PDJ 2

Druga zadaća

1.) Vidi predavanja [DISTRIBUCIJE I SLABE DER., KOROLAR 5, str. 14]

2.) a) $m \in \mathbb{N}$,

$x^m T = 0$.

Dokažimo matematičkom indukcijom da je rješenje dano s

$$T = \sum_{k=0}^{m-1} C_k \delta_0^{(k)}.$$

BAZA $(m=1)$

Na vježbama smo pokazali da je rješenje $xT=0$, $T = C\delta_0$, $C = $ konst.

Pretpostavimo da tvrdnja vrijedi za neki $m \in \mathbb{N}$.

$x^{m+1} T = 0$

$U := x^m T$

$xU = 0 \Leftrightarrow U = C\delta_0$, C ... konst. $(C \in \mathbb{C})$

Sad imamo:

$$x^m T = C\delta_0$$

Po pretpostavci indukcije znamo da su sva rješenja pripadne homogene j. dana s

$$T_H = \sum_{k=0}^{m-1} C_k \delta_0^{(k)}.$$

Moramo još pronaći jedno partikularno rješenje.

Pokušajmo s $T_P = D\delta_0^{(m)}$, $D \in \mathbb{C}$.

$$\langle x^m (D\delta_0^{(m)}), \varphi \rangle = D \langle \delta_0^{(m)}, x^m \varphi \rangle = (-1)^m D \langle \delta_0, (x^m \varphi)^{(m)} \rangle$$

$$= (-1)^m D \sum_{k=0}^{m} \binom{m}{k} \frac{m!}{(m-k)!} \underbrace{\langle \delta_0, x^{m-k} \varphi^{(m-k)} \rangle}_{=0 \text{ za } k<m}$$

$$= (-1)^m D\, m! \langle \delta_0, \varphi \rangle$$

$$= \langle (-1)^m D\, m!\, \delta_0, \varphi \rangle$$

$$\Rightarrow D = \frac{C}{(-1)^m m!} =: C_m$$

$$\Rightarrow T = T_H + T_P = \sum_{k=0}^{m} C_k \delta_0^{(k)}$$

2.) b)

$$(1-x^3)^2 T = 1$$

$$(1-x)^2 \underbrace{(1+x+x^2)^2 T}_{=:U} = 1$$

$$(x-1)^2 U = 1$$

$$(\tau_1 x)^2 U = 1$$

$$(\tau_1 x^2) U = 1$$

$$\tau_1 \left(x^2 (\tau_{-1} U) \right) = 1$$

$$x^2 (\tau_{-1} U) = 1$$

Po a) dijelu znamo da je rješenje pripadne homogene

$$\tau_{-1} U_H = C_0 \delta_0 + C_1 \delta_0'$$

$$U_H = C_0 \delta_1 + C_1 \delta_1'$$

Znamo da je $\mathrm{pv}\left(\frac{1}{x^2}\right)$ rješenje nehomogene:

$$\tau_{-1} U_P = \mathrm{pv}\left(\frac{1}{x^2}\right)$$

$$\Rightarrow U_P = \tau_1 \mathrm{pv}\left(\frac{1}{x^2}\right) = \mathrm{pv}\left(\frac{1}{(x-1)^2}\right)$$

$$\Rightarrow U = \mathrm{pv}\frac{1}{(x-1)^2} + C_0 \delta_1 + C_1 \delta_1', \quad C_0, C_1 \in \mathbb{C}$$

$$\Rightarrow (1+x+x^2)^2 T = \mathrm{pv}\left(\frac{1}{(x-1)^2}\right) + C_0 \delta_1 + C_1 \delta_1'$$

$$\Rightarrow T = \frac{1}{(1+x+x^2)^2} \mathrm{pv}\left(\frac{1}{(x-1)^2}\right) + \frac{C_0}{(1+x+x^2)^2} \delta_1 + \frac{C_1}{(1+x+x^2)^2} \delta_1',$$

$C_0, C_1 \in \mathbb{C}$

jer je $\frac{1}{(1+x+x^2)^2} \in C^\infty(\mathbb{R})$.

