POPRAVNI

1.) $$f(x) = \begin{cases} x & , \quad x < 0 \\ 2x-1 & , \quad 0 \leq x < 1 \\ -x & , \quad x \geq 1 \end{cases}$$

a) $$\langle f', \varphi \rangle = -\langle f, \varphi' \rangle = -\int_{-\infty}^{0} x\,\overline{\varphi'(x)}\,dx - \int_{0}^{1} (2x-1)\,\overline{\varphi'(x)}\,dx + \int_{1}^{+\infty} x\,\overline{\varphi'(x)}\,dx$$

$$= \underbrace{-x\,\overline{\varphi(x)}\Big|_{-\infty}^{0}}_{=0} + \int_{-\infty}^{0} \overline{\varphi(x)}\,dx - (2x-1)\overline{\varphi(x)}\Big|_{0}^{1} + \int_{0}^{1} 2\,\overline{\varphi(x)}\,dx +$$

$$+ x\,\overline{\varphi(x)}\Big|_{1}^{+\infty} - \int_{1}^{\infty} \overline{\varphi(x)}\,dx$$

$$= \int_{-\infty}^{0} \overline{\varphi(x)}\,dx + \int_{0}^{1} 2\,\overline{\varphi(x)}\,dx + \int_{1}^{\infty} (-1)\,\overline{\varphi(x)}\,dx - \overline{\varphi(1)} - \overline{\varphi(0)} - \overline{\varphi(1)}$$

$$= \int_{-\infty}^{+\infty} g\,\overline{\varphi(x)}\,dx + \langle -2\delta_1, \varphi \rangle + \langle -\delta_0, \varphi \rangle$$

gdje je $$g(x) = \begin{cases} 1 & , \quad x < 0 \\ 2 & , \quad 0 \leq x < 1 \\ -1 & , \quad x \geq 1 \end{cases}$$

b) $$\langle f'', \varphi \rangle = -\langle f', \varphi' \rangle = -\int_{-\infty}^{0} \overline{\varphi'(x)}\,dx - \int_{0}^{1} 2\,\overline{\varphi'(x)}\,dx + \int_{1}^{+\infty} \overline{\varphi'(x)}\,dx$$

$$- \langle -2\delta_1, \varphi' \rangle - \langle -\delta_0, \varphi' \rangle$$

$$= - \overline{\varphi(x)}\Big|_{-\infty}^{0} - 2\,\overline{\varphi(x)}\Big|_{0}^{1} + \overline{\varphi(x)}\Big|_{1}^{+\infty}$$

$$+ \langle -2\delta_1', \varphi \rangle + \langle -\delta_0', \varphi \rangle$$

$$= -\overline{\varphi(0)} - 2\,\overline{\varphi(1)} + 2\,\overline{\varphi(0)} - \overline{\varphi(1)} + \langle -2\delta_1', \varphi \rangle + \langle -\delta_0', \varphi \rangle$$

$$= \langle \delta_0 - 3\delta_1 - 2\delta_1' - \delta_0', \varphi \rangle$$

b) $f'' + 2\delta_1' + \delta_0' = \delta_0 - 3\delta_1$

Rastavljamo:

$xT_1 = \delta_0$

$T_1^H = C\delta_0, \ C \in \mathbb{C}$

Pretp. $T_1^P = D\delta_0'$

$$\langle x(D\delta_0'), \varphi\rangle = D\langle \delta_0', x\varphi\rangle$$
$$= -D\langle \delta_0, \varphi + x\varphi'\rangle$$
$$= -D\overline{\varphi(0)}$$
$$= \langle -D\delta_0, \varphi\rangle$$

$\Rightarrow -D = 1 \Rightarrow \boxed{D = -1}$

$$\boxed{T_1 = -\delta_0' + C\delta_0, \ C \in \mathbb{C}}$$

$xT_2 = -3\delta_1$

$T_2^H = C\delta_0, \ C \in \mathbb{C}$

$T_2^P = D\delta_1$ ... pretp.

