rightarrow\quad \Phi'\in C([0,+\infty\rangle)\ \&\ \Phi'\equiv 0 \text{ na } \langle M,+\infty\rangle$$

$$\langle C_\lambda,\varphi\rangle = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^M \cos(\lambda x)\overline{\Phi(x)}\,dx \overset{\text{P.I.}}{=} \underbrace{\lim_{\varepsilon\to 0}\frac{\sin(\lambda x)}{\lambda}\overline{\Phi(x)}\Big|_\varepsilon^M}_{①} - \underbrace{\frac{1}{\lambda}\lim_{\varepsilon\to 0}\int_\varepsilon^M \sin(\lambda x)\overline{\Phi'(x)}\,dx}_{②}$$

$$|①| \leq \frac{\|\Phi\|_{L^\infty}}{\lambda} \xrightarrow{\lambda\to\infty} 0$$

$$|②| \leq \frac{1}{\lambda}\|\Phi'\|_{L^\infty} M \xrightarrow{\lambda\to\infty} 0 \qquad\Rightarrow\quad |\langle C_\lambda,\varphi\rangle| \xrightarrow{\lambda\to\infty} 0$$

3.) a) $u_1(x,y) = \dfrac{1}{(a^2+x^2)(b^2+y^2)} = f_1(x)\, f_2(y)$, $f_1(x) = \dfrac{1}{a^2+x^2}$

$f_2(y) = \dfrac{1}{b^2+y^2}$

$$\hat{u}_1(\xi,\eta) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i\,(x,y)\cdot(\xi,\eta)} f_1(x) f_2(y)\, dx\, dy$$

$$= \int_{\mathbb{R}} e^{-2\pi i x\xi} f_1(x)\, dx \int_{\mathbb{R}} e^{-2\pi i y\eta} f_2(y)\, dy$$

$$= \hat{f}_1(\xi)\, \hat{f}_2(\eta)$$

$$= \frac{\pi}{a} e^{-2\pi a|\xi|} \frac{\pi}{b} e^{-2\pi b|\eta|}$$

$$= \frac{\pi^2}{ab} e^{-2\pi(a|\xi|+b|\eta|)}$$

b) $u_{tt} - \Delta u = 0$

$\Rightarrow \hat{u}_{tt} + 4\pi^2(\xi^2+\eta^2)\,\hat{u} = 0$

$\Rightarrow \hat{u}(t;\xi,\eta) = C_1(\xi,\eta)\sin\left(2\pi\sqrt{\xi^2+\eta^2}\,t\right) + C_2(\xi,\eta)\cos\left(2\pi\sqrt{\xi^2+\eta^2}\,t\right)$

$\hat{u}(0;\xi,\eta) = 0$

$\hat{u}_t(0;\xi,\eta) = \hat{u}_1(\xi,\eta)$

$\Rightarrow C_2 \equiv 0$

& $C_1(\xi,\eta)\, 2\pi\sqrt{\xi^2+\eta^2} = \hat{u}_1(\xi,\eta)$

$\Rightarrow C_1(\xi,\eta) = \dfrac{\hat{u}_1(\xi,\eta)}{2\pi\sqrt{\xi^2+\eta^2}}$

$\Rightarrow \hat{u}(t;\xi,\eta) = \dfrac{\hat{u}_1(\xi,\eta)}{2\pi\sqrt{\xi^2+\eta^2}} \sin\left(2\pi\sqrt{\xi^2+\eta^2}\,t\right)$

$$\Rightarrow u(t;x,y) = \int_{\mathbb{R}^2} \frac{\sin\left(2\pi\sqrt{\xi^2+\eta^2}\,t\right)}{2\pi\sqrt{\xi^2+\eta^2}}\, e^{-2\pi\left((\xi x+\eta y)i + a|\xi| + b|\eta|\right)}\, d\xi\, d\eta$$

4.) a) VJEŽBE ... $\Phi(t,x) = H(t) \frac{1}{(4\pi\nu t)^{d/2}} e^{-\frac{|x|^2}{4\nu t}}$

b) Znamo $\int_{\mathbb{R}^d} e^{-|x|^2} dx = \pi^{d/2}$

$$\left|\langle \Phi(t,x), \varphi\rangle - \langle \delta_0, \varphi\rangle\right| = \left| \int_{\mathbb{R}^d} \underbrace{H(t)}_{\substack{=1 \\ \text{jer je} \\ t>0}} \frac{1}{(4\pi\nu t)^{d/2}} e^{-\frac{|x|^2}{4\nu t}} \overline{\varphi(x)}\, dx - \overline{\varphi(0)} \right|$$

$y = \frac{x}{2\sqrt{\nu t}}$

$dy = \left(\frac{1}{\sqrt{4\nu t}}\right)^d dx$

$$= \left| \frac{1}{\pi^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-|y|^2} \overline{\varphi(2\sqrt{\nu t}\, y)}\, dy - \underbrace{\frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} e^{-|y|^2} dy}_{=1} \overline{\varphi(0)}\, dy \right|$$

$$\leq \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} e^{-|y|^2} \left| \overline{\varphi(2\sqrt{\nu t}\, y)} - \overline{\varphi(0)} \right| dy$$

$\downarrow$

0 po teoremu o dominiranoj konv.

c) $\langle T_n * f, \varphi\rangle = \langle T_n, \overline{\langle f, \underbrace{\varphi(x+\cdot)}_{\in \mathcal{S}} \rangle}\rangle \longrightarrow \langle T, \overline{\langle f, \varphi(x+\cdot)\rangle}\rangle$

$= \langle T * f, \varphi\rangle$ ✓

d) VJEŽBE:

$u(t,x) = (\Phi(t,\cdot) * u_0)(x)$

e) $\lim_{t \downarrow 0} (\Phi(t,\cdot) * u_0)(x) = (\delta_0 * u_0)(x) = u_0(x)$ ✓