3.) a)

$$\Phi' = H'f + Hf' = \delta_0 f + Hf' = \underbrace{f(0)}_{=0}\delta_0 + Hf' = Hf'$$

$$\Phi'' = H'f' + Hf'' = \delta_0 f' + Hf'' = \underbrace{f'(0)}_{=0}\delta_0 + Hf'' = Hf''$$

$$\vdots$$

$$\Phi^{(k)} = Hf^{(k)}$$

$$\Phi^{(k+1)} = H'f^{(k)} + Hf^{(k+1)} = f^{(k)}\delta_0 + Hf^{(k+1)} = Hf^{(k+1)}$$

$$\vdots$$

$$\Phi^{(m-1)} = Hf^{(m-1)}$$

$$\Phi^{(m)} = f^{(m-1)}(0)\,\delta_0 + Hf^{(m)} = \frac{1}{a_m}\delta_0 + Hf^{(m)}$$

$$\Rightarrow P\Phi = H(\underbrace{Pf}_{=0}) + \delta_0 = \delta_0$$

gdje $H \neq 0$

b) $u''' + 2u'' - u' + 2u = f$

$$\begin{cases} v''' - 2v'' - v' + 2v = 0, & x > 0 \\ v(0) = v'(0) = 0 \\ v''(0) = 1 \end{cases}$$

$$\left.\begin{aligned} &\lambda^3 - 2\lambda^2 - \lambda + 2 = 0 \\ &(\lambda - 1)(\lambda^2 - \lambda - 2) = 0 \\ &(\lambda - 1)(\lambda + 1)(\lambda - 2) = 0 \\ &\lambda_1 = \cancel{1} - 1 \\ &\lambda_2 = 1 \\ &\lambda_3 = 2 \end{aligned}\right\} \Rightarrow$$

$$v(x) = C_1 e^{-x} + C_2 e^{x} + C_3 e^{2x}$$

$$0 = v(0) = C_1 + C_2 + C_3$$

$$0 = v'(0) = -C_1 + C_2 + 2C_3$$

$$1 = v''(0) = C_1 + C_2 + 4C_3$$

$$\Rightarrow 3C_3 = 1$$

$$C_3 = \frac{1}{3}$$

$$C_1 + C_2 = -\frac{1}{3}$$

$$-C_1 + C_2 = -\frac{2}{3} \quad \Big|+$$

$$2C_2 = -1 \Rightarrow C_2 = -\frac{1}{2}$$

$$C_1 = -\frac{1}{3} - C_2 = \frac{-2+3}{6} = \frac{1}{6}$$

$$\Rightarrow \boxed{v(x) = \frac{e^{-x}}{6} - \frac{e^{x}}{2} + \frac{e^{2x}}{3}}$$

Elementarno rješenje je $\Phi(x) = H(x)\, v(x)$.

$$\Rightarrow \quad u(x) = (\Phi * f)(x) = \int_{-\infty}^{+\infty} \Phi(x-y)\, f(y)\, dy = \int_{-\infty}^{x} v(x-y)\, f(y)\, dy$$

$$= \frac{1}{6}\int_{-\infty}^{x} e^{y-x} f(y)\,dy - \frac{1}{2}\int_{-\infty}^{x} e^{x-y} f(y)\,dy + \frac{1}{3}\int_{-\infty}^{x} e^{2x-2y} f(y)\,dy$$

$$= \frac{e^{-x}}{6}\int_{-\infty}^{x} e^{y} f(y)\,dy - \frac{e^{x}}{2}\int_{-\infty}^{x} e^{-y} f(y)\,dy + \frac{e^{2x}}{3}\int_{-\infty}^{x} e^{-2y} f(y)\,dy .$$