$$\langle x(D\delta_1), \varphi\rangle = D\langle \delta_1, x\varphi\rangle$$
$$= D\overline{\varphi(1)}$$
$$= \langle D\delta_1, \varphi\rangle$$

$\Rightarrow \boxed{D = -3}$

$$\Rightarrow \boxed{T_2 = -3\delta_1 + C\delta_0, \ C \in \mathbb{C}}$$

$$\Rightarrow \boxed{T = -\delta_0' - 3\delta_1 + C\delta_0, \ C \in \mathbb{C}}$$

2)

a) $f(x,y) = \begin{cases} 1 & , \quad x \in [a_1, a_2],\ y \in [b_1, b_2] \\ 0 & , \quad \text{inače} \end{cases}$

$$\hat{f}(\xi,\eta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2\pi i(\xi x + \eta y)} f(x,y)\, dx\, dy$$

$$= \int_{b_1}^{b_2}\int_{a_1}^{a_2} e^{-2\pi i \xi x} e^{-2\pi i \eta y}\, dx\, dy$$

$$= \left(\int_{b_1}^{b_2} e^{-2\pi i \eta y}\, dy\right)\left(\int_{a_1}^{a_2} e^{-2\pi i \xi x}\, dx\right)$$

$$= \frac{e^{-2\pi i \eta y}}{-2\pi i \eta}\Bigg|_{b_1}^{b_2} \frac{e^{-2\pi i \xi x}}{-2\pi i \xi}\Bigg|_{a_1}^{a_2} \quad , \quad \eta \neq 0,\ \xi \neq 0$$

$$\cancel{= \frac{1}{-2\pi i \eta}\left(e^{-2\pi i \eta b_2} - 1\right) \frac{1}{-2\pi i \xi}}$$

$$= \frac{\sin(\pi(b_2 - b_1)\eta)}{\pi\eta} e^{-i\pi(b_1 + b_2)\eta} \frac{\sin(\pi(a_2 - a_1)\xi)}{\pi\xi} e^{-i\pi(a_1 + a_2)\xi}$$

$a_1 = -a_2$, $a_2 = a$

$b_1 = -b_2$, $b_2 = b$

$$\boxed{= \frac{\sin(2\pi b\eta)}{\pi\eta} \frac{\sin(2\pi a\xi)}{\pi\xi}}$$

b) $b = \frac{1}{2\pi}$, $a = \frac{1}{2\pi}$

jer je f jin parna pa je $\wedge = \vee$

$$\left(\frac{\sin(x)\sin(y)}{\pi^2 xy}\right)^{\wedge}(\xi,\eta) = \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]^2}(\xi,\eta)$$

$$\Rightarrow \left(\frac{\sin x \sin y}{xy}\right)^{\wedge}(\xi,\eta) = \pi^2 \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]^2}(\xi,\eta).$$

$$f(x)=\begin{cases} x, & 0\le x\le 1\\ 0, & \text{inače}\end{cases}$$

$$\hat{f}(\xi)=\int_0^1 e^{-2\pi i x\xi}\,x\,dx=\frac{e^{-2\pi i x\xi}}{-2\pi i\xi}\,x\,\Big|_0^1+\frac{1}{2\pi i\xi}\int_0^1 e^{-2\pi i x\xi}\,dx$$

$$=\frac{e^{-2\pi i\xi}}{-2\pi i\xi}+\frac{1}{2\pi i\xi}\,\frac{e^{-2\pi i x\xi}}{-2\pi i\xi}\Big|_0^1$$

$$=\frac{e^{-2\pi i\xi}}{-2\pi i\xi}+\frac{1}{4\pi^2\xi^2}\left(e^{-2\pi i\xi}-1\right)$$

$$=e^{-2\pi i\xi}\,\frac{i}{2\pi\xi}\left(1-\frac{i}{2\pi\xi}\right)-\frac{1}{4\pi^2\xi^2}$$

2.) c) $\left(\frac{\sin(mx)}{x}\right)^{\wedge}(\xi) = \pi \chi_{[-\frac{m}{2\pi}, \frac{m}{2\pi}]}(\xi)$

$\pi \chi_{[-\frac{m}{2\pi}, \frac{m}{2\pi}]} \xrightarrow{\mathcal{S}'} \pi$ (po Leb. izr. o dom. konv.)

