PDJ 1 (vježbe 2015/2016)

LINEARNA JEDNADŽBA 1. REDA S KONSTANTNIM KOEFICIJENTIMA

$$(1) \begin{cases} u_t + c u_x = 0 & \text{na } \mathbb{R}^+ \times \mathbb{R}, \ c \in \mathbb{R} \\ u(0,\cdot) = g \in C^1(\mathbb{R}) \end{cases}$$

TEOREM 1. (1) ima jedinstveno rješenje $u \in C^1(\mathbb{R}^2; \mathbb{C})$ koje je dano formulom $u(t,x) = g(x-ct)$.

Dz.

Iz jednadžbe imamo

$$\underbrace{\begin{bmatrix} u_t \\ u_x \end{bmatrix}}_{\nabla u} \cdot \begin{bmatrix} 1 \\ c \end{bmatrix} = 0 \iff u \text{ je konstanta duž pravaca}$$

čiji je vektor smjera $\begin{bmatrix} 1 \\ c \end{bmatrix}$, odnosno duž pravaca

$$t = \frac{1}{c}x + \text{const} \Rightarrow \underbrace{x - ct = \text{const}}_{\text{pravci ovog oblika prekrivaju cijeli prostor!}}.$$

Neka je $(t_0, x_0) \in \mathbb{R}^2$ proizvoljna. Ta točka leži na pravcu $x - ct = D$, pri čemu $D = x_0 - ct_0$.

Budući da je u nužno konstantno na tom pravcu, imamo

$$u(t_0,x_0) = \underbrace{u(0,D)}_{(0,D) \text{ leži na istom pravcu}} \underset{(1_2)}{=} g(D) = g(x_0 - ct_0).$$

Iz toga smo dobili da je jedini kandidat za rješenje upravo $u(t,x) = g(x-ct)$.

PROVJERA: • $g \in C^1 \Rightarrow u \in C^1$

• $\left.\begin{array}{l} \partial_t u(t,x) = g'(x-ct)(-c) \\ \partial_x u(t,x) = g'(x-ct) \end{array}\right\} \Rightarrow u_t + c u_x = 0$ ✓

• $u(0,x) = g(x)$ ✓

ZAD 1. (nehomogena jed. ; ZAD 5 iz skripte) Pokazati da za $c \in \mathbb{R}$ i $f \in C^1(\mathbb{R}^2)$ postoji točno jedno rješenje početne zadaće

$$\begin{cases} u_t + c u_x = f \\ u(0,\cdot) = g \end{cases}.$$

Naći formulu za rješenje. Primjerom pokazati da u ne mora biti iz $C^2(\mathbb{R}^2)$.

Rj. JEDINSTVENOST (standardni postupak kod linearnih jednadžbi)

u_1, u_2 neka su dva rješenja.

Tada $u := u_1 - u_2$ zadovoljava

$$\begin{cases} u_t + c u_x = 0 \\ u(0,\cdot) = 0 \end{cases}$$

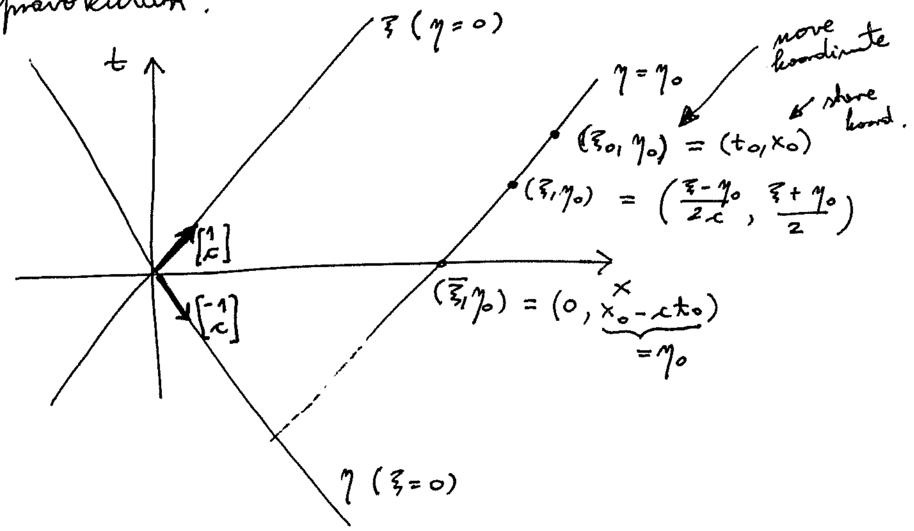
$$\overset{T-1}{\Longrightarrow} \quad u \equiv 0 \Rightarrow \boxed{u_1 = u_2}$$

POSTOJANJE Pretpostavimo da je $c \neq 0$.

Želimo uvesti zamjenu varijabli t.d. PDJ $u_t + c u_x$ zamijenimo s pripadnom ODJ.

$$\left.\begin{aligned} \xi &= x + ct \\ \eta &= x - ct \end{aligned}\right\} \Rightarrow \begin{aligned} x &= \frac{\xi + \eta}{2} \\ t &= \frac{\xi - \eta}{2c} \end{aligned}$$

Time smo dobili novi koordinatni sustav ($\eta = 0$ definira ξ-os, a $\xi = 0$ definira η-os) koji naravno nije nužno ~~koordinat~~ pravokutan.



Želimo odrediti vrijednost f-je u u proizvoljnoj točki (t_0, x_0). Ta točka u novom koordinatnom sustavu ima koordinate $(\xi_0, \eta_0) = (x_0 + ct_0, x_0 - ct_0)$.

Definiramo funkcije u novim varijablama:

$$v(\xi, \eta) := u(t, x)$$
$$\tilde{f}(\xi, \eta) := f(t, x).$$

Pogledajmo koju jednadžbu zadovoljava v.

$$\partial_t u = \partial_\xi v \frac{d\xi}{dt} + \partial_\eta v \frac{\partial \eta}{dt} = c\,\partial_\xi v - c\,\partial_\eta v$$

$$\partial_x u = \partial_\xi v \frac{d\xi}{dx} + \partial_\eta v \frac{d\eta}{dx} = \partial_\xi v + \partial_\eta v$$

$$\Rightarrow \quad \partial_t u + c\,u_x = 2c\,\partial_\xi v$$

$$\Rightarrow \quad 2c\,\partial_\xi v(\xi,\eta) = \tilde{f}(\xi,\eta)$$

$$\Rightarrow \quad \boxed{\partial_\xi v(\xi,\eta) = \frac{1}{2c}\tilde{f}(\xi,\eta)} \quad \text{ODJ prvog reda}$$

Za čvrsti $\eta_0 \in \mathbb{R}$, integriramo po ξ od $\bar{\xi}$ do ξ_0 (integriramo na segmentu $[(\bar{\xi},\eta_0),(\xi_0,\eta_0)]$).

$$v(\xi_0,\eta_0) = \frac{1}{2c}\int_{\bar{\xi}}^{\xi_0} \tilde{f}(\xi,\eta_0)\,d\xi + v(\bar{\xi},\eta_0)$$

Još je preostalo vratiti sve u (t,x) koordinatni sustav.

- $v(\xi_0,\eta_0) = u(t_0,x_0)$
- $v(\bar{\xi},\eta_0) = u(0, x_0 - ct_0)$
- točka (ξ,η_0) u starim koordinatama je: $t = \frac{\xi-\eta_0}{2c}$, $x = \frac{\xi+\eta_0}{2}$

$$\Rightarrow \quad \tilde{f}(\xi,\eta_0) = f\left(\frac{\xi-\eta_0}{2c}, \frac{\xi+\eta_0}{2}\right)$$

Sada imamo

$$u(t_0,x_0) = \frac{1}{2c}\int_{\bar{\xi}}^{\xi_0} f\left(\frac{\xi-\eta_0}{2c}, \frac{\xi+\eta_0}{2}\right)d\xi + \underbrace{u(0,x_0-ct_0)}_{=\,g(x_0-ct_0)}$$

$$\left[\begin{array}{l} t = \frac{\xi-\eta_0}{2c} \Rightarrow dt = \frac{1}{2c}\,d\xi \\ \bullet\ \xi = \bar{\xi} \Rightarrow t = \frac{\bar{\xi}-\eta_0}{2c} = 0 \\ \bullet\ \xi = \xi_0 \Rightarrow t = \frac{\xi_0-\eta_0}{2c} = t_0 \end{array}\right] \rightsquigarrow \frac{\xi+\eta_0}{2} = \frac{2ct+\eta_0+\eta_0}{2} = ct+\eta_0 = ct + x_0 - ct_0$$

$$u(t_0,x_0) = \int_0^{t_0} f(t, ct + x_0 - ct_0)\,dt + g(x_0 - ct_0)$$

Lako se provjeri da je gornjom formulom uistinu dano rj.
Nadalje, lako se provjeri da je gornjom formulom također dano rj. za slučaj $\boxed{c=0}$

Pokažimo primjerom da u ne mora biti iz $C^2(\mathbb{R}^2)$.

Npr. $f \equiv 0$ i $\underbrace{g \in C^1(\mathbb{R}), \text{ ali } g \notin C^2(\mathbb{R})}$

$$g(x) = \begin{cases} -\frac{x^2}{2}, & x<0 \\ \frac{x^2}{2}, & x \geq 0 \end{cases} \quad \Rightarrow \quad g'(x) = |x| \notin C^1(\mathbb{R})$$

$\Rightarrow$ $u(t,x) = g(x-ct)$ nije iz $C^2(\mathbb{R}^2)$ jer je u u točkama na pravcu $t = \frac{1}{c}x$ samo klase C^1.

ZAD. 2. Pokazati da za $\vec{c} \in \mathbb{R}^d$, $g \in C^1(\mathbb{R}^d)$ i $f \in C^1(\mathbb{R}^{1+d})$ postoji točno jedno rješenje početne zadaće:

$$\begin{cases} u_t + \vec{c} \cdot \nabla u = f \\ u(0, \cdot) = g \end{cases},$$

te naći formulu za rješenje.

(Poopćenje ZAD 1)

Rj: ~~Pretpostavimo da je~~

Provjerimo da je

$$u(t, x) := g(x - \vec{c}t) + \int_0^t f(s, x - \vec{c}t + \vec{c}s)\, ds$$

rješenje.

$$g : \mathbb{R}^2 \to \mathbb{R}^{1+d}$$
$$g(t,s) := (s, x - \vec{c}t + \vec{c}s)$$
$$f : \mathbb{R}^{1+d} \to \mathbb{R}$$
$$f \circ g : \mathbb{R}^2 \to \mathbb{R},$$
$$\cancel{\nabla(f \circ g)(t,s) = \nabla f(g(t,s)) \nabla g(t,s)}$$
$$\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_1}, \dots\right)$$
$$\partial_t (f \circ g)(t,s) = \sum_{i=0}^{d} \partial_i f(g(t,s))\, \partial_t g_i(t,s)$$

$$\begin{aligned}
&\partial_t u(t,x) = -\vec{c} \cdot \nabla g(x - \vec{c}t) + f(t,x) - \int_0^t \vec{c} \cdot \nabla f(s, x - \vec{c}t + \vec{c}s)\, ds \\
&\partial_{x_i} u(t,x) = \partial_{x_i} g(x - \vec{c}t) + \int_0^t \partial_{x_i} f(s, x - \vec{c}t + \vec{c}s)\, ds \\
&\Rightarrow \quad \vec{c} \cdot \nabla u(t,x) = \vec{c} \cdot \nabla g(x - ct) + \int_0^t \vec{c} \cdot \nabla f(s, x - \vec{c}t + \vec{c}s)\, ds \\
&\Rightarrow \quad \partial_t u + \vec{c} \nabla u = f \checkmark
\end{aligned}$$

$$u(0, x) = g(x) \checkmark$$

Dovoljno je provjeriti da je rješenje zadaće

$$\begin{cases} u_t + \vec{c} \cdot \nabla u = 0 \\ u(0, \cdot) = 0 \end{cases}$$

jedinstveno.

$$u_t + \vec{c} \cdot \nabla u = 0 \Rightarrow \begin{bmatrix} u_t \\ \nabla u \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \vec{c} \end{bmatrix} = 0 \Rightarrow$$ rješenje je konstantno duž pravaca smjera $\begin{bmatrix} 1 \\ \vec{c} \end{bmatrix}$.

Za proizvoljne $(t_0, x_0) \in \mathbb{R}^{1+d}$ postoji konstanta $\vec{\alpha}$ t.d. (t_0, x_0) leži na pravcu $x - \vec{c}t = \vec{\alpha}$ $(\vec{\alpha} := x_0 - \vec{c}t_0)$. Budući da je rješenje duž pravca $x - \vec{c}t = \vec{\alpha}$ konstantno, to imamo

$$u(t_0, x_0) = u(0, \vec{\alpha}) = g(\vec{\alpha}) = 0 \Rightarrow u \equiv 0.$$

ZAD. 3. Za nelinearnu početnu zadaću ($c\in\mathbb{R}$)

$$\begin{cases} u_t + c u_x + u^2 = 0 \\ u(0,\cdot) = g \end{cases} ,\qquad \text{(polulinearna j.)}$$

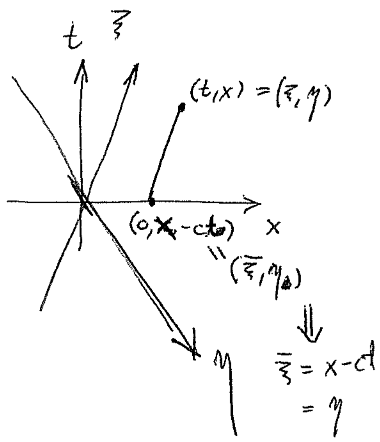
pokazati da za funkciju $g\in C_c^\infty(\mathbb{R})$, koja nije identički jednaka nuli, postoji lokalno rješenje $u \in C^\infty(\langle -\delta,\delta\rangle \times \mathbb{R})$, ali da se to rješenje ne može proširiti do C^∞ rješenja na čitavoj ravnini.
Usporediti pojavu s nelinearnom običnom diferencijalnom jednadžbom $u' = u^2$.