4.) a) $$\left(\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]} * \sin\right)(x) = \int_{-\infty}^{+\infty} \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(y)\, \sin(x-y)\, dy$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x-y)\, dy = \cos(x-y)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \cos\left(x-\frac{\pi}{2}\right) - \cos\left(x+\frac{\pi}{2}\right)$$

$$= \sin x + \sin x$$

$$= 2\sin x$$

b) $$\widehat{\chi}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi)\, \widehat{\sin}(\xi) = \widehat{\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]} * \sin}\,(\xi)$$

$$= 2\, \widehat{\sin}(\xi)$$

$$= \cancel{2}\, \frac{1}{\cancel{2}i}\left(\delta_{\frac{1}{2\pi}}(\xi) - \delta_{-\frac{1}{2\pi}}(\xi)\right)$$

$$= i\left(\delta_{-\frac{1}{2\pi}}(\xi) - \delta_{\frac{1}{2\pi}}(\xi)\right).$$

5.) $$\left(\frac{x^2}{(x^2+x+1)(x+1)} * e^{-x^2}\right)^\wedge(\xi) = \underbrace{\left(\frac{x^3}{(x^2+x+1)(x+1)}\right)^\wedge(\xi)}_{*} \underbrace{\left(e^{-x^2}\right)^\wedge(\xi)}_{**}$$

$*$: $$\frac{x^2}{(x^2+x+1)(x+1)} = \frac{-1}{x^2+x+1} + \frac{1}{x+1}$$

greška u zadaći

- $$\left(\frac{1}{x+1}\right)^\wedge(\xi) = \left(\tau_{-1}\frac{1}{x}\right)^\wedge(\xi) = e^{2\pi i \xi}(-i\pi \operatorname{sign}(\xi))$$

- $$\left(\frac{-1}{x^2+x+1}\right)^\wedge(\xi) = -\left(\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\right)^\wedge(\xi)$$

$$= -\left(\tau_{-\frac{1}{2}}\frac{1}{x^2+(\frac{\sqrt{3}}{2})^2}\right)^\wedge(\xi)$$

$$= -e^{\pi i \xi}\frac{\pi}{\frac{\sqrt{3}}{2}}e^{-\cancel{2}\pi\frac{\sqrt{3}}{\cancel{2}}|\xi|}$$

$$= -\frac{2\pi}{\sqrt{3}}e^{i\pi\xi}e^{-\pi\sqrt{3}|\xi|}$$

$**$: $$\left(e^{-x^2}\right)^\wedge(\xi) = \sqrt{\pi}\, e^{-\pi^2\xi^2}$$

$\Rightarrow$ Rezultat je : $$-\left(e^{2\pi i\xi}\, i\pi \operatorname{sign}(\xi) + \frac{2\pi}{\sqrt{3}}e^{i\pi\xi}e^{-\pi\sqrt{3}|\xi|}\right)\sqrt{\pi}\, e^{-\pi^2\xi^2}$$

6.) a) $\left|\left\langle Pf\,\frac{1}{|x|^2},\varphi\right\rangle\right| \leq \int\limits_{|x|<1} \frac{|\varphi(x)-\varphi(0)|}{|x|^2}\,dx + \int\limits_{|x|\geq 1}\frac{|\varphi(x)|}{|x|^2}\,dx$

<u>$\operatorname{supp}\varphi \subseteq K$</u>

(▲)

$$\leq \int\limits_{|x|<1}\frac{|x|\,|\nabla\varphi(y_x)|}{|x|^2}\,dx + \|\varphi\|_{L^\infty}\underbrace{\int\limits_{\{|x|\geq 1\}\cap \operatorname{supp}\varphi}\frac{dx}{|x|^2}}$$

$$\leq \|\nabla\varphi\|_{L^\infty}\int\limits_{|x|<1}\frac{dx}{|x|} + C_K\|\varphi\|_{L^\infty}$$ konstanta koja ovisi o kompaktu

$$= 2\pi\|\nabla\varphi\|_{L^\infty} + C_K\|\varphi\|_{L^\infty}$$

$$\leq \max\{2\pi, C_K\}\max_{|\alpha|\leq 1}\|\partial^\alpha\varphi\|_{L^\infty}$$

$$\int\limits_{|x|<1}\frac{dx}{|x|} = \int_0^1 \frac{1}{r}\,2\pi r\,dr = 2\pi$$

$\Rightarrow$ $Pf\,\frac{1}{|x|^2}$ je distribucija reda 1.

