METODA REFLEKSIJE

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{na } \mathbb{R}^+ \times \langle 0, l\rangle \\ u(t,0) = u(t,l) = 0 & , \ t \in \mathbb{R}^+ \\ u(0,x) = u_0(x) & \\ u_t(0,x) = u_1(x) & , \ x \in [0,l] \end{cases}$$

Na vježbama smo dobili:

$$(1) \quad u(t,x) = F(x+ct) + G(x-ct) \ , \quad F \in C^2(\langle 0, +\infty\rangle), \quad G \in C^2(\langle -\infty, l\rangle)$$

$$(2) \quad G(z) = -F(-z) \ , \quad z \in \langle -\infty, 0\rangle$$
$$F(z) = -G(2l - z) \ , \quad z \in \langle l, +\infty \rangle$$

$$(3) \quad F(x) = \frac{1}{2} u_0(x) + \frac{1}{2c}\int_0^x u_1(\xi)\,d\xi + D$$
$$G(x) = \frac{1}{2} u_0(x) - \frac{1}{2c}\int_0^x u_1(\xi)\,d\xi - D$$

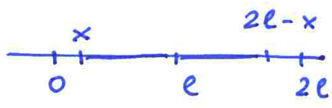
Iz (2) & (3) slijedi:

$$(4) \quad \frac{1}{2}\left(\tilde{u}_0(x) + \tilde{u}_0(-x)\right) - \frac{1}{2c}\int_{-x}^{x} u_1(\xi)\,d\xi = 0 \ ,$$

$$(5) \quad \frac{1}{2}\left(\tilde{u}_0(x) + \tilde{u}_0(2l-x)\right) - \frac{1}{2c}\int_{x}^{2l-x} u_1(\xi)\,d\xi = 0 \ ,$$

gdje su $\tilde{u}_0$ i $\tilde{u}_1$ proširenja redom od u_0 i u_1 koje su definirane na $[0,l]$.

Ako je
$$\tilde{u}_0(x) = -\tilde{u}_0(-x)$$
$$\tilde{u}_1(x) = -\tilde{u}_1(-x) \ , \ \forall x \ ,$$
tj. ako su $\tilde{u}_0$ i $\tilde{u}_1$ neparne, onda je (4) zadovoljeno.



Ako je
$$\tilde{u}_0(x) = -\tilde{u}_0(2l-x)$$
$$\tilde{u}_1(x) = -\tilde{u}_1(2l-x) \ , \ \forall x \ ,$$
tj. ako su $\tilde{u}_0$ i $\tilde{u}_1$ neparne s obzirom na l (l je polovište intervala $\langle x, 2l-x\rangle$),

onda je (5) zadovoljeno.

Da bismo imali egzistenciju i jedinstvenost rješenja, f-je $\tilde{u}_0$ i $\tilde{u}_1$ moraju zadovoljavati izvedeni uvjet kompatibilnosti:

$$\begin{cases} \tilde{u}_0(l) = \tilde{u}_1(l) = \tilde{u}_0''(l) = 0 \\ \tilde{u}_0(0) = \tilde{u}_1(0) = \tilde{u}_0''(0) = 0 \end{cases}$$

Pokažimo da su tako definirane f-je periodičke s