Rj: ~~Uzmimo~~ Neka je $c\neq 0$.

$$\begin{cases} \xi := x + ct \\ \eta := x - ct \end{cases} \Longrightarrow \quad x = \frac{\xi+\eta}{2}, \quad t = \frac{\xi-\eta}{2c}$$

$$\partial_t = \frac{d\xi}{dt}\partial_\xi + \frac{d\eta}{dt}\partial_\eta = c\,\partial_\xi - c\,\partial_\eta$$

$$\partial_x = \partial_\xi + \partial_\eta$$



$v(\xi,\eta) := u(t,x)$

$$0 = u_t + c u_x + u^2 = (c\partial_\xi - c\partial_\eta)v + c(\partial_\xi + \partial_\eta)v + v^2$$
$$= c\partial_\xi - \cancel{c\partial_\eta v} + c\partial_\xi v + \cancel{c\partial_\eta v} + v^2$$
$$= 2c\,\partial_\xi v + v^2$$

$$\Longrightarrow \boxed{\partial_\xi v + \frac{1}{2c} v^2 = 0}$$ ← jednadžba ima samo derivaciju po ξ pa je zapravo riječ o ODJ (η je samo parametar)

$$\Longrightarrow v(\xi,\eta) = -\frac{1}{-\frac{1}{2c}\xi + D(\eta)}$$ OVO JE ZA SLUČAJ DA JE $v\neq 0$, TJ. TAMO GDJE JE $v(\eta,\eta) = g(\eta) \neq 0$

Trebamo odrediti $D(\eta)$.

$$v(\xi,\eta) \overset{\xi=\eta}{=} v(\eta,\eta) = u(0,x-ct) = g(x-ct) = g(\eta)$$ AKO JE $g(\eta)=0$, ONDA JE $v(\xi,\eta)=0$

$$u' = au^2$$
$$\int\frac{du}{u^2} = \int a\,dt$$
$$-\frac{1}{u} = at + b$$
$$\Rightarrow u = -\frac{1}{at+b}$$
OVO JE ZA SLUČAJ DA JE $u\neq 0$

bitno je da u nazivniku nemamo nulu

$$\Longrightarrow v(\xi,\eta) = v(\eta,\eta) = -\frac{1}{-\frac{1}{2c}\eta + D(\eta)}$$
/ OVO JE RJ. U SLUČAJU $g(\eta) = g(x-ct) \neq 0$ /

može se i sve izračunati dalje u koordinatama x i t

$$\Longrightarrow -\frac{1}{g(\eta)} = -\frac{1}{2c}\eta + D(\eta) \Rightarrow D(\eta) = \frac{1}{2c}\eta - \frac{1}{g(\eta)}$$

$$\Longrightarrow v(\xi,\eta) = \frac{1}{\frac{1}{g(\eta)} + \frac{1}{2c}(\xi-\eta)}$$

U OVOM OBLIKU JE OBUHVAĆENO I RJ. $u=0$ KAD JE $g=0$ (odnosno $g(x-ct)=0$)

$$\Longrightarrow u(t,x) = \frac{1}{\frac{1}{g(x-ct)} + t} \Longrightarrow \boxed{u(t,x) = \frac{g(x-ct)}{1+t\,g(x-ct)}}$$

SAMO TREBAMO OSIGURATI DA NAZIVNIK NIKAD NIJE NULA, TJ. DA POSTOJI NEKA KONSTANTA $\varepsilon > 0$ T.D.

$$\underset{\text{iz odgovarajućih skupova}}{(\forall x, t)} \qquad |1 + t g(x - ct)| > \varepsilon$$

$$|1 + t g(x-ct)| \geq 1 - |t| \overbrace{|g(x-ct)|}^{\leq M \text{ (jer je } g \text{ s kompaktnim nosačem pa je omeđena)}} \geq \underbrace{1 - |t| M > \varepsilon}_{}$$

$$\Rightarrow |t| < \frac{1-\varepsilon}{M} =: \delta \quad \text{(uz naravno } \varepsilon < 1\text{)}$$

$\Rightarrow \quad u \in C^\infty(\langle -\delta, \delta \rangle \times \mathbb{R})$ je rješenje.

Postoji li mogućnost da u proširimo na cijelu ravninu $\mathbb{R} \times \mathbb{R}$?

NE! Naime, $g \not\equiv 0$ pa $\exists x_0$ t.d. $g(x_0) \neq 0$. Tada za $t = -\frac{1}{g(x_0)}$ izaberemo x t.d. $x - ct = x_0$ $\left(x = x_0 - \frac{c}{g(x_0)}\right)$.

$$1 + t g(x - ct) = 1 - \frac{1}{g(x_0)} g(x_0) = 0,$$

pa uočavamo da u točki $(t, x) = \left(-\frac{1}{g(x_0)}, x_0 - \frac{c}{g(x_0)}\right)$ imamo singularitet.

$\boxed{DZ}$ Provjerite da je $u(t,x) = \dfrac{g(x-ct)}{1 + t g(x-ct)}$ uistinu rješenje početne ~~jednadžbe~~ zadaće.

KVAZILINEARNE JEDNADŽBE 1. REDA

– METODA KARAKTERISTIKA –

Promatramo kvazilinearnu jednadžbu 1. reda

$$\vec{a}(x, u(x)) \cdot \nabla u(x) = b(x, u(x)).$$

Ova metoda daje <u>lokalno</u> rješenje i primjenjiva je isključivo na jednadžbe 1. reda.

<u>TEOREM</u>. Neka je S hiperploha klase C^1 u $\mathbb{R}^d$. Ako

(i) $\vec{a}$, b i u_0 klase C^1 na okolini S, i

(ii) polje $\vec{a}$ nije ni u jednoj točki tangencijalno na S, tj.

$\vec{a}(x, u_0(x)) \cdot \nu(x) \neq 0$, $x \in S$, pri čemu je $\nu(x)$ normala na S u točki x.

tada Cauchyjeva zadaća

$$\begin{cases} \vec{a}(x, u(x)) \cdot \nabla u(x) = b(x, u(x)) \\ u|_S = u_0 \end{cases},$$

ima <u>lokalno</u> rješenje na okolini S.

↳ $\exists U \subseteq \mathbb{R}^d$ t.d. $S \subseteq U$ & $\exists u \in C^1(U)$ t.d. zadovoljava gornju j. na U.

<u>MOGUĆI PROBLEMI</u> (za dobivanje globalnog rj.)

- sijeku se karakteristike ($\Rightarrow$ ne znamo koju vrijednost uzeti)
- karakteristike ne prekrivaju cijelu domenu
- karakteristične točke $\vec{a}(x, u_0(x)) \cdot \nu(x) = 0$, $x \in S$.

Kod linearnih jednadžbi karakteristike se <u>ne sijeku</u> i u većini slučajeva prekrivaju cijeli prostor.

ZAD. 4. $\begin{cases} \partial_1 u + \partial_2 u = u \\ u = \cos x_1 \quad \text{na} \quad x_2 = 0 \end{cases}$

Rj:

$$\vec{a}(x_1, x_2, z) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad b(x_1, x_2, z) = z$$

$$\Rightarrow \quad \vec{a}(x_1, x_2, u(x_1,x_2)) \cdot \nabla u(x_1,x_2) = b(x_1,x_2,z)$$

$$S \dots \{(x_1, 0) : x_1 \in \mathbb{R}\}$$

Pogledajmo imamo li karakterističnih točaka:

$$\nu(x_1, 0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dots \text{ normala na } S \text{ u točki } (x_1, 0)$$

$$\vec{a}(x_1, 0, \cos x_1) \cdot \nu(x_1, 0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \neq 0$$

$\Rightarrow$ nema karakterističnih točaka

$$\left.\begin{aligned} \frac{dx_1}{d\tau} &= a_1(x_1,x_2,z) = 1 \Rightarrow x_1(\tau) = \tau + C_1 \\ \frac{dx_2}{d\tau} &= a_2(x_1,x_2,z) = 1 \Rightarrow x_2(\tau) = \tau + C_2 \end{aligned}\right\} \text{ projicirane karakteristike}$$

$$\frac{dz}{d\tau} = b(x_1,x_2,z) = z \Rightarrow \int \frac{dz}{z} = \int d\tau \Rightarrow \ln|z| = \tau + C \Rightarrow z(\tau) = C_3 e^{\tau}$$

rješenje parametrizirano projiciranim karak.

C_1, C_2, C_3 odredimo t.d. tražimo $(x_1(0), x_2(0)) \in S$.

$$\begin{aligned} x_1(0) = x_1^0 &\Rightarrow C_1 = x_1^0 \\ x_2(0) = 0 &\Rightarrow C_2 = 0 \end{aligned} \Rightarrow \boxed{\begin{aligned} x_1(\tau) &= \tau + x_1^0 \\ x_2(\tau) &= \tau \end{aligned}}$$

projicirane karakteristike ovise o parametru x_1^0

Za $\tau = 0$ se nalazimo u točki $(x_1^0, 0)$ pa $z(0)$ mora biti jednako $\cos x_1^0$.

$$z(0) = C_3 = \cos x_1^0 \Rightarrow \boxed{z(\tau) = \cos x_1^0 \, e^{\tau}}$$

Neka je $(x_1, x_2) \in \mathbb{R}^2$ proizvoljna točka. Odredimo τ i x_1^0 t.d.

$$\begin{cases} \tau + x_1^0 = x_1 \\ \tau = x_2 \end{cases}$$

tražimo projicirane karakteristike koje prolaze točkom (x_1, x_2).

$$\Rightarrow \boxed{\tau = x_2} \quad \boxed{x_1^0 = x_1 - x_2}$$

Sada imamo:

$$\boxed{u(x_1,x_2) = z(\tau) = \cos x_1^\circ e^{\tau} = \cos(x_1-x_2)\, e^{x_2}}$$

<u>NAPOMENA</u>. Ostavimo jednadžbu, ali promijenimo S na kojem je rješenje zadano.

$$\begin{cases} \partial_1 u + \partial_2 u = u \\ u = u_0 \quad \text{na} \quad x_1 = x_2 \end{cases}$$

$\vdots$

$$x_1(\tau) = \tau + C_1$$
$$x_2(\tau) = \tau + C_2$$

$$\vec{x}(0) = (x_1(0), x_2(0)) = (x_1^\circ, x_1^\circ) \Rightarrow \boxed{\begin{aligned} x_1(\tau) &= \tau + x_1^\circ \\ x_2(\tau) &= \tau + x_1^\circ \end{aligned}}$$ karakteristike

Neka je $(x_1,x_2) \in \mathbb{R}^2$ proizvoljan. Odredimo τ i x_1° t.d.

$$\left.\begin{aligned} x_1 &= \tau + x_1^\circ \\ x_2 &= \tau + x_1^\circ \end{aligned}\right\} \Rightarrow x_1 - x_2 = 0 \Rightarrow \underline{\underline{x_1 = x_2}}$$

$\Rightarrow \tau$ i x_1° postoje samo u slučaju $\underline{\underline{x_1 = x_2}}$

Time smo dobili da projicirane karakteristike $(x_1(\tau), x_2(\tau))$ ne prekrivaju cijeli prostor!

RAZLOG: $\quad S \ldots \{(x,x) : x \in S\} \Rightarrow \nu(x,x) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\vec{a}(x_1,x_1, u(x_1,x_1)) \cdot \nu(x_1,x_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$\Rightarrow$ sve točke krivulje (pravca) S su karakteristične.

<u>NAPOMENA</u>. Zadatke 1, 2, 3 možete riješiti i ovom metodom.

ZAD. 5. $\begin{cases} x_1^2\, \partial_1 u + x_2^2\, \partial_2 u = u^2 \\ u = 1 \quad \text{na} \quad x_2 = 2x_1 \end{cases}$

Rj.

$$\vec{a}(x, z) = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}, \quad b(x, z) = z^2$$

$$S = \{(x, 2x) : x \in \mathbb{R}\} \Rightarrow \vec{\nu}(x, 2x) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

karakteristične točke:

$$\vec{a}(x, 2x, 1) \cdot \vec{\nu}(x, 2x) = \begin{bmatrix} x^2 \\ 4x^2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2x^2 - 4x^2 = -2x^2$$

$\Rightarrow (0,0)$ je jedina karakteristična točka

To je i trivijalni slučaj jer vektorsko polje u ishodištu iščezava.

karakteristike:

$$\frac{dx_1}{d\tau} = x_1^2 \Rightarrow x_1(\tau) = \frac{1}{c_1 - \tau}$$

$$\frac{dx_2}{d\tau} = x_2^2 \Rightarrow x_2(\tau) = \frac{1}{c_2 - \tau}$$

$$\frac{dz}{d\tau} = z^2 \Rightarrow z(\tau) = \frac{1}{c_3 - \tau}$$

ovo su rješenja uz uvjet da funkcija nije trivijalna, odnosno da u početnom trenutku f-je nisu trivijalne

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$(x_1(0), x_2(0)) = (x_0, 2x_0) \in S \quad (x_0 \neq 0)$

$$\Rightarrow \left.\begin{aligned} x_1(0) &= \frac{1}{c_1} = x_0 \Rightarrow c_1 = \frac{1}{x_0} \\ x_2(0) &= \frac{1}{c_2} = 2x_0 \Rightarrow c_2 = \frac{1}{2x_0} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x_1(\tau) &= \frac{1}{\frac{1}{x_0} - \tau} = \frac{x_0}{1 - \tau x_0} \\ x_2(\tau) &= \frac{1}{\frac{1}{2x_0} - \tau} = \frac{2x_0}{1 - 2\tau x_0} \end{aligned}}$$

ovdje sad ima smisla $x_0 = 0$ i tada dobijemo trivijalne f-je što odgovara rj. koje bismo dobili da smo računali za $x_0 = 0$ pa onda gdje karakteristike imaju smisla za $x_0 \in \mathbb{R}$ s tim da za $x_0 = 0$ dobivamo stacionarnu karakteristiku $(x_1(\tau), x_2(\tau)) = (0,0)$

Također imamo $\tau \in \langle -\infty, \frac{1}{2x_0} \rangle$ za $x_0 > 0$ i $\tau \in \langle \frac{1}{2x_0}, +\infty \rangle$ za $x_0 < 0$.

$$z(0) = u(x_1(0), x_2(0)) = u(\underbrace{x_0, 2x_0}_{\in S}) = 1$$

$$\Rightarrow \frac{1}{c_3} = 1 \Rightarrow c_3 = 1$$

$$\Rightarrow \boxed{z(\tau) = \frac{1}{1 - \tau}}$$

Neka je $(x_1, x_2) \in \mathbb{R}^2$ proizvoljna točka. Pokušajmo odrediti τ i x_0 t.d. $(x_1(\tau; x_0), x_2(\tau; x_0)) = (x_1, x_2)$.

$$\frac{x_0}{1-\tau x_0} = x_1 \quad \Big/ \cdot (1-\tau x_0) \quad \Rightarrow \quad x_1(1-\tau x_0) = x_0$$

$$x_1 - \tau x_1 x_0 = x_0$$

$$\boxed{\tau = \frac{x_1 - x_0}{x_1 x_0}}$$

$$\frac{2x_0}{1-2\tau x_0} = x_2$$

Očito račun vrijedi za $x_1 \neq 0$. Ako je $x_1 = 0$, tada je nužno $x_0 = 0$ pa onda i $x_2 = 0$. Iz toga slijedi da na ordinati jedino ishodište leži na nekoj karakteristici (i to trivijalnoj). Analogno za $x_2 = 0$.

$$2x_0 = x_2\left(1 - \frac{2(x_1 - x_0)}{x_1 \cancel{x_0}}\cancel{x_0}\right)$$

$$2x_0 = x_2 - 2\frac{x_2}{x_1}(x_1 - x_0)$$

$$2x_0 = x_2 - 2x_2 + 2\frac{x_2}{x_1}x_0$$

$$2\left(1 - \frac{x_2}{x_1}\right)x_0 = -x_2$$

$$x_0 = \frac{-x_2}{2 - 2\frac{x_2}{x_1}} = \boxed{\frac{x_1 x_2}{2x_2 - 2x_1}}$$

Očito račun vrijedi za $x_1 \neq x_2$, a lako se pokaže da jedino ishodište leži na nekoj karakt. s pravca $x_1 = x_2$.

$$\Rightarrow \tau = \frac{x_1 - \frac{x_1 x_2}{2x_2 - 2x_1}}{x_1 \frac{x_1 x_2}{2x_2 - 2x_1}} \begin{array}{l} / \cdot 2x_2 - 2x_1 \\ / \cdot 2x_2 - 2x_1 \end{array} = \frac{2x_1x_2 - 2x_1^2 - x_1x_2}{x_1^2 x_2} \begin{array}{l} / : x_1 \\ / : x_1 \end{array} = \frac{x_2 - 2x_1}{x_1 x_2}$$

$$\Rightarrow \boxed{u(x_1, x_2) = z(\tau) = \frac{1}{1 - \frac{x_2 - 2x_1}{x_1 x_2}} = \frac{x_1 x_2}{x_1 x_2 - x_2 + 2x_1}}$$

lako se provjeri da u zadovoljava jednadžbu za sve točke za koje je u dobro def., a to su $\{(x_1, x_2) : x_1x_2 - x_2 + 2x_1 \neq 0\}$.