$$\left\langle \widetilde{Pf\,\frac{1}{|x|^2}},\varphi\right\rangle = \left\langle Pf\,\frac{1}{|x|^2},\check{\varphi}\right\rangle = \int\limits_{|x|<1}\frac{\bar{\varphi}(-x)-\bar{\varphi}(0)}{|x|^2}\,dx + \int\limits_{|x|\geq 1}\frac{\bar{\varphi}(-x)}{|x|^2}\,dx$$

$$= \left\{\begin{matrix} y=-x \\ dy = dx\end{matrix}\right\}$$

$$= \int\limits_{|y|<1}\frac{\bar{\varphi}(y)-\bar{\varphi}(0)}{\underset{\substack{\parallel \\ |y|^2}}{|-y|^2}}\,dy + \int\limits_{|y|\geq 1}\frac{\bar{\varphi}(y)}{\underset{\substack{\parallel \\ |y|^2}}{|-y|^2}}\,dy$$

$$= \left\langle Pf\,\frac{1}{|x|^2},\varphi\right\rangle \checkmark$$

$\Rightarrow$ $Pf\,\frac{1}{|x|^2}$ je parna distribucija

Da dokažemo da je $Pf\,\frac{1}{|x|^2}$ temperirana distribucija moramo (▲) malo drugačije ocijeniti jer nemamo kompaktan nosač. Međutim, za $\varphi\in\mathcal{S}(\mathbb{R}^2)$ znamo da je $|x|\varphi$ omeđena f-ja pa imamo:

$$\int\limits_{|x|\geq 1}\frac{|\,|x|\varphi(x)|}{|x|^3}\,dx \leq \underbrace{\|\,|x|\varphi\|_{L^\infty}}_{\text{polunorma u } \mathcal{S}}\int_1^\infty \frac{1}{r^3}\,2\pi r\,dr = \|\,|x|\varphi\|_{L^\infty}\underbrace{2\pi\left(-\frac{1}{r}\right)\Big|_1^\infty}_{=2\pi} = 2\pi\|\,|x|\varphi\|_{L^\infty} \checkmark$$

6.) b) $\langle |x|^2 \mathrm{Pf} \frac{1}{|x|^2}, \varphi \rangle = \langle \mathrm{Pf} \frac{1}{|x|^2}, |x|^2 \varphi \rangle$

greška u zadaći

$$= \int_{|x|<1} \frac{|x|^2 \bar{\varphi}(x) - (|x|^2 \varphi)\big|_{x=0}}{|x|^2} dx + \int_{|x| \geq 1} \frac{|x|^2 \bar{\varphi}(x)}{|x|^2} dx$$

$$= \int_{|x|<1} \frac{|x|^2 \bar{\varphi}(x)}{|x|^2} dx + \int_{|x| \geq 1} \bar{\varphi}(x)\, dx$$

$$= \int_{\mathbb{R}^2} \bar{\varphi}(x)\, dx$$

$$= \langle 1, \varphi \rangle .$$

7.) $f : \mathbb{R}^3 \to \mathbb{R}, \quad f(\mathbf{x}) = \dfrac{1}{|\mathbf{x}|^2}$.