$\Rightarrow$ (ker je F. pretvorba zvezna na $\mathcal{S}'$)

$\frac{\sin(mx)}{x} \xrightarrow{\mathcal{S}'} (\pi)^{\vee}(x) = \pi \delta_0(x)$.

3.) a) $u_0 = \dfrac{x\cos x - \sin x}{x^2} = \left(\dfrac{\sin x}{x}\right)' = (\operatorname{sinc}(x))'$

$$(\operatorname{sinc})^{\wedge}(\xi) = \pi\, \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}(\xi)$$

$$\hat{u}_0(\xi) = \widehat{\operatorname{sinc}'}(\xi) = (2\pi i \xi)\, \widehat{\operatorname{sinc}}(\xi) = 2\pi^2 i \xi\, \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}(\xi)$$

b) $\begin{cases} u_t - \Delta u = 0 \\ u(0,\cdot) = u_0 \end{cases}$

$\Rightarrow \begin{cases} \hat{u}_t + 4\pi^2\xi^2 \hat{u} = 0 \\ \hat{u}(0,\cdot) = \hat{u}_0 \end{cases}$

$\Rightarrow \hat{u}(t,\xi) = C e^{-4\pi^2\xi^2 t} \qquad (C = \hat{u}_0)$

$\Rightarrow \hat{u}(t,\xi) = 2\pi^2 i \xi\, \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}(\xi)\, e^{-4\pi^2\xi^2 t}$

$\Rightarrow u(t,x) = \mathcal{F}\left(-2\pi^2 i \xi\, \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}\, e^{-4\pi^2\xi^2 t}\right)(x)$

neparna f-ja.

4.) a) $\begin{cases} \partial_t u_m - \frac{i}{m^2} \Delta u_m = 0 \\ u_m(0,\cdot) = \sin(mx) =: u_0(x) \end{cases}$

$\vdots$

$\Rightarrow \quad u_m(t,x) = \left(e^{-4\pi^2 \frac{1}{m^2} it \xi^2} \hat{u}_0(\xi)\right)^{\vee}(x)$

$$\boxed{(\sin(mx))^{\wedge}(\xi) = \frac{1}{2} i \left(\delta_{-\frac{m}{2\pi}}(\xi) - \delta_{\frac{m}{2\pi}}(\xi)\right)}$$

$$\Rightarrow \quad e^{-4\pi^2 \frac{1}{m^2} it\xi^2} \hat{u}_0(\xi) = \frac{i}{2} e^{-4\pi^2 \frac{1}{m^2} it\xi^2} \left(\delta_{-\frac{m}{2\pi}}(\xi) - \delta_{\frac{m}{2\pi}}(\xi)\right)$$

$$= \frac{i}{2} e^{-4\pi^2 \frac{1}{m^2} it \frac{m^2}{4\pi^2}} \left(\delta_{-\frac{m}{2\pi}}(\xi) - \delta_{\frac{m}{2\pi}}(\xi)\right)$$

$$= \frac{i}{2} e^{-it} \left(\delta_{-\frac{m}{2\pi}}(\xi) - \delta_{\frac{m}{2\pi}}(\xi)\right)$$

$$\boxed{\hat{\delta}_a = \widehat{\tau_a \delta_0} = e^{-2\pi i a \xi} \hat{\delta}_0 = e^{-2\pi i a\xi}}$$

$$\Rightarrow \quad \tilde{u}_m(t,x) = \frac{i}{2} e^{-it} \left(e^{imx} - e^{-imx}\right)$$

$$= \frac{i}{2} e^{-it} \, 2i \sin(mx)$$

$$= - e^{-it} \sin(mx)$$

$$\Rightarrow \quad \boxed{u_m(t,x) = e^{-it} \sin(mx)}$$

b) $\quad u_m \xrightarrow{\mathcal{S}'} 0 \quad$ jen $\quad \sin(mx) \longrightarrow 0$.