Iz gornjeg računa smo vidjeli da ne postoje karakteristike kroz točke $(x,x), (0,x), (x,0)$, $x \in \mathbb{R} \setminus \{0\}$. Međutim, u tim točkama je ipak rješenje dobro definirano, a za to je najviše zaslužna (konstantna) početna f-ja pa se rješenje moglo glatko proširiti.

Iz gornje formule vidimo da je rješenje jednako 0 na koordinatnim osima, dok je u ishodištu 1 (po početnom uvjetu) pa rješenje nije proširivo po neprekinutosti na nekoj okolini ishodišta.

Presjek krivulje $x_1x_2 - x_2 + 2x_1 = 0$ i $x_2 = 2x_1$ je samo ishodište pa rezultat nije u kontradikciji s TEOREMOM jer je ishodište karakteristična točka, a ~~na~~ u svim ostalim točkama pravca $x_2 = 2x_1$ postoji okolina gdje je rješenje u dobro def.

- - - - -

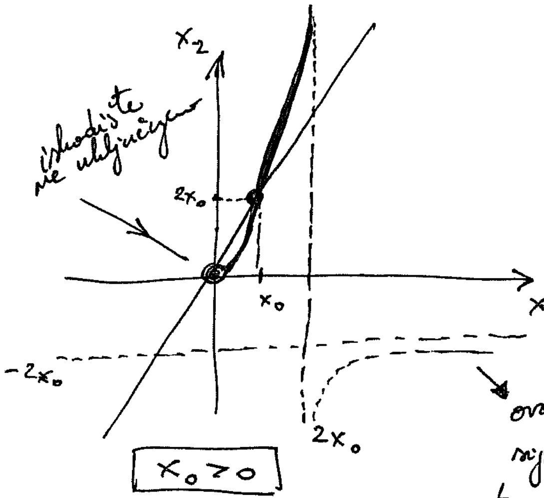
Skicirajmo neke karakteristike.

Crtati parametarski zadanu krivulju je nezgodno pa ćemo izraziti x_2 preko x_1.

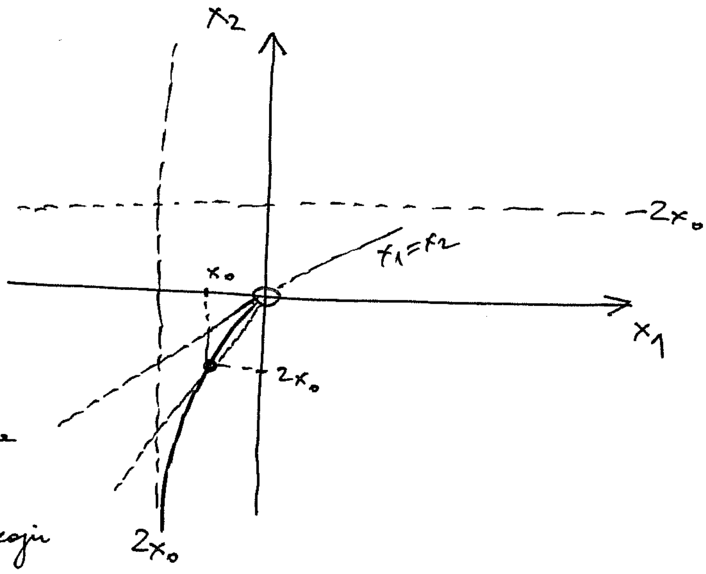
Ranije smo izračunali $\tau = \frac{x_1 - x_0}{x_1 x_0}$ pa uvrstimo to u $x_2(\tau)$:

$$x_2 = \frac{2x_0}{1 - 2\frac{x_1 - x_0}{x_1 x_0} x_0} = \frac{2x_0 x_1}{x_1 - 2(x_1 - x_0)}$$

$$= \frac{2x_0 x_1}{2x_0 - \cancel{4}x_1} \qquad (x_1 = x_0 \Rightarrow x_2 = 2x_0 \checkmark)$$



ova grana ne siječe $x_2 = 2x_1$ pa to ne gledamo (to je grana za koju je $\tau > \frac{1}{2x_0}$)

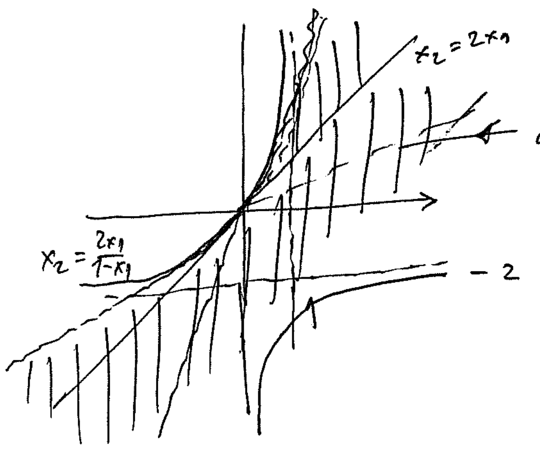


- - - - - -

$$x_1 x_2 - x_2 + 2x_1 = 0$$

$$x_2(x_1 - 1) = -2x_1$$

$$x_2 = \frac{2x_1}{1 - x_1}$$ ← tu nemamo rješenje



dio pravca $x_2 = 2x_1$ gdje imamo rj.

ZAD. 6. $\begin{cases} u_y = x u u_x \\ u(x,0) = x \end{cases}$

Rj:

$$\vec{a}(x,y,z) = \begin{bmatrix} xz \\ -1 \end{bmatrix}, \quad b(x,y,z) = 0$$

$$S = \{(x,0) : x \in \mathbb{R}\} \Rightarrow \vec{\nu}(x,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{a}(x,0,x) \cdot \vec{\nu}(x,0) = \begin{bmatrix} x^2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 \neq 0$$ (nema karak. točaka)

$$\frac{dx}{d\tau} = xz \Rightarrow \frac{dx}{d\tau} = C_3 x \Rightarrow x(\tau) = C_1 e^{C_3 \tau}$$

$$\frac{dy}{d\tau} = -1 \Rightarrow y(\tau) = -\tau + C_2$$

$$\frac{dz}{d\tau} = 0 \Rightarrow z(\tau) = C_3$$

$$(x(0), y(0)) = (x_0, 0) \in S \Rightarrow \left.\begin{matrix} C_1 = x_0 \\ C_2 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x(\tau) = x_0 e^{C_3 \tau} = x_0 e^{x_0 \tau} \\ y(\tau) = -\tau \end{matrix}$$

$$z(0) = u(x(0), y(0)) = u(x_0, 0) = x_0 \Rightarrow C_3 = x_0 \Rightarrow z(\tau) = x_0$$

$$\boxed{\tau = -y}, \ \boxed{x_0 = u} \Rightarrow \boxed{x = u(x,y)\, e^{-u(x,y)\, y}}$$

ZAD. 7. $\begin{cases} x u_y - y u_x = u \\ u(\cdot, 0) = u_0 \end{cases}$

Rj:

$$\vec{a}(x,y,z) = \begin{bmatrix} -y \\ x \end{bmatrix}, \quad b(x,y,z) = z$$

$$S = \{(x,0) : x \in \mathbb{R}\} \Rightarrow \vec{\nu}(x,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

karakteristične točke:

$$\vec{a}(x,0,u_0(x)) \cdot \vec{\nu}(x,0) = \begin{bmatrix} 0 \\ x \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x$$

$\Rightarrow$ ishodište je jedina karakteristična točka

karakteristike:

$$\frac{dx}{d\tau} = -y \;\Big/\; \frac{d}{d\tau} \Rightarrow \frac{d^2x}{d\tau^2} = -\frac{dy}{d\tau} = -x$$

$$\frac{dy}{d\tau} = x$$

$$\frac{dz}{d\tau} = z \Rightarrow z(\tau) = Ce^{\tau}$$

$$\ddot{x} + x = 0 \Rightarrow x(\tau) = A \sin\tau + B\cos\tau$$

periodičke f-je pa ćemo poslije odlučiti koji ćemo uzeti interval za τ

$$(\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i) \qquad y(\tau) = -\frac{dx}{d\tau}(\tau) = +B\sin\tau - A\cos\tau$$

$$(x(0), y(0)) = (x_0, 0) \Rightarrow \left.\begin{matrix} B = x_0 \\ A = 0 \end{matrix}\right\} \Rightarrow \boxed{\left.\begin{matrix} x(\tau) = x_0\cos\tau \\ y(\tau) = +x_0\sin\tau \end{matrix}\right\}}$$

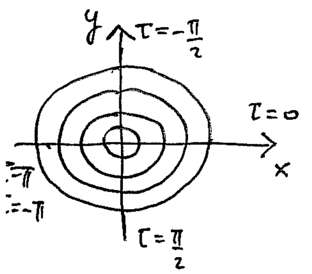
$\to$ kružnice radijusa $|x_0|$

$$z(0) = u_0(x_0) \Rightarrow C = u_0(x_0) \Rightarrow \boxed{z(\tau) = u_0(x_0)e^{\tau}}$$

$x(\tau) = x_0\cos\tau$

$y(\tau) = +x_0\sin\tau$ za $\tau = 0$ je $(x(0), y(0)) = (x_0, 0) \in S$

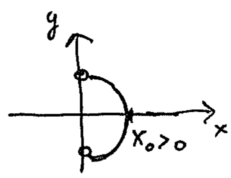
za $\tau = \pi$ je $(x(\pi), y(\pi)) = (-x_0, 0) \in S$



karakteristike sijeku S u dvije točke pa ćemo imati problem koju vrijednost odabrati za vrijednost na karakteristici ($u_0(x_0)e^{\tau}$ ili $u_0(-x_0)e^{\tau}$)

↳ razdvojit ćemo karakteristike na dva dijela

$$\boxed{\begin{matrix} x(\tau) = x_0\cos\tau \\ y(\tau) = +x_0\sin\tau \end{matrix}, \quad \tau \in \langle -\tfrac{\pi}{2}, \tfrac{\pi}{2} \rangle}$$



↳ time ćemo imati glatko rj. za $x > 0$ i $x < 0$, dok na ordinati imamo singularitet

↳ druga je grana tada dana karakteristikom s parametrom $-x_0$

$(x,y) \in \mathbb{R}^2$, $\quad x = x_0 \cos\tau \;|^2$, $\quad y = +x_0 \sin\tau \;|^2$ $\quad \Rightarrow \quad x^2 + y^2 = x_0^2$

$\Rightarrow |x_0| = \sqrt{x^2+y^2}$

↳ predznak od x_0 određujemo po predznaku od x, tj.

$\operatorname{sign} x_0 = \operatorname{sign} x$

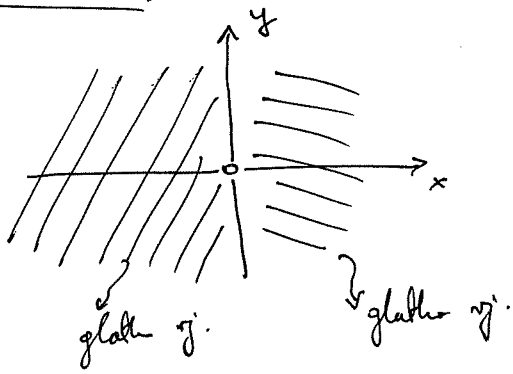
$\Rightarrow x_0 = (\operatorname{sign} x)\sqrt{x^2+y^2}$

$\Rightarrow \operatorname{tg}\tau = \frac{y}{x}$ $\quad$ ($\tau \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$ pa je $\operatorname{arctg}(\operatorname{tg}\tau) = \tau$)

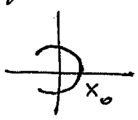
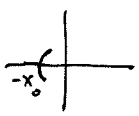
$\Rightarrow \tau = \operatorname{arctg}(+\frac{y}{x})$

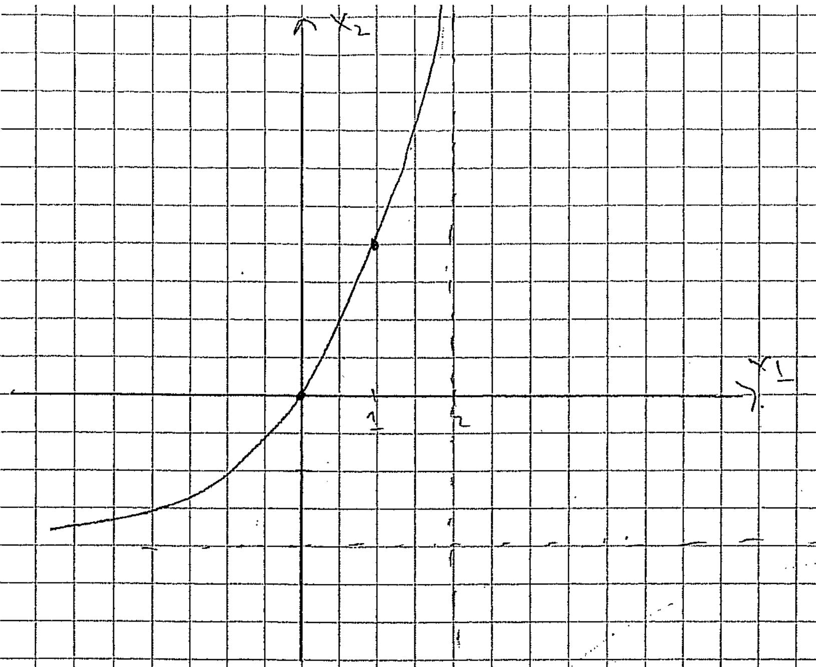
$\Rightarrow \tau = \operatorname{arctg}(\frac{y}{x})$

$$\Rightarrow \boxed{u(x,y) = u_0\left(\operatorname{sign}(x)\sqrt{x^2+y^2}\right) e^{\operatorname{arctg}\left(\frac{y}{x}\right)}}$$



NAP. Mogli smo i drugačije izabrati karakteristike, npr.

 i  pa bi tada singulariteti bili na drugim polupravcima.