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-2\pi i \mathbf{x}\cdot\xi} f(\mathbf{x})\, d\mathbf{x}$$

Prelazimo na sferične koordinate, međutim kako je f-ja f radijalna možemo BSO pretpostaviti da je $\xi = (0, 0, |\xi|)$, tj. nalazi se na z-osi, a to možemo jer je Jacobijan rotacija 1, a kako je f radijalna samo bi se vidjele promjene u $\mathbf{x}\cdot\xi$ što i želimo pojednostaviti.

$$\begin{aligned} x_1 &= r\sin\theta\cos\varphi \\ x_2 &= r\sin\theta\sin\varphi \\ x_3 &= r\cos\theta \end{aligned}, \quad \mathbf{x} = (x_1, x_2, x_3), \quad \begin{aligned} \theta &\in [0, \pi] \\ \varphi &\in [0, 2\pi] \end{aligned}$$

$$\Rightarrow \int_{|\mathbf{x}|<R} \frac{e^{-2\pi i \mathbf{x}\cdot\xi}}{|\mathbf{x}|^2}\, d\mathbf{x} = \int_0^R\int_0^\pi\int_0^{2\pi} \frac{e^{-2\pi i |\xi| r\cos\theta}}{\cancel{r^2}} \overbrace{\cancel{r^2}\sin\theta}^{\text{Jacobijan}}\, d\varphi\, d\theta\, dr$$

$$= 2\pi \int_0^R\int_0^\pi e^{-2\pi i |\xi| r\cos\theta} \sin\theta\, d\theta\, dr$$

$$= \left\{ \begin{aligned} s &= \cos\theta \\ ds &= -\sin\theta\, d\theta \\ \theta = 0 &\to s = 1 \\ \theta = \pi &\to s = -1 \end{aligned} \right\} = 2\pi \int_0^R \int_{-1}^1 e^{-2\pi i |\xi| r s}\, ds\, dr$$

$$= 2\pi \int_0^R \left. \frac{e^{-2\pi i |\xi| r s}}{-2\pi i |\xi| r} \right|_{-1}^{1} dr$$

$$= \frac{i}{|\xi|} \int_0^R \left(e^{-2\pi i |\xi| r} - e^{2\pi i |\xi| r} \right) \frac{1}{r}\, dr$$

$$= \frac{2}{|\xi|} \int_0^R \frac{\sin(2\pi|\xi| r)}{r}\, dr = \left\{ \begin{aligned} t &= 2\pi|\xi| r \\ dt &= 2\pi|\xi|\, dr \\ r &= \frac{t}{2\pi|\xi|} \end{aligned} \right\}$$

$$= \frac{2}{|\xi|} \int_0^R \frac{\sin(t)}{\frac{t}{\cancel{2\pi|\xi|}}} \frac{dt}{\cancel{2\pi|\xi|}}$$

$$= \frac{2}{|\xi|} \int_0^R \frac{\sin t}{t}\, dt \xrightarrow{R\to\infty} \frac{2}{|\xi|}\cdot\frac{\pi}{2} = \frac{\pi}{|\xi|}$$

$$\boxed{\hat{f}(\xi) = \frac{\pi}{|\xi|}}$$

8.) a) $f(x) = \frac{\sin^2 x}{x^2}$

$g(x) := \frac{\sin x}{x} \Rightarrow f(x) = g(x) \cdot g(x) \quad /\,\hat{}$

$\hat{f}(\xi) = \hat{g}(\xi) * \hat{g}(\xi)$

Znamo iz ~~tablice~~ tablice:

$\hat{g}(\xi) = \pi \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}(\xi)$