$$\frac{2x_1}{2-x_1} = \frac{-2(2-x_1)+4}{2-x_1} = -2 + \frac{4}{2-x_1}$$

SKALARNI ZAKONI SAČUVANJA

$$(*)\begin{cases} u_t + F(u)_x = 0 & u \ \langle 0,+\infty\rangle \times \mathbb{R} \\ u(0,\cdot) = g \end{cases} \qquad F(u)_x = F'(u)\cdot u_x$$

Ovo je kvazilinearna jednadžba. Rješavat ćemo metodom karakteristika, metodom kojoj se mogu pojaviti svi od prije navedeni problemi. (osim karakterističnih točaka: $\begin{bmatrix} 1 \\ F'(g(x_0)) \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \neq 0$)

$$\frac{dt}{d\tau} = 1 \Rightarrow t(\tau) = \tau + C_1$$

$$\frac{dx}{d\tau} = F'(y) \Rightarrow \frac{dx}{d\tau} = F'(C_2) \Rightarrow x(\tau) = F'(C_2)\tau + C_3$$

ne ovisi o τ

$$\frac{dz}{d\tau} = 0 \Rightarrow z(\tau) = C_2$$

$(0,x_0)$ ... točka s pravca na kojem je zadano rješenje.

$$t(0) = 0 \Rightarrow C_1 = 0 \Rightarrow t(\tau) = \tau$$

$$\underline{x(0) = x_0} \Rightarrow C_3 = x_0 \Rightarrow x(t) = F'(C_2)t + x_0 = F'(g(x_0))t + x_0$$

$$z(0) = g(x_0) \Rightarrow z(t) = g(x_0) = C_2$$

Primjer 1. Rješavamo (*) uz $F(u) = \frac{1}{2}u^2$ ($F'(u) = u$)

$\Rightarrow$ Burgersova jednadžba

$$g(x) = \begin{cases} 1, & x \leq 0 & (1) \\ 1-x, & x \in \langle 0,1\rangle & (2) \\ 0, & x \geq 1 & (3) \end{cases}$$

Pogledajmo kako glavne karakteristike po područjima

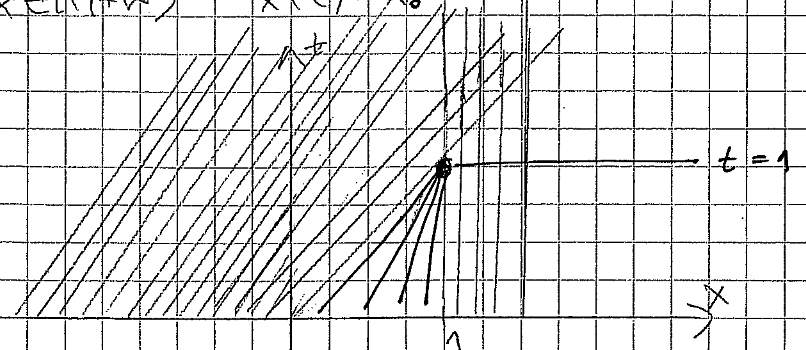
(1) $x_0 \in \langle -\infty, 0]$ $\rightarrow x(t) = g(x_0)\,t + x_0$

$x(t) = t + x_0$

(2) $x \in \langle 0, 1 \rangle$

$x(t) = (1 - x_0)\,t + x_0 =$

(3) $x \in [1, +\infty\rangle$ $\quad x(t) = x_0$



Odredimo rj. do vremena $t = 1$ (područje na kojem se ne sijeku karakteristike).

$$u(t,x) = \begin{cases} 1, & x \le \cancel{0}\, t \\ \dfrac{1-x}{1-t}, & 0 < x < \cancel{x}\, t \\ 0, & x \ge 1 \end{cases}$$

5. vj.

Moramo proširiti pojam rješenja u kojem ćemo dopustiti ne samo neglatko rj., već i prekidno rješenje.

Uvodimo pojam slabog rješenja.

$v \in C_c^\infty(\mathbb{R}_0^+ \times \mathbb{R})$ ... proizvoljna test fja

Pomnožimo s v jednadžbu i parcijalno integriramo (uz pp da je u glatko rj.)

preskočiti izvod (pozvati se na predavanja)

$$0 = \int_{-\infty}^{\infty}\int_0^{\infty} \left(u_t + F(u)_x\right) v \, dt\, dx = \int_{-\infty}^{\infty}\int_0^{\infty} u_t v \, dt\, dx + \int_0^{\infty}\int_{-\infty}^{\infty} F(u)_x v \, dx\, dt$$

$$= -\int_{-\infty}^{\infty}\int_{0}^{\infty} u v_t \,dt\,dx - \int_{-\infty}^{\infty} uv\big|_{t=0}\,dx - \int_{0}^{\infty}\int_{-\infty}^{\infty} F(u) v_x \,dx\,dt$$

$$= -\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(u v_t + F(u) v_x\right) dt\,dx - \int_{-\infty}^{\infty} uv\big|_{t=0}\,dx$$

Ako je u glatko rješenje dane dif. jednadžbe onda zadovoljava i integralnu relaciju za svaku probnu fju v. Međutim, ako u zadovoljava samo integralnu relaciju ne mora biti glatka jer se uopće ne pojavljuju derivacije od u. Time smo definirali slabo rj.

• $u \in L^\infty(\mathbb{R}_0^+ \times \mathbb{R})$ t.d.

$$\forall v \in C_c^\infty(\mathbb{R}_0^+ \times \mathbb{R}) \quad \int_{-\infty}^{\infty}\int_{0}^{\infty} u v_t + F(u) v_x \,dt\,dx + \int_{-\infty}^{\infty} uv\big|_{t=0}\,dx = 0$$

ZOVEMO SLABO RJEŠENJE

Pretpostavimo da je u glatko rješenje na podskupovima V_l i V_r ($V_l \cup V_r = V$) koji su razdvojeni nekom krivuljom $\mathcal{C}$. Na $\mathcal{C}$ možemo imati prekide.
Uvjet koji mora biti zadovoljen na $\mathcal{C}$ izvodimo iz integralne relacije

RASPISANO U EVANSU §3 I U BILJEŠKAMA IZ PERUGIAE

① $v \in C_c^\infty(\mathbb{R}_0^+ \times \mathbb{R})$, supp $v \subseteq V_l$

U integralnoj relaciji možemo ići unatrag s parcijalnom integracijom jer je u glatko na V_l.

$$\Rightarrow \left.\begin{aligned} u_t + F(u)_x &= 0 \\ u(0,\cdot) &= g \end{aligned}\right\} \text{ na } V_l$$

② Analogno, $v \in C_c^\infty(\mathbb{R}_0^+ \times \mathbb{R})$ supp $v \subseteq V_r$

$$\Rightarrow \left.\begin{aligned} u_t + F(u)_x &= 0 \\ u(0,\cdot) &= g \end{aligned}\right\} \text{ na } V_r$$

(3) Uzmimo sada $v \in C_c^\infty(\mathbb{R}_0^+ \times \mathbb{R})$, supp $v \subseteq V$. (može sadržavati i $\mathcal{C}$)

$$0 = \iint_V u v_t + F(u) v_x = \iint_{V_l} -\|- + \iint_{V_r} -\|-$$

$$= -\iint_{V_l} (u_t + F(u)_x) v \, dx\, dt + \int_{\mathcal{C}} u_l v \nu_2 + F(u_l) v \nu_1 \, d\sigma$$

$$+ \iint_{V_r} (u_t + F(u)_x) v \, dx\, dt - \int_{\mathcal{C}} u_r v(-\nu_2) + F(u_r) v \cdot (-\nu_1) \, d\sigma$$

u_r ... limes fje u na $\mathcal{C}$ desna (iz skupa V_r)

u_l ... limes fje u na $\mathcal{C}$ slijeva (iz skupa V_l)

$$\Rightarrow \int_{\mathcal{C}} (u_l - u_r) v \cdot \nu_2 + (F(u_l) - F(u_r)) v \cdot \nu_1 \, d\sigma = 0$$

Budući da gornje vrijedi za svaki $v \in C_c^\infty(\mathbb{R}_0^+ \times \mathbb{R})$, supp $v \subseteq V \Rightarrow$

$$(u_l - u_r) v \cdot \nu_2 + (F(u_l) - F(u_r)) v \cdot \nu_1 = 0 \quad \text{na } \mathcal{C}$$

Iz gornje relacije mi zapravo možemo izračunati $\mathcal{C}$.

Pp da je $\mathcal{C}$ dana kao $x = s(t)$, $s \in C^1$

$$\nu = (\nu_1, \nu_2) = \frac{1}{\sqrt{1 + \dot{s}^2}} (1, -\dot{s})$$

$$-\dot{s}(u_l - u_r) + F(u_l) - F(u_r) = 0$$

$[F(u)] = \sigma [u]$ - RANKINE - HUGENOIT UVJET

$[u] = u_l - u_r$

$[F(u)] = F(u_l) - F(u_r)$

$\sigma = \dot{s}$

Primjer 1. U točki (1,1) smo imali problem pa trebamo primijeniti R-H uvjet

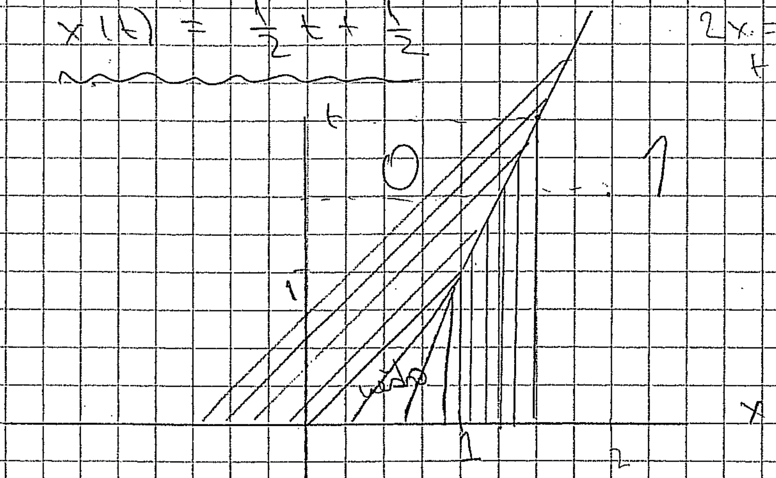
$$\left.\begin{array}{l} u_l = 1 \\ u_r = 0 \\ F(u_l) = \frac{1}{2} \\ F(u_r) = 0 \end{array}\right\} \Rightarrow \dot{s} = \frac{1}{2}, \quad s(1) = 1$$

$$s(t) = \frac{1}{2}t + A \Rightarrow A = \frac{1}{2}$$

$$x(t) = \frac{1}{2}t + \frac{1}{2}$$

$2x = t+1$

$t = 2x-1$

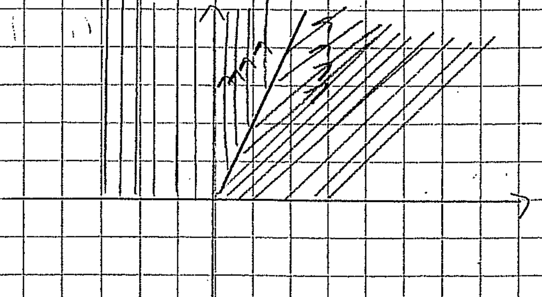


6. tj.

Pr. 2.

$$\begin{cases} u_t + u u_x = 0 \\ u = g \end{cases} \qquad F(u) = \frac{1}{2}u^2$$

$$g = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$



I. NAČIN PROŠIRENJA

Proširiti dio koji je 0 i dio koji je 1 te na spoju primijeniti R-H uvjet

$u_l = 0$

$u_r = 1$ } $[u] = -1$

$F_l = 0$

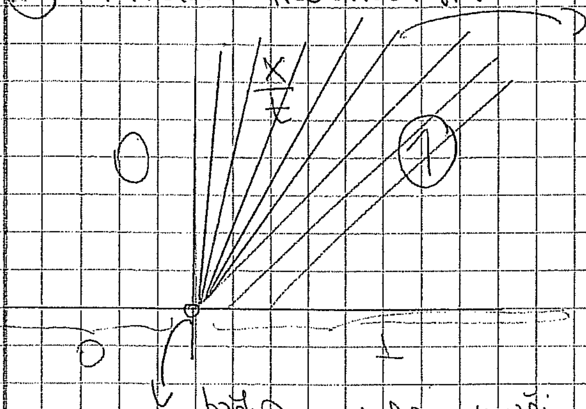
$F_r = \frac{1}{2}$ $\quad [F] = -\frac{1}{2}$

$[u]\dot{s} = [F]$

$\Rightarrow s(t) = \frac{1}{2}t$

$\Rightarrow$ dobili smo rj. koji nije dobro u smislu fizikalne interpretacije jer krivulje izlaze iz krivulje skoka.

2.) NAČIN PROŠIRENJA



→ uzeli smo da je rješenje rj. na neodređenom dijelu dimenzije $\frac{x}{t}$

tako nas muči, ali to je nebitno

Možemo provjeriti da je $u = \frac{x}{t}$ rj. početne jednadžbe na području gdje smo rješenje tako definirali.

Očito vidimo da rješenje nije jedinstveno. Jedinstvenost ćemo dobiti tako da uvedemo dodatni uvjet koji u mora zadovoljavati.

Entropijski uvjet

$$F'(u_l) > \dot{s} > F'(u_r)$$

Međutim, ako je F strogo konveksna, $F(u) = \frac{1}{2}u^2$ tada je gornji uvjet ekvivalentan s $u_l > u_r$.

Takvo rj. zovemo ENTROPIJSKO RJEŠENJE.

ZAD. 1. $u_t + u u_x = 0$

$$u(0,\cdot) = g$$

$$g(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

Odredite entropijsko rješenje.

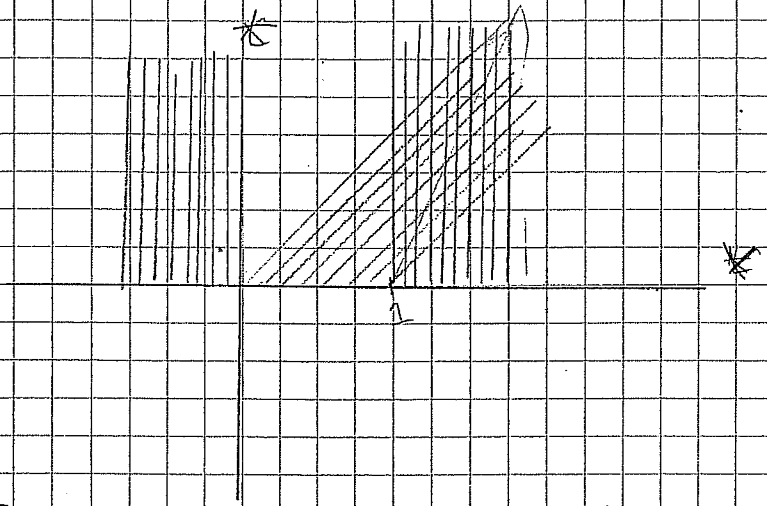
Rj: karakteristike po područjima

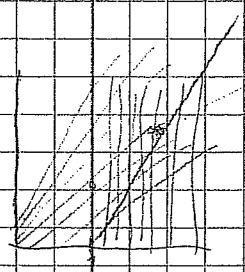
(1) $x_0 \in \langle -\infty, 0\rangle \qquad x(t) = x_0$

(2) $x_0 \in [0,1\rangle \qquad x(t) = t + x_0$

(3) $x_0 \in [1, +\infty\rangle \qquad x(t) = x_0$

$t = x - x_0$





(1) Dodefiniramo ju u $\{x > 0\} \cap \{x < t\}$ tako da bude zadovoljen entropijski uvjet $\quad u(t,x) = \frac{x}{t}$

(2) Karakteristike se sijeku u $(0,1)$ tražimo šok

$\Rightarrow$ min t.

R-H uvjet

$$\left.\begin{array}{l} u_l = 0 \\ u_r = 1 \end{array}\right\} [u] = -1 \qquad \Rightarrow s(t) = \frac{1}{2}t + 1$$

$$\left.\begin{array}{l} F(u_l) = 0 \\ F(u_r) = \frac{1}{2} \end{array}\right\} [F] = -\frac{1}{2} \qquad x = \frac{1}{2}t + 1$$

Nakon toga imamo problem u $(2,2)$

$(x = t \quad i \quad x = \frac{1}{2}t + 1 \Rightarrow x = \frac{1}{2}x + 1 \Rightarrow x = 2, t = 2$

R-H uvjet u točki $(2,2)$

s_1 = tražena krivulja šoka

$$\left.\begin{array}{l} u_l = \frac{s_1(t)}{t} \\ u_r = 0 \end{array}\right\} \Rightarrow [u] = \frac{s_1(t)}{t}$$

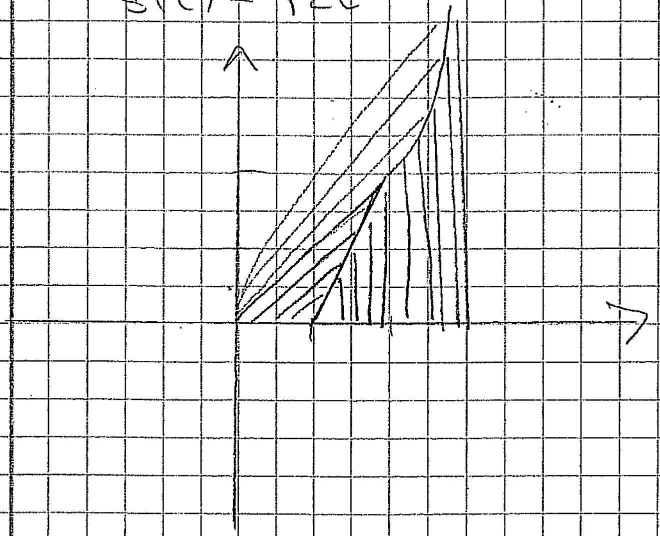
$$F_l = \frac{1}{2}\frac{s_1^2(t)}{t^2}, \quad F_r = 0 \Rightarrow [F] = \frac{1}{2}\frac{s_1^2(t)}{t^2}$$

$$\dot{s}\,\cancel{\frac{s}{t}} = \frac{1}{2}\frac{s^{\cancel{2}}}{t^{\cancel{2}}} \Rightarrow \frac{ds}{dt} = \frac{1}{2}\frac{s}{t} \Rightarrow \ln|s| = \frac{1}{2}\ln|t|$$

$|s| = |t|^{1/2} \cdot C \Rightarrow \quad s = C\sqrt{t}$

$s(2) = 2 \Rightarrow \quad 2 = C\sqrt{2} \Rightarrow C = \sqrt{2}$

$s(t) = \sqrt{2t}$



LAGRANGEOV POSTUPAK ZA RJEŠAVANJE PDJ 1. REDA

$\vec{a}(x, u(x)) \cdot \nabla u(x) = b(x, u(x))$

Pokušati konstruirati sva rješenja (nemamo rubni uvjet zato sva rješenja).

Promotrimo 2D slučaj:

$$a_1(x, y, u)u_x + a_2(x, y, u)u_y = b(x, y, u)$$

Uvodimo sustav ODJ:

$$\frac{dx}{a_1} = \frac{dy}{a_2} = \frac{du}{b}$$

Primjer 1. $xuu_x + yuu_y = -(x^2+y^2)$

Pripadni ODJ glasi

$$\frac{dx}{xu} = \frac{dy}{yu} = \frac{du}{-(x^2+y^2)}$$

1° $\frac{dx}{x\cancel{u}} = \frac{dy}{y\cancel{u}} \Rightarrow \frac{dx}{x} = \frac{dy}{y} \Rightarrow$

$x = cy \Rightarrow c = \frac{x}{y}$

$\varphi(x, y, u) = \frac{x}{y}$

2° $\frac{dx}{xu} = \frac{du}{-(x^2+y^2)} \quad / \cdot x^2u$

$$x\,dx = \frac{du \cdot u}{-\left(1 + \frac{1}{c_1^2}\right)} \quad / \int$$

$$\Rightarrow x^2\left(1 + \frac{1}{c_1^2}\right) = -u^2 + c_2, \quad c_2 = x^2 + y^2 + u^2 =: \psi(x, y, u)$$

Sada je rj. dano s $F(\varphi, \psi) = 0$ gdje je F neka C^1 f-ja. Zašto? Rj. dano je $u(x,y) = f(x,y)$

Deriviramo gornju jednakost po x i y

$$0 = \partial_x F = \frac{\partial F}{\partial \varphi}\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial u}\cdot\frac{\partial f}{\partial x}\right) + \frac{\partial F}{\partial\psi}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial u}\cdot\frac{\partial f}{\partial x}\right)$$

$$0 = \partial_y F = \frac{\partial F}{\partial \varphi}\left(\frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial u}\cdot\frac{\partial f}{\partial y}\right) + \frac{\partial F}{\partial\psi}\left(\frac{\partial\psi}{\partial y}+\frac{\partial\psi}{\partial u}\cdot\frac{\partial f}{\partial y}\right)$$

neka je $F_\psi \neq 0$, podijelimo jednadžbe s F_ψ.

$$\frac{F_\varphi}{F_\psi}(\varphi_x + \varphi_u\cdot f_x) = -(\psi_x + \psi_u\cdot f_x)$$ → izrazi se $\frac{F_\varphi}{F_\psi}$ i uvrsti u 2. j.

$$\frac{F_\varphi}{F_\psi}(\varphi_y + \varphi_u\cdot f_y) = -(\psi_y + \psi_u\cdot f_y)$$

$$\Rightarrow (\varphi_u\psi_y - \varphi_y\psi_u)f_x + (\varphi_x\psi_u - \varphi_u\cdot\psi_x)f_y + \varphi_x\psi_y - \psi_x\varphi_y = 0$$

$$\underbrace{\begin{vmatrix}\varphi_u & \varphi_y\\ \psi_u & \psi_y\end{vmatrix}}_{\frac{\partial(\varphi,\psi)}{\partial(u,y)}} f_x + \underbrace{\begin{vmatrix}\varphi_x & \varphi_u\\ \psi_x & \psi_u\end{vmatrix}}_{\frac{\partial(\varphi,\psi)}{\partial(x,u)}} f_y + \underbrace{\begin{vmatrix}\varphi_x & \varphi_y\\ \psi_x & \psi_y\end{vmatrix}}_{\frac{\partial(\varphi,\psi)}{\partial(x,y)}} = 0$$

možemo računati

$$\frac{\partial(\varphi,\psi)}{\partial(u,y)} = \begin{vmatrix}0 & \frac{-x}{y^2}\\ 2u & 2y\end{vmatrix} = \frac{2xu}{y^2}$$

$$\frac{\partial(\varphi,\psi)}{\partial(y,x)} = \begin{vmatrix}\frac{-x}{y^2} & \frac{1}{y}\\ 2y & 2x\end{vmatrix} = \frac{-2x^2}{y^2} - 2$$

$$\frac{\partial(\varphi,\psi)}{\partial(x,u)} = \begin{vmatrix}\frac{1}{y} & 0\\ 2x & 2u\end{vmatrix} = \frac{2u}{y}$$

$$\Rightarrow xu f_x + yu f_y = -(x^2+y^2)$$

$F\left(\frac{x}{y}, x^2+y^2+u\right) = 0$ ⇐ definira j. koju god je $\frac{\partial(\varphi,\psi)}{\partial(x,y,u)}$ punog ranga (ili barem jedna od gornje tri determinante nije 0)

Ako imamo $F_\psi \neq 0$ postoji $g \in C^1$ t.d.

$$x^2+y^2+u^2 = g_2\left(\frac{x}{y}\right)$$

$$\Rightarrow u^2 = g_2\left(\frac{x}{y}\right) - x^2 - y^2$$

Ako je sustav ODJ prekompliciran možemo
izvesti veze $\frac{A}{B}=\frac{C}{D}=\frac{\lambda A+\mu C}{\lambda B+\mu D}=\frac{A}{B}$

Za proizvoljne λ,μ,ν također vrijedi

$$\frac{\lambda dx+\mu dy+\nu du}{\lambda a_1+\mu a_2+\nu b}=\frac{dx}{a_1}=\frac{dy}{a_2}=\frac{du}{b}$$

Ako možemo naći λ,μ,ν td. je $\lambda a_1+\mu a_2+\nu b=0$, onda mora biti i $\lambda dx+\mu dy+\nu du=0$.
Ako nađemo fju φ td. je $d\varphi=\lambda dx+\mu dy+\nu du$, onda je $\varphi(x,y,u)=c_1$ integral prvog sustava ODJ.

zad.1. $(y-x)u_x+(y+x)u_y=\frac{x^2+y^2}{u}$

Rj: $\frac{dx}{y-x}=\frac{dy}{y+x}=\frac{du}{\frac{x^2+y^2}{u}}$

1. $\lambda=1,\ \mu=1,\ \nu=0$

$(y-x)+(y+x)=2y$

$$\frac{dx+dy}{2y}=\frac{dy}{y+x}$$

$\Rightarrow (x+y)\,d(x+y)=2y\,dy$

$\Rightarrow x^2+2xy-y^2=c=\varphi(x,y,u)$

2. $\lambda=-x,\ \mu=y,\ \nu=-u$

$-x(y-x)+y(y+x)-\cancel{u}\frac{x^2+y^2}{\cancel{u}}=0$

Trebamo pokušati odrediti ψ td.

$\psi_x=-x$ $\quad\psi(x,y,z)$

$\psi_y=y$

$\psi_u=-u$

$\psi(x,y,u) = -\frac{x^2}{2} + \frac{y^2}{2} - \frac{u^2}{2} \ (= C)$

Na kraju trebamo provjeriti jesu li φ i ψ nezavisni, tj. da je pripadni Jacobijan različit od 0 gotovo u svim točkama, tj. da je punog ranga matrica

$$\frac{\partial(\varphi,\psi)}{\partial(x,y,u)} = \begin{bmatrix} 2x+2y & 2x-2y & 0 \\ -x & y & -u \end{bmatrix}$$

det ove podmatrice $2x^2+2y^2=0$

$\iff (x,y)=(0,0)$

$F(\varphi,\psi)=0$ je konačno rješenje

$F\left(x^2+2xy+y^2, \frac{x^2}{2}+\frac{y^2}{2}-\frac{u^2}{2}\right)=0$

$F_\psi \neq 0 \Rightarrow$ po tm. o implicitnoj fji slijedi da postoji g t.d. je

$$g(x^2+2xy-y^2) = -\frac{x^2}{2}+\frac{y^2}{2}-\frac{u^2}{2}$$

$$u^2 = -x^2+y^2-2g(x^2+2xy-y^2)$$

KOMENTAR: Mogli smo uzeti

(1) $\left.\begin{array}{l} \lambda = y+x \\ \mu = x-y \\ \nu = 0 \end{array}\right\} \Rightarrow \lambda a_1 + \mu a_2 + \nu b = 0$

2. zad. $uu_x + yu_y = x$

(1) $\lambda = x, \ \mu = 0, \ \nu = -u$

(2) $\lambda = y, \ \mu = -u, \ \nu = 0$

(1) $\left.\begin{array}{l} \varphi_x = x \\ \varphi_y = 0 \\ \varphi_u = -u \end{array}\right|$ $\varphi(x,y,u) = \frac{x^2}{2} - \frac{u^2}{2} \ (= C)$, tj. $\varphi(x,y,u) = x^2-y^2$

(2) $\begin{array}{l} \psi_x = y \\ \psi_y = -u \\ \psi_u = 0 \end{array} \Rightarrow \psi(x,y,u) = xy + A(y,u)$

$-u \neq x + A_y$

$\Rightarrow$ ne postoji takva fja ψ

(2) $\lambda = \frac{1}{y}$, $\mu = -\frac{x+u}{y^2}$, $\nu = \frac{1}{y}$

$$\varphi_x = \frac{1}{y} \Rightarrow \varphi(x,y,u) = \frac{x}{y} + A(y,u)$$

$$\varphi_y = -\frac{x}{y^2} - \frac{u}{y^2} \qquad -\frac{x}{y^2} - \frac{u}{y^2} = -\frac{x}{y^2} + A_y(y,u) \Rightarrow A(y,u) = +\frac{u}{y} + B(u)$$

$$\varphi_u = \frac{1}{y} \Rightarrow B'(u) = 0$$

$$\Rightarrow \psi(x,y,u) = \frac{x}{y} + \frac{u}{y} \quad (= C)$$

...

<u>Zad. 3.</u> $x(y-u)u_x + y(u-x)u_y = u(x-y)$

(1) $\lambda = \mu = \nu = 1$

$$xy - xu + yu - \cancel{yx} + ux - uy = 0$$

$$\psi(x,y,u) = x+y+u$$

(2) $\lambda = \frac{1}{x}$, $\mu = \frac{1}{y}$, $\nu = \frac{1}{u}$

$$y - \cancel{u} + \cancel{u} - \cancel{x} + \cancel{x} - y = 0$$

$$\Rightarrow \psi(x,y,u) = \ln(xyu)$$

...

LAPLACEOVA JEDNADŽBA

(§2 Evans od sad do kraja)

$$-\Delta u = 0 \quad , \quad \Delta u = \sum_{i=1}^{d} \partial_i^2 u$$

① Jednadžba na $\mathbb{R}^d$ (nema ruba pa ni rubnog uvjeta)

$-\Delta u = f$ ~~na~~ $\mathbb{R}^d$ ⇝ (nehomogena) Poissonova jed.

② Jednadžba na domeni s rubom. Moramo zadati uvjet (vrijednost f-je ili derivacije) na rubu.

$$\begin{cases} -\Delta u = f & u \ \Omega \subseteq \mathbb{R}^d \\ u|_{\partial\Omega} = g \end{cases}$$

prvi dio 33. stranice tu ubaciti (do zadatka)

②a Ω omeđen

Tipičan primjer: Ω je kugla ili kocka ⇝ to se rješavalo na kolegiju MMF koristeći Fourierove redove

②b Ω neomeđen

Metoda s Fourierovim redovima ne prolazi u ovom slučaju.

Može se rješavati koristeći Greenovu f-ju, s tim da je ona komplicirnija u ②a slučaju.