$(\hat{g} * \hat{g})(\xi) = \int_{-\infty}^{+\infty} \hat{g}(\xi - \eta)\, \hat{g}(\eta)\, d\eta = \pi^2 \int_{-\frac{1}{2\pi}}^{\frac{1}{2\pi}} \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}(\xi - \eta)\, d\eta$

$$= \pi^2 \cdot \begin{cases} \frac{1}{\pi} + \xi, & -\frac{1}{\pi} < \xi \leq 0 \\ \frac{1}{\pi} - \xi, & 0 < \xi < \frac{1}{\pi} \\ 0, & \text{inače} \end{cases}$$

b) $\begin{cases} u_t = 4u_{xx} \\ u(0,x) = \frac{\sin^2 x}{x^2} \end{cases}$,

$u_t = 4u_{xx} \quad /\,\hat{}$

$\hat{u}_t = -16\pi^2 \xi^2 \hat{u}$

$\Rightarrow \hat{u}(t,\xi) = C(\xi)\, e^{-16\pi^2 \xi^2 t}$

$\hat{u}(0,\xi) = C(\xi) = \left(\frac{\sin^2 x}{x^2}\right)^{\wedge}(\xi) = \hat{f}(\xi)$ ... a) dio zadatka

$\Rightarrow u(t,x) = \bar{\mathcal{F}}\left(e^{-16\pi^2\xi^2 t} \hat{f}(\xi)\right)(x)$ ~~$= \bar{\mathcal{F}}(e^{-16\pi^2\xi^2 t} \hat{f}(\xi))(x)$~~

jer su obe parne f-je

$= \bar{\mathcal{F}}\left(e^{-16\pi^2\xi^2 t} \hat{f}(\xi)\right)(x) = \int_{-\frac{1}{\pi}}^{\frac{1}{\pi}} e^{-2\pi i x \xi} e^{-16\pi^2 \xi^2 t} \hat{f}(\xi)\, d\xi$

$= \left(e^{-16\pi^2\xi^2 t}\right)^{\vee}(x) * f(x)$

$= \sqrt{\frac{\pi}{16\pi^2 t}}\, e^{-\frac{x^2}{16t}} * f(x)$

$= \sqrt{\frac{\pi}{16\pi^2 t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{16t}} f(y)\, dy$

TREBA SE POZVATI NA VJEŽBE GDJE SMO POKAZALI DA JE $\frac{1}{\sqrt{16\pi t}} e^{-\frac{x^2}{16t}}$ ~~[illegible]~~ ELEM. RJ.

9.) $\partial_{tt}\hat{u} + 4\pi^2\xi^2\hat{u} = 0 \quad \Rightarrow \quad \hat{u}(t,\xi) = C_1(\xi)\sin(2\pi\xi t) + C_2(\xi)\cos(2\pi\xi t)$

$\hat{u}(0,\xi) = \frac{1}{1+\pi^2\xi^2} \quad \Rightarrow \quad C_2(\xi) = \frac{1}{1+\pi^2\xi^2}$

$\partial_t\hat{u}(0,\xi) = 0 \quad \Rightarrow \quad 2\pi\xi\, C_1(\xi) = 0 \Rightarrow \boxed{C_1(\xi) \equiv 0}$

$$\Rightarrow \boxed{\hat{u}(t,\xi) = \frac{\cos(2\pi\xi t)}{1+\pi^2\xi^2}}$$

- $\left(\cos(2\pi\xi t)\right)^{\wedge}(x) = \frac{1}{2}\left(\delta_t(x) + \delta_{-t}(x)\right)$

$\Rightarrow \left(\cos(2\pi\xi t)\right)^{\vee}(x) = \frac{1}{2}\left(\delta_t(x) + \delta_{-t}(x)\right)$ ... jer je cos parna

- $\frac{1}{1+\pi^2\xi^2} = \frac{1}{\pi^2}\,\frac{1}{\xi^2 + \left(\frac{1}{\pi}\right)^2}$

$\Rightarrow \left(\frac{1}{1+\pi^2\xi^2}\right)^{\wedge}(x) = \frac{1}{\pi^2}\,\pi^2 e^{-2\pi\frac{1}{\pi}|x|} = e^{-2|x|}$