Na predavanjima je izvedeno elementarno rješenje t.d. se ~~prvetj~~ ustvrdilo da je Laplaceov operator $-\Delta$ invarijantan na rotacije, pa smo tražili radijalno rješenje $\Phi(x) = \phi(|x|)$. Dokažimo tu tvrdnju.

ZAD.1. Laplaceov operator je invarijantan na rotacije. Precizније, $-\Delta u = 0 \Rightarrow (\forall R \ldots \text{rotacija}) \ -\Delta v = 0$, gdje je $v(x) := u(\mathbb{R}x)$.

Rj. Operator rotacije možemo shvatiti kao produkt ortogonalne matrice $\mathbb{R}$ ($\mathbb{R}^T\mathbb{R} = \mathbb{R}\mathbb{R}^T = I$) i danog vektora.

$$r_{ij} := [\mathbb{R}]_{ij}$$

$$\partial_i v(x) = \sum_{j=1}^{d} \partial_j u(\mathbb{R}x)\, r_{ji} \qquad / \partial_i$$

$$\begin{aligned}
\partial_i^2 v(x) &= \sum_{j=1}^{d} \partial_i \big((\partial_j u)(\mathbb{R}x)\big)\, r_{ji} \\
&= \sum_{j=1}^{d} \sum_{k=1}^{d} (\partial_k \partial_j u)(\mathbb{R}x)\, r_{ki}\, r_{ji}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \Delta v(x) &= \sum_{j,k=1}^{d} (\partial_k \partial_j u)(\mathbb{R}x) \underbrace{\sum_{i=1}^{d} r_{ki} r_{ji}}_{\delta_{kj} \text{ (jer je } \mathbb{R}\mathbb{R}^T = I)} \\
&= \sum_{j=1}^{d} (\partial_j^2 u)(\mathbb{R}x) \\
&= \Delta u(\mathbb{R}x) = 0 .
\end{aligned}$$

← najprije napraviti ZADATAK na str. 33.

ZAD. 3: Dokažite da postoji konstanta C neovisna o f, g i d tako da

$$\max_{K([0,1])} |u| \leq C \left(\max_{S(0,1)} |g| + \max_{K[0,1]} |f| \right)$$

za sve glatke u koji zadovoljavaju

$$(*) \begin{cases} -\Delta u = f & \text{u } K(0,1) \\ u|_{S(0,1)} & \text{na } S(0,1) \end{cases}$$

Rj: Rastavimo naš problem na dvije zadaće

$$① \begin{cases} -\Delta u = 0 \\ u|_{S(0,1)} = g \end{cases} , \qquad ② \begin{cases} -\Delta u = f \\ u|_{S(0,1)} = 0 \end{cases} .$$

Ako je u_1 rješenje od ① i u_2 rj. od ② tada je očito $u := u_1 + u_2$ rj. od (*) (posljedica linearnosti jednadžbe i rubnog uvjeta).

Iz principa maksimuma slijedi

$$|u_1(x)| \leq \|g\|_{L^\infty(S(0,1))}$$

$$\Rightarrow |u(x)| \leq |u_1(x)| + |u_2(x)| \leq \|g\|_{L^2(S(0,1))} + \underbrace{|u_2(x)|}_{\text{još ovo moramo ocijeniti}}$$

$M := \|f\|_{L^\infty(K[0,1])}$.

Definirajmo $v(x) := u_2(x) + \frac{M}{2d}\|x\|^2$.

Kako je

$$\Delta\left(\frac{M}{2d}\|x\|^2\right) = \frac{M}{2d}\cdot 2d = M \quad ,$$

slijedi

$$\Delta v(x) = \Delta u_2(x) + M = -f(x) + M \geq 0$$

$\Rightarrow$ $\boxed{-\Delta v \leq 0}$ $\rightsquigarrow$ v je <u>podharmonička</u> f-ja

(za takve f-je vrijedi princip maksimuma i dalje)

$$\Rightarrow \quad v|_{S(0,1)} = u_2|_{S(0,1)} + \frac{M}{2d} = \frac{M}{2d} \leq \frac{M}{2}$$

$$\Rightarrow \quad u_2(x) \leq v(x) \leq \frac{M}{2d} \leq \frac{M}{2}.$$

Još je potrebno pokazati da $u_2(x) \geq -\frac{M}{2}$.

Definirajmo $\tilde{v}(x) := -u_2(x) + \frac{M}{2d}\|x\|^2$.

$\Delta\tilde{v}(x) = f(x) + M \geq 0$ $\rightsquigarrow$ $\tilde{v}$ je <u>podharmonička</u> f-ja

Analogno dobivamo $\tilde{v}|_{S(0,1)} = \frac{M}{2d} \leq \frac{M}{2}$ pa imamo

$$u_2(x) = -\tilde{v}(x) + \frac{M}{2d}\|x\|^2 \geq -\tilde{v}(x) \geq -\frac{M}{2}$$

$$\Rightarrow \quad |u_2(x)| \leq \frac{M}{2} \leq M.$$

Konačno dobivamo

$$\max_{K[0,1]} |u| \leq \max_{S(0,1)} |g| + \max_{K[0,1]} |f| \quad , \text{ tj. } \quad C = 1.$$

GREENOVA FUNKCIJA

Elementarno rješenje Laplaceove jednadžbe je dano s

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln|x| & , d=2 \\ \frac{1}{d(d-2)\omega_d} \frac{1}{|x|^{d-2}} & , d \geq 3 \end{cases} ,$$

te je rješenje Poissonove jednadžbe na cijelom $\mathbb{R}^d$ dano s

$$u(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y)\, dy = (\Phi * f)(x) .$$

konstante kod f-je Φ su upravo namještene t.d. ovo vrijedi.

Iako Φ u ishodištu ima singularitet, funkcija Φ je integrabilna na okolini ishodišta.

ZAD. ~~3~~ 2. Pokažite da je Φ integrabilna na okolini ishodišta.

Rj. Dovoljno je pokazati da je za neki $R>0$ integral

$$\int_{K[0,R]} \Phi(x)\, dx$$

konačan.

$\boxed{d=2}$

polarne koordinate

$$\int_{K[0,R]} \Phi(x)\, dx = \int_0^R \int_{S(0,r)} \Phi(y)\, dS_y\, dr$$

$$= \int_0^R \int_{S(0,r)} -\frac{1}{2\pi} \ln|y|\, dS_y\, dr$$

$$= -\frac{1}{2\pi} \int_0^R \ln r \left(\int_{S(0,r)} dS_y \right) dr$$

$$= -\frac{1}{2\pi} \int_0^R 2\pi r \ln r\, dr = -\int_0^R r \ln r\, dr$$

$$\overset{PI}{=} -\frac{r^2}{2} \ln r \Big|_0^R + \int_0^R \frac{r}{2}\, dr$$

$$= -\frac{R^2}{2} \ln R + \frac{R^2}{4}$$

limes $\lim_{r \to 0} r^2 \ln r$ je jednak nuli.

GREENOVA FUNKCIJA

⇝ Poissonova jednadžba za $\Omega \subseteq \mathbb{R}^d$ $(\Omega \neq \mathbb{R}^d)$

$$\begin{cases} -\Delta u = f & \text{u } \Omega \\ u|_\Gamma = g & \text{na } \partial\Omega \end{cases}$$

Pomoću Greenove funkcije moći ćemo eksplicitno izraziti rješenje u. Naime (za dovoljno dobre f i g) rješenje je dano s

$$u(x) = -\int_{\partial\Omega} \frac{\partial G}{\partial \vec{n}_y}(x,y)\, g(y)\, dS_y + \int_\Omega G(x,y)\, f(y)\, dy ,$$

pri čemu je G Greenova f-ja koja je za fiksni $x \in \cancel{\mathbb{R}^d}\ \Omega$ dana s

$$\begin{cases} -\Delta G(x,\cdot) = 0 & \text{u } \Omega \setminus \{x\} \\ G(x,\cdot) = 0 & \text{na } \partial\Omega \end{cases} .$$

Može i jednostavnije : $G(x,y) = \Phi(|x-y|) + w(x,y)$ pri čemu je

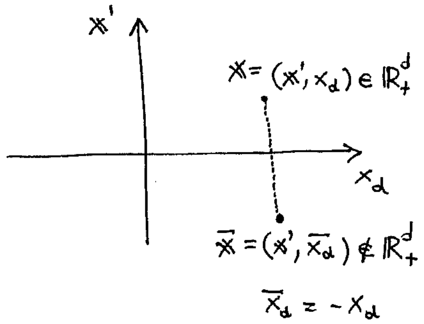
$$\begin{cases} \Delta w(x,\cdot) = 0 \\ w(x,y) = -\Phi(|x-y|) \end{cases} , \quad y \in \partial\Omega ,$$

~~pri~~ dok je Φ elementarno rješenje Laplaceove j.

Mi nećemo tražiti Greenovu f-ju rješavanjem jednadžbi, već metodom refleksije.

PRIMJER 1. Poluprostor (vidi predavanja)

$$\mathbb{R}^d_+ = \{(x', x_d) : x_d > 0\}$$



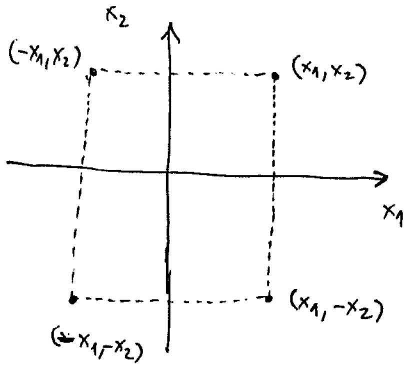
izbacili smo singularitet iz $\Omega = \mathbb{R}^d_+$

⇝ $G(x,y) = \Phi(|x-y|) - \Phi(|\bar{x}-y|)$

NAP. Kod „ravnih" rubova uvijek radimo refleksiju kao u prethodnom primjeru, s tim da predznak od Φ ovisi o broju refleksija pripadne točke: ako je taj broj paran predznak je $+$, a inače $-$ (u prethodnom primjeru na x smo primijenili 0 refleksija pa je zato $\Phi(|x-y|)$ s $+$, a na $\bar{x}$ jednu pa je zato $\Phi(|\bar{x}-y|)$ s $-$). Nadalje, možemo imati $x=\bar{x}$ za $x\in\partial\Omega$.

PRIMJER 2. I kvadrant u $\mathbb{R}^2$

$$\Omega=\{(x_1,x_2)\in\mathbb{R}^2 : x_1,x_2>0\}$$



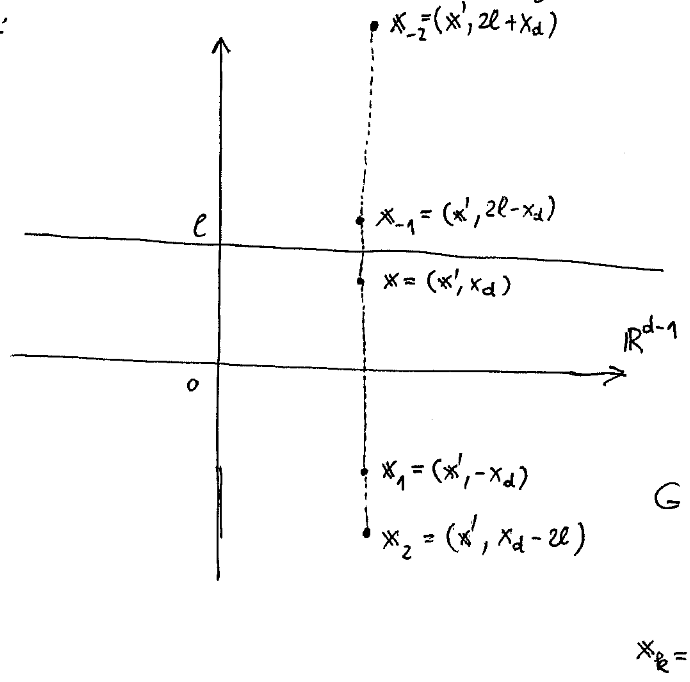
Konačno:
$$G(x,y)=\Phi(|(x_1,x_2)-(y_1,y_2)|)-\Phi(|(x_1,-x_2)-(y_1,y_2)|)+\Phi(|(-x_1,-x_2)-(y_1,y_2)|)-\Phi(|(-x_1,x_2)-(y_1,y_2)|)$$
$$=-\frac{1}{2\pi}\Big[\ln\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}-\ln\sqrt{(x_1-y_1)^2+(x_2+y_2)^2}+\ln\sqrt{(x_1+y_1)^2+(x_2+y_2)^2}-\ln\sqrt{(x_1+y_1)^2+(x_2-y_2)^2}\Big]$$

Očito imamo: $G(x_1,x_2,0,y_2)=G(x_1,x_2,y_1,0)=0$,

tj. G je 0 na $\partial\Omega$. $((y_1,y_2)\in\partial\Omega)$.

ZAD. 4. Odredite Greenovu funkciju za prugu $\mathbb{R}^{d-1}\times\langle 0,\ell\rangle$

Rj:



Lako vidimo da točke x, x_{-1}, x_1 ne čine zatvoren sustav obzirom na promatrane refleksije. Dapače, dobit ćemo beskonačno mnogo točaka (~~osim za~~ čak i u slučajevima $x_d=0,\frac{\ell}{2},\ell$).

$$G(x,y)=\Phi(|x-y|)+w(x,y),$$
$$w(x,y)=\sum_{k=1}^{\infty}(-1)^k\left(\Phi(|x_k-y|)+\Phi(|x_{-k}+y|)\right)$$
$$x_k=\left(x',(-1)^k x_d-2\left\lfloor\tfrac{k}{2}\right\rfloor\ell\right)$$
$$x_{-k}=\left(x',(-1)^k x_d+2\left\lfloor\tfrac{k+1}{2}\right\rfloor\ell\right)$$

PRIMJER 3. Greenova funkcija za kuglu

$$\Omega = K(0,R)$$

$$\tilde{x} = \frac{R^2}{|x|^2}\, x \qquad \left[\; |x| < R \Rightarrow |\tilde{x}| = \frac{R^2}{|x|^2}|x| = \frac{R^2}{|x|} > \frac{R^2}{R} = R \;\Rightarrow\; \boxed{x \in K(0,R) \Rightarrow \tilde{x} \notin K(0,R)} \right.$$

$$G(x,y) = \Phi(|x-y|) - \Phi\left(\frac{|x|}{R}|\tilde{x}-y|\right).$$

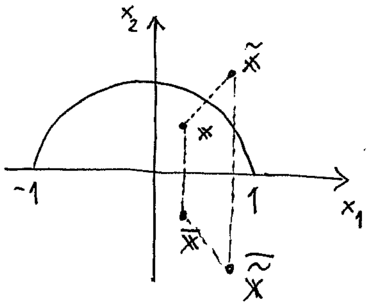
za $|y| = R$ je $\frac{|x|}{R}|\tilde{x}-y| = |x-y|$

(isti raspis kao u Evans na str. 39.)

ZAD. 5. Konstruirajte Greenovu f-ju za polukuglu u $\mathbb{R}^2$ danu s

$$\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1,\ x_2 > 0\}$$

Rj:



Dobili smo 4 točke: $x, \tilde{x}, \bar{x}, \tilde{\bar{x}}$.

$$G(x,y) = \Phi(|x-y|) - \Phi(|x|\,|\tilde{x}-y|) + \Phi(|\bar{x}|\,|\tilde{\bar{x}}-y|) - \Phi(|\bar{x}-y|)$$

ZAD. 6 Nađite rješenje problema

$$\begin{cases} -\Delta u(x_1,x_2,x_3) = f(x_1,x_2,x_3) & , x \in D \\ u(x_1,x_2,0) = g(x_1,x_2) \end{cases}$$

gdje je D poluprostor. $(\{x_3>0\})$

a) $f(x) = 0$, $g(x) = \cos x_1 \cos x_2$

b) $f(x) = e^{-x_3} \sin x_1 \cos x_2$, $g(x) = 0$

Rj.

$$G(x_1,x_2,x_3;y_1,y_2,y_3) = \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}} - \frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2}}\right)$$

$$u(x) = -\int_{\partial D} \frac{\partial G}{\partial n_y}(x,y)\, g(y)\, dS_y + \int_D G(x,y) f(y)\, dy$$

a) Drugi integral je nula, a da izračunamo prvi moramo najprije izračunati $\frac{\partial G}{\partial n_y}$.

n_y je normala na rub od D, a to je ravnina $x_3=0$ i kako je ona orijentirana izvan skupa D, onda je

$$n_y = -e_3 .$$

Općenito : $\frac{\partial F}{\partial v} = \nabla_v F = \nabla F \cdot v$ ← skalarno množenje

$$\Rightarrow \frac{\partial G}{\partial n_y} = -\frac{\partial G}{\partial y_3}$$

$$\frac{\partial G}{\partial y_3}(x,y) = \frac{1}{4\pi}\left(\frac{x_3-y_3}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right)^{\frac{3}{2}}} + \right.$$

$$\left. + \frac{x_3+y_3}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2\right)^{\frac{3}{2}}}\right)$$

Budući da je integral samo po ravnini $x_3 = 0$, imamo:

$$u(x_1,x_2,x_3) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial G}{\partial y_3}(x, y_1, y_2, 0)\, g(y_1,y_2)\, dy_1\, dy_2$$

$$= \frac{x_3}{2\pi}\int_{-\infty}^{+\infty} \cos y_2 \int_{-\infty}^{+\infty} \frac{\cos y_1}{\left((x_1-y_1)^2+(x_2-y_2)^2+x_3^2\right)^{\frac{3}{2}}} dy_1\, dy_2$$

riječ je o Besselovoj funkciji 2. vrste i poprilično komplicirano za računati

⋮

$$= \underline{e^{-\sqrt{2}x_3} \cos x_1 \cos x_2}$$

b) $$u(x_1,x_2,x_3) = \int_{\mathbb{R}^3_+} \frac{1}{4\pi}\left(\frac{1}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2}} - \frac{1}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3+x_3)^2}}\right) e^{-y_3} \sin y_1 \sin y_2\, dy_1 dy_2 dy_3$$

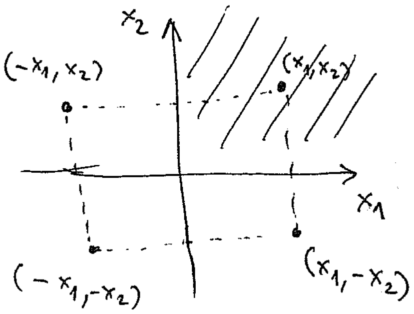
⋮

Rj.: $u(x_1,x_2,x_3) = \left(e^{-\sqrt{x_3}} - e^{\sqrt{x_2}}\right)\cos x_1 \cos x_2$.

Nađite rješenje Dirichletove zadaće na I kvadrantu ($x_1>0, x_2>0$), $\begin{cases} \ldots = 0 \\ u(\cdot,0)=f \\ u(0,\cdot)=0 \end{cases}$

gdje je $f(x) = x$ za $0<x<1$, a 0 inače.

ZAD. 7. Na satu smo izračunali Greenovu funkciju na prvom kvadrantu:



$$G((x_1,x_2),(y_1,y_2)) = -\frac{1}{2\pi}\ln\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} + \frac{1}{2\pi}\ln\sqrt{(x_1+y_1)^2+(x_2-y_2)^2} + \frac{1}{2\pi}\ln\sqrt{(x_1-y_1)^2+(x_2+y_2)^2} - \frac{1}{2\pi}\ln\sqrt{(x_1+y_1)^2+(x_2+y_2)^2}$$

Sada je rješenje u dano formulom:

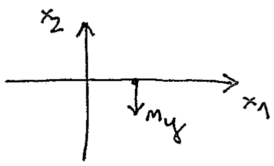
$$u(x_1,x_2) = -\int_{\partial D} \frac{\partial G}{\partial n_y}((x_1,x_2),(y_1,y_2))\, g(y_1,y_2)\, dS_y$$

$D \ldots$ prvi kvadrant $\Rightarrow \partial D \ldots$ pozitivni dio koordinatnih osi

g je nula na x_2-osi, a jednaka je f-ji f na x_1-osi

$$u(x_1,x_2) = -\int_0^1 \underbrace{\frac{\partial G}{\partial n_y}((x_1,x_2),(y_1,0))}_{\text{ovo trebamo izračunati}}\, y_1\, dy_1$$

$$-\frac{\partial G}{\partial n_y} = \frac{\partial G}{\partial y_2}$$



$$\frac{\partial G}{\partial y_2}((x_1,x_2),(y_1,y_2)) = \frac{1}{2\pi}\frac{x_2-y_2}{(x_1-y_1)^2+(x_2-y_2)^2} - \frac{1}{2\pi}\frac{x_2-y_2}{(x_1+y_1)^2+(x_2-y_2)^2} + \frac{1}{2\pi}\frac{x_2+y_2}{(x_1-y_1)^2+(x_2+y_2)^2} - \frac{1}{2\pi}\frac{x_2+y_2}{(x_1+y_1)^2+(x_2+y_2)^2}$$

$$\Rightarrow \frac{\partial G}{\partial y_2}((x_1,x_2),(y_1,0)) = \frac{1}{\pi}\frac{x_2}{(x_1-y_1)^2+x_2^2} - \frac{1}{\pi}\frac{x_2}{(x_1+y_1)^2+x_2^2}$$

$$u(x_1,x_2) = \frac{x_2}{\pi}\int_0^1 \frac{y_1}{(x_1-y_1)^2+x_2^2}\,dy_1 - \frac{x_2}{\pi}\int_0^1 \frac{y_1}{(x_1+y_1)^2+x_2^2}\,dy_1$$

- $$\int_0^1 \frac{y_1}{(x_1-y_1)^2+x_2^2}\,dy_1 = \frac{1}{2}\ln\left((x_1 - y_1)^2+x_2^2\right)\Big|_0^1 - \frac{x_1}{x_2}\,\mathrm{arctg}\left(\frac{x_1-y_1}{x_2}\right)\Big|_0^1$$

$$= \frac{1}{2}\ln\left((x_1-1)^2+x_2^2\right) - \frac{1}{2}\ln\left(x_1^2+x_2^2\right) - \frac{x_1}{x_2}\,\mathrm{arctg}\left(\frac{x_1-1}{x_2}\right) + \frac{x_1}{x_2}\,\mathrm{arctg}\left(\frac{x_1}{x_2}\right)$$

- $$\int_0^1 \frac{y_1}{(x_1+y_2)^2+x_2^2}\,dy_1 = \frac{1}{2}\ln\left((x_1+y_2)^2+x_2^2\right)\Big|_0^1 - \frac{x_1}{x_2}\,\mathrm{arctg}\left(\frac{x_1+y_1}{x_2}\right)\Big|_0^1$$

$$= \frac{1}{2}\ln\left((x_1+1)^2+x_2^2\right) - \frac{1}{2}\ln\left(x_1^2+x_2^2\right) - \frac{x_1}{x_2}\,\mathrm{arctg}\left(\frac{x_1+1}{x_2}\right) + \frac{x_1}{x_2}\,\mathrm{arctg}\left(\frac{x_1}{x_2}\right)$$

$$\Rightarrow\quad u(x_1,x_2) = \frac{x_2}{\pi}\,\ln\sqrt{\frac{(x_1-1)^2+x_2^2}{(x_1+1)^2+x_2^2}} - \frac{x_1}{\pi}\left(\mathrm{arctg}\left(\frac{x_1-1}{x_2}\right) - \mathrm{arctg}\left(\frac{x_1+1}{x_2}\right.\right.$$

<u>Provjera rubnog uvjeta:</u>

- $x_1=0,\ x_2\neq 0$: $\quad u(0,x_2) = \frac{x_2}{2\pi}\ln\left(\frac{x_2^2+1^2}{x_2^2+1^2}\right) = 0$
- $x_2=0,\ x_1\neq 0$: $\quad \lim_{x_2\to 0^+} u(x_1,x_2) = \begin{cases} x_1, & x_1<1 \\ 0, & x_1>1 \end{cases}$

JEDNADŽBA PROVOĐENJA

$$\begin{cases} u_t - \Delta u = f & \text{u } \mathbb{R}^+ \times \mathbb{R}^d \\ u(0,\cdot) = g & \text{na } \{t=0\} \times \mathbb{R}^d \end{cases}$$

Rj. je dano s

$$\rightsquigarrow \quad u(t,x) = \int_{\mathbb{R}^d} \Phi(t, x-y)\, g(y)\,dy + \int_0^t \int_{\mathbb{R}^d} \Phi(t-s, x-y)\, f(s,y)\,dy\,ds,$$

gdje je

$$\Phi(t,x) = \begin{cases} \frac{1}{(\sqrt{4\pi t})^d}\, e^{-\frac{|x|^2}{4t}} & , \ t>0, \ x \in \mathbb{R}^d \\ 0 & , \ t<0, \ x \in \mathbb{R}^d \end{cases}$$

... elementarno rj. (Gaussian)

Vrijedi: $\boxed{\int_{\mathbb{R}^d} \Phi(t,x)\,dx = 1}$.

Kod računanja također često koristimo sljedeći integral:

$$\int_{-\infty}^{+\infty} e^{-ax^2} \cos(bx)\,dx = \sqrt{\frac{\pi}{a}}\, e^{-b^2/4a}.$$

$$\begin{cases} u_t - \Delta u = e^t \\ u(0,x_1,x_2) = \cos x_1 \sin x_2 \end{cases}$$

Rj:

$$u(t,x_1,x_2) = \underbrace{\int_{\mathbb{R}^2} \Phi(t, x-y) \cos y_1 \sin y_2 \, dy}_{\mathrm{I}} + \underbrace{\int_0^t \int_{\mathbb{R}^2} \Phi(t-s, x-y)\, e^s \, dy\, ds}_{\mathrm{II}}$$

$$\mathrm{I} = \frac{1}{4\pi t} \int_{\mathbb{R}^2} e^{-\frac{(x_1-y_1)^2 + (x_2-y_2)^2}{4t}} \sin y_2 \cos y_1 \, dy_1 dy_2$$

$$= \frac{1}{4\pi t} \Big(\underbrace{\int_{\mathbb{R}} e^{-\frac{(x_1-y_1)^2}{4t}} \cos y_1 \, dy_1}_{I_1} \Big) \Big(\underbrace{\int_{\mathbb{R}} e^{-\frac{(x_2-y_2)^2}{4t}} \sin y_2 \, dy_2}_{I_2} \Big)$$

$$I_1 = \left| \begin{array}{l} z = y_1 - x_1 \\ dz = dy_1 \\ \Rightarrow y_1 = z + x_1 \end{array} \right| = \int_{\mathbb{R}} e^{-\frac{z^2}{4t}} \cos(z+x_1)\, dz = \cos x_1 \int_{\mathbb{R}} e^{-\frac{z^2}{4t}} \cos z \, dz -$$

$$- \sin x_1 \underbrace{\cancel{\int_{\mathbb{R}} e^{-\frac{z^2}{4t}} \sin z \, dz}}_{=0 \text{ (neparne f-je)}}$$

$$= \cos x_1 \int_{\mathbb{R}} e^{-\frac{z^2}{4t}} \cos z\, dz \overset{\text{formula}}{=} \cos x_1 \sqrt{4\pi t}\, e^{-t}$$

$$I_2 = \left| \begin{array}{l} z = y_2 - x_2 \\ dz = dy_2 \\ \Rightarrow y_2 = z + x_2 \end{array} \right| = \underbrace{\cancel{\cos x_2 \int_{\mathbb{R}} e^{-\frac{z^2}{4t}} \sin z\, dz}}_{=0} + \sin x_2 \int_{\mathbb{R}} e^{-\frac{z^2}{4t}} \cos z \, dz$$

$$= \sin x_2 \sqrt{4\pi t}\, e^{-t}$$

$$\Rightarrow \mathrm{I} = \frac{1}{4\pi t} \cos x_1 \sqrt{4\pi t}\, e^{-t} \sin x_2 \sqrt{4\pi t}\, e^{-t} = \underline{e^{-2t} \cos x_1 \sin x_2}$$

$$\mathrm{II} = \int_0^t e^s \Big(\underbrace{\int_{\mathbb{R}^2} \Phi(t-s, x-y)\, dy}_{=1} \Big) ds = \int_0^t e^s \, ds = e^s \Big|_0^t = e^t - 1$$

$$\Rightarrow \boxed{u(t,x_1,x_2) = e^{-2t} \cos x_1 \sin x_2 + e^t - 1}$$

ZAD. 1 Neka je $g : [0,\infty\rangle \to \mathbb{R}$, $g(0)=0$, izvedite formulu ~~za~~

(EVANS ZAD. 15)

$$u(t,x) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)\, ds$$

za rj. početno-rubne zadaće

$$\begin{cases} u_t - u_{xx} = 0 & u\ \langle 0,\infty\rangle \times \mathbb{R}^+ \\ u = 0 & \text{na } \{t=0\} \times \mathbb{R}^+ \\ u = g & \text{na } [0,\infty\rangle \times \{x=0\} \end{cases}$$

Rj: Problem je što integriramo po $\mathbb{R}^+$ pa nemamo formulu za rj.

$$v(t,x) := \begin{cases} u(t,x) - g(t) & , \ x > 0 \\ -u(t,-x) + g(t) & , \ x \leq 0 \end{cases}$$

CILJ: izvesti formulu t.d. proširimo f-ju na cijeli $\mathbb{R}$ i koristimo poznate rezultate

$$\Rightarrow \quad v_t(t,x) = \begin{cases} u_t(t,x) - g'(t) & , \ x > 0 \\ -u_t(t,-x) + g'(t) & , \ x \leq 0 \end{cases}$$

$$v_{xx} = \begin{cases} u_{xx}(t,x) & , \ x > 0 \\ -u_{xx}(t,-x) & , \ x \leq 0 \end{cases}$$

$$\Rightarrow \left\{ \begin{aligned} & v_t(t,x) - v_{xx}(t,x) = \begin{cases} -g'(t) & , \ x > 0 \\ g'(t) & , \ x \leq 0 \end{cases} \\ & v(x,0) = 0 \\ & v(0,t) = 0 \end{aligned} \right\} \Rightarrow$$

v zadovoljava j. provođenja na cijelom $\mathbb{R}$

$$\Rightarrow v(t,x) = \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} \left(\int_{-\infty}^{0} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} g'(\tau)\, d\xi\, d\tau - \int_0^{\infty} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} g'(\tau)\, d\xi\, d\tau \right)$$

$\boxed{x \geq 0}$

$$u(t,x) = v(t,x) + g(t)$$

$$= v(t,x) + \int_0^t g'(\tau)\, d\tau \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}}\, d\xi}_{=1}$$

dalje parcijalna integracija ⋮

ZAD. 2 Napišite eksplicitnu ~~rj.~~ formulu za rj. početne zadaće:

$$\begin{cases} u_t - \Delta u + cu = f & u \ \langle 0, \infty\rangle \times \mathbb{R}^d \\ u = g & \text{na} \ \{t=0\} \times \mathbb{R}^d \end{cases}$$

$c \in \mathbb{R}$.