$\Rightarrow \left(\frac{1}{1+\pi^2\xi^2}\right)^{\vee}(x) = e^{-2|x|}$ ... parna

$$\begin{aligned}
\Rightarrow u(t,x) &= \frac{1}{2}\left(\delta_t(x) + \delta_{-t}(x)\right) * e^{-2|x|} \\
&= \frac{1}{2}\left(\delta_t * e^{-2|x|} + \delta_{-t} * e^{-2|x|}\right) \\
&= \frac{1}{2}\left(\tau_t e^{-2|x|} + \tau_{-t} e^{-2|x|}\right) \\
&= \frac{1}{2} e^{-2|x-t|} + \frac{1}{2} e^{-2|x+t|}
\end{aligned}$$

$$\boxed{\begin{aligned} &\delta_0 * f = f \\ &\delta_a * f = \tau_a f \end{aligned}}$$

Ovdje smo koristili:

~~cos(2πtξ)~~

$\left(\cos(2\pi t\xi)\,\frac{1}{1+\pi^2\xi^2}\right)^{\vee}(x) = \left(\cos(2\pi t\cdot)\right)^{\vee}(x) * \left(\frac{1}{1+\pi^2\xi^2}\right)^{\vee}(x)$

Međutim, to vrijedi ako je $\left(\cos(2\pi t\cdot)\right)^{\vee} \in \mathcal{S}'$ što je istina i $\left(\frac{1}{1+\pi^2\xi^2}\right)^{\vee} \in \mathcal{S}$ što ipak nije istina jer $e^{-2|x|}$ nije derivabilna!

Zato je račun formalan i rezultat nije točan (u ② dobivamo problem)

$\boxed{t=0}$ $\quad u(0,x) = \frac{e^{-2|x|} + e^{-2|x|}}{2} = e^{-2|x|}$

① $\boxed{x \leq -t}$

$u(t,x) = \frac{e^{2(x-t)} + e^{2(x+t)}}{2} = e^{2x}\,\frac{e^{-2t} + e^{2t}}{2} = e^{2x}\,\mathrm{ch}(2t)$

$u_{tt} = 4e^{2x}\,\mathrm{ch}(2t)$ ✓

$u_{xx} = 4e^{2x}\,\mathrm{ch}(2t)$

$u_t(t,x) = 2e^{2x}\,\mathrm{sh}(2t) \Rightarrow u_t(0,x) = 0$ ✓

② $\boxed{-t < x < t}$

$$u(t,x) = \frac{e^{2(x-t)} + e^{-2(x+t)}}{2} = e^{-2t}\, ch(2x)$$

$$u_{tt}(t,x) = 4 e^{-2t} ch(2x)$$

$$u_{xx}(t,x) = 4 e^{-2t} ch(2x) \quad \| \checkmark$$

$$u_t(t,x) = -2 e^{-2t} ch(2x) \quad , \quad u_t(0,x) \neq 0$$

razlog: $e^{-2|x|}$ se "čudno ponaša oko" nule

③ $\boxed{x \geq t}$

$$u(t,x) = \frac{e^{-2(x-t)} + e^{-2(x+t)}}{2} = e^{-2x} \frac{e^{2t} + e^{-2t}}{2} = e^{-2x} ch(2t)$$

$$u_t(t,x) = 2 e^{-2x} sh(2t) \Rightarrow u_t(0,x) = 0$$

$$u_{tt}(t,x) = 4 e^{-2x} ch(2t)$$

$$u_{xx}(t,x) = 4 e^{-2x} ch(2t) \quad \| \checkmark$$

Mogli smo napisati rješenje na ovaj način:

$$u(t,x) = \int_{-\infty}^{+\infty} e^{2\pi i x \xi} \underbrace{\cos(2\pi \xi t) \frac{1}{1+\pi^2 \xi^2}}_{\text{za fiksan } t \text{ je u } L^1 \text{ pa je } u(t,\cdot) \text{ iz } C_0.} d\xi$$

OVO JE KOREKTNO I LAKO SE PROVJERI DA JE ZADOVOLJENA JEDNADŽBA, KAO I POČ. UVJETI.