Rj:

$$v(t,x) = u(t,x)\, e^{ct}$$

$$\Rightarrow \quad v_t = u_t e^{ct} + u e^{ct} c$$

$$v_{x_i x_i} = u_{x_i x_i} e^{ct}$$

$$\Rightarrow \quad v_t - \Delta v = e^{ct} f$$

$$\Rightarrow \quad v \text{ je rj. } \begin{cases} v_t - \Delta v = e^{ct} f & u \ \langle 0, \infty\rangle \times \mathbb{R}^d \\ v = g & \text{na} \ \{t=0\} \times \mathbb{R}^d \end{cases}$$

$$\Rightarrow \quad v(t,x) = \int_{\mathbb{R}^d} \Phi(x-y,t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^d} \Phi(x-y, t-\tau)\, e^{c\tau} f(y,\tau)\, dy\, d\tau$$

fundamentalno rj.

$$\Rightarrow \quad u(t,x) = e^{-ct} \left(\qquad \right)$$

~~4.7~~ Nađite eksplicitnu formulu za rješenje Cauchyjeve zadaće
ZAD. 3. u $\mathbb{R}^+ \times \mathbb{R}$:

$$\begin{cases} u_t - k u_{xx} + b u_x + c u = 0 \\ u(0,\cdot) = g \,; \end{cases}$$

gdje su $k>0$, $b, c \in \mathbb{R}$ konst.

Pokažite da za $c>0$ i g omeđenu imamo $u(t,x) \to 0$ kad $t \to +\infty$.

Rj. Uvodimo supstituciju $v(t,x) = u(t,x)\, e^{dt+\sigma x}$, pri čemu ćemo odrediti d i σ t.d. $v_t - k v_{xx} = 0$.

$$v_t = u_t e^{dt+\sigma x} + u e^{dt+\sigma x} d$$

$$v_{xx} = u_{xx} e^{dt+\sigma x} + 2u_x e^{dt+\sigma x} \sigma + u e^{dt+\sigma x} \sigma^2$$

$$\Rightarrow v_t - k v_{xx} = \Big(\underbrace{u_t - k u_{xx} - 2\sigma k u_x + (d - \sigma^2 k) u}_{\text{želimo da je ovo nula pa namještamo koef.}}\Big) e^{dt+\sigma x}$$

- $-2\sigma k = b \Rightarrow \boxed{\sigma = -\dfrac{b}{2k}}$
- $d - \sigma^2 k = c \Rightarrow \boxed{d = c + \dfrac{b^2}{4k}}$

$\Rightarrow$ za takve d i σ imamo $\begin{cases} v_t - k v_{xx} = 0 \\ v(0,x) = u(0,x)\, e^{\sigma x} = g(x)\, e^{\sigma x} \end{cases}$

$$\Rightarrow v(t,x) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4kt}} e^{\sigma\xi} g(\xi)\, d\xi$$

$$\Rightarrow u(t,x) = \frac{1}{\sqrt{4\pi k t}}\, e^{-dt-\sigma x} \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4kt}} e^{\sigma\xi} g(\xi)\, d\xi$$

$$\Rightarrow \quad u(t,x) = \frac{1}{\sqrt{4\pi k t}}\, e^{-dt} e^{-\sigma x} \int_{-\infty}^{+\infty} e^{-\frac{\xi^2 - 2(x+2kt\sigma)\xi + x^2}{4kt}}\, g(\xi)\, d\xi$$

$x^2 + 4k^2t^2\sigma^2 + 4xkt\sigma$

$$= \frac{1}{\sqrt{4\pi k t}}\, e^{-dt} e^{-\sigma x} \int_{-\infty}^{+\infty} e^{-\frac{(\xi-(x+2kt\sigma))^2}{4kt}}\, e^{x\sigma + kt\sigma^2}\, g(\xi)\, d\xi$$

$$= \frac{\cancel{e^{\sigma x}}}{\sqrt{4\pi k t}}\, e^{-\underbrace{(d-k\sigma^2)}_{=c}t}\, \cancel{e^{-\sigma x}} \int_{-\infty}^{+\infty} e^{-\frac{(\xi-(x+2kt\sigma))^2}{4kt}}\, g(\xi)\, d\xi$$

$$= \frac{\cancel{e^{\sigma x}}}{\sqrt{4\pi k t}}\, e^{-ct}\, \cancel{e^{-\sigma x}} \int_{-\infty}^{+\infty} e^{-\frac{(\xi-(x+2kt\sigma))^2}{4kt}}\, g(\xi)\, d\xi$$

$$\underset{(\text{za fiksni } x)}{\Longrightarrow} \quad |u(t,x)| \leq \cancel{e^{\sigma x}} e^{-ct}\, \cancel{e^{-\sigma x}}\, \|g\|_{L^\infty}\, \underbrace{\frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{+\infty} e^{-\frac{(\xi-(x+2kt\sigma))^2}{4kt}}\, d\xi}_{=1}$$

$$= e^{-ct}\, \cancel{e^{\sigma x}}\, \cancel{e^{-\sigma x}}\, \|g\|_{L^\infty}$$

Budući da je $c>0$ & $\|g\|_{L^\infty} < \infty$ imamo,

$$|u(t,x)| \xrightarrow[t\to\infty]{} 0 .$$

VALNA JEDNADŽBA

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{u} \quad \langle 0,\infty\rangle \times \mathbb{R}^d \\ u(0,\cdot) = g \\ u_t(0,\cdot) = h \end{cases}$$

već je i Cauchyjeva zadaća sama po sebi komplicirana tako da ćemo uglavnom samo nju riješiti

Za razliku od Poissonove j. i jednadžbe provođenja, nemamo formulu za rješenje neovisnu o dimenziji, a i sama formula je nešto kompliciranija.

$\boxed{1D}$ D'Alambertova formula

$$u(t,x) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi$$

radilo se u MMF-u

$\boxed{2D}$ Poissonova formula

$$u(t,x) = \frac{1}{2} \fint_{K(x,t)} \frac{t g(y) + t^2 h(y) + t\nabla g(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}}\,dy$$

$\boxed{3D}$ Kirchhoffova formula

$$u(t,x) = \fint_{S(x,t)} t h(y) + g(y) + \nabla g(y)\cdot(y-x)\; dS_y$$

Nehomogenu zadaću rješavamo Duhamelovim načelom.
Za zadaću

$$\begin{cases} u_{tt} - \Delta u = f \\ u(0,\cdot) = 0 \\ u_t(0,\cdot) = 0 \end{cases}, \quad \text{u } \langle 0,\infty\rangle \times \mathbb{R}^d$$

rješenje je dano s $\quad u(t,x) = \int_0^t v(t,x;s)\,ds$, pri čemu je $v(\cdot,\cdot;s)$ rješenje zadaće

$$\begin{cases} v_{tt}(\cdot;s) - \Delta v(\cdot;s) = 0 & \text{u } \langle 0,\infty\rangle \times \mathbb{R}^d \\ v(s,\cdot;s) = 0 \\ v_t(s,\cdot;s) = f(\cdot,s) \end{cases}$$

ZAD. 1. Riješite sljedeću Cauchyjevu zadaću

$$\begin{cases} u_{tt} - \Delta u = 2 & u \quad \langle 0,\infty\rangle \times \mathbb{R}^2 \\ u(0,x_1,x_2) = x_1 \\ u_t(0,x_1,x_2) = x_2 \end{cases}$$

Rj: Rastavit ćemo problem na dva:

$$① \begin{cases} u_{tt} - \Delta u = 2 \\ u(0,x_1,x_2) = 0 \\ u_t(0,x_1,x_2) = 0 \end{cases} \qquad ② \begin{cases} u_{tt} - \Delta u = 0 \\ u(0,x_1,x_2) = x_1 \\ u_t(0,x_1,x_2) = x_2 \end{cases},$$

a konačno je rješenje dano kao zbroj rješenja problema ① i ②.

① Koristit ćemo Duhamelovo načelo pa rješimo najprije

$$\begin{cases} v_{tt} - \Delta v = 0 \\ v(0,x) = 0 \\ v_t(0,x) = 2 \end{cases}$$

$$\Rightarrow v(t,x) = \frac{1}{2}\int_{K(x,t)} \frac{2t^2}{(t^2-|y-x|^2)^{1/2}}\,dy = \frac{1}{\pi}\int_{K(x,t)} \frac{dy}{(t^2-|y-x|^2)^{1/2}}$$

zamjena varijabli

$$= \frac{1}{\pi}\int_{K(0,t)} \frac{dy}{(t^2-|y|^2)^{1/2}} = \frac{1}{\pi}\int_0^t\int_{S(0,r)} \frac{dS_y\,dr}{(t^2-r^2)^{1/2}}$$

$$= \frac{1}{\pi}\int_0^t 2\pi r\,\frac{1}{(t^2-r^2)^{1/2}}\,dr = \left\{ \begin{array}{l} s = t^2-r^2 \\ ds = -2r\,dr \end{array} \right\}$$

$$= -\int_{t^2}^0 \frac{ds}{\sqrt{s}} = \int_0^{t^2}\frac{ds}{\sqrt{s}} = 2\sqrt{s}\,\Big|_0^{t^2} = 2t$$

$$\Rightarrow v(t,x_1,x_2;s) = 2(t-s)$$

u Duhamelovom načelu je rješenje translatirano za s po t

$$\Rightarrow u_1(t,x) = \int_0^t v(t,x;s)\,ds = 2\int_0^t (t-s)\,ds = 2\left(ts - \frac{s^2}{2}\right)\Big|_0^t = t^2$$

rješenje problema ①

② Po Poissonovoj formuli imamo

$$u(t,x) = \frac{1}{2} \int_{K(x,t)} \frac{t y_1 + t^2 y_2 + t(y_1 - x_1)}{(t^2 - |y - x|^2)^{1/2}} \, dy$$

(POLARNE KOORDINATE:
$$\left.\begin{aligned} y_1 &= r\cos\varphi + x_1 \\ y_2 &= r\sin\varphi + x_2 \end{aligned}\right\} \text{Jacobijan je } r$$
)

$$= \frac{1}{2\pi t^2} \int_0^t \int_0^{2\pi} r \, \frac{tr\cos\varphi + tx_1 + t^2 r\sin\varphi + t^2 x_2 + tr\cos\varphi + \cancel{tx_1} - \cancel{tx_1}}{(t^2 - r^2)^{1/2}} \, d\varphi \, dr$$

$$= \frac{1}{2\pi t^2} \int_0^t \frac{tr}{(t^2 - r^2)^{1/2}} \int_0^{2\pi} (\underbrace{2r\cos\varphi}_{=0} + \underbrace{tr\sin\varphi}_{=0} + x_1 + t x_2) \, d\varphi \, dr$$

$$= \frac{1}{2\pi t^2} \int_0^t \frac{2\pi t r}{(t^2 - r^2)^{1/2}} (x_1 + t x_2) \, dr$$

$$= \frac{x_1 + t x_2}{t} \underbrace{\int_0^t \frac{r}{(t^2 - r^2)^{1/2}} \, dr}_{= t \text{ (iz ① jer smo tamo to računali)}} = x_1 + t x_2$$

$$\Rightarrow \quad u_2(t,x) = x_1 + t x_2 \quad \rightsquigarrow \text{rješenje problema ②}$$

$$\Rightarrow \boxed{u(t,x) = x_1 + t x_2 + t^2}$$

② Po Poissonovoj formuli imamo

$$u(t,x) = \frac{1}{2} \fint_{K(x,t)} \frac{t y_1 + t^2 y_2 + t(y_1 - x_1)}{(t^2 - |y - x|^2)^{1/2}} \, dy$$

$$\left(\begin{array}{l} \text{POLARNE KOORDINATE:} \\ \left.\begin{array}{l} y_1 = r\cos\varphi + x_1 \\ y_2 = r\sin\varphi + x_2 \end{array}\right\} \text{Jacobijan je } r \end{array} \right)$$

$$= \frac{1}{2\pi t^2} \int_0^t \int_0^{2\pi} r \, \frac{tr\cos\varphi + tx_1 + t^2 r\sin\varphi + t^2 x_2 + tr\cos\varphi + \cancel{tx_1} - \cancel{tx_1}}{(t^2 - r^2)^{1/2}} \, d\varphi \, dr$$

$$= \frac{1}{2\pi t^2} \int_0^t \frac{tr}{(t^2 - r^2)^{1/2}} \int_0^{2\pi} \Big(\underbrace{2r\cos\varphi}_{=0} + \underbrace{tr\sin\varphi}_{=0} + x_1 + tx_2 \Big) \, d\varphi \; dr$$

$$= \frac{1}{2\pi t^2} \int_0^t \frac{2\pi t r}{(t^2 - r^2)^{1/2}} (x_1 + tx_2) \, dr$$

$$= \frac{x_1 + tx_2}{t} \underbrace{\int_0^t \frac{r}{(t^2 - r^2)^{1/2}} \, dr}_{= t \text{ (iz ① jer smo tamo to računali)}} = x_1 + tx_2$$

$\Rightarrow \; u_2(t,x) = x_1 + tx_2 \;\rightsquigarrow$ rješenje problema ②

$$\Rightarrow \boxed{u(t,x) = x_1 + tx_2 + t^2}$$

ZAD. 2. Ako su g, h i f_2 harmoničke f-je na $\mathbb{R}^d$, a f_1 klase C^1 na $\langle 0, \infty \rangle$, pokažite da je jedinstveno rj. Cauchyjeve zadaće

(ostaviti možda za zadaću)

$$\begin{cases} u_{tt} - \Delta u = f_1(t) f_2(x) \\ u(0,\cdot) = g \\ u(0,\cdot) = h \end{cases}$$

dano s

$$u(t,x) = g(x) + th(x) + f_2(x) \int_0^t (t - \tau) f_1(\tau) \, d\tau .$$

Po ZAD. 2. sada lako možemo potvrditi rezultat ZAD. 1. Naime, $g(x_1,x_2) = x_1$, $h(x_1,x_2) = x_2$ i $f_2(x_1,x_2) = 2$ su očito harmoničke, a $f_1(x_1,x_2) = 1$ je klase C^1

$$\Rightarrow \; u(t,x) = x_1 + tx_2 + 2\int_0^t (t - \tau) \, d\tau = x_1 + tx_2 + t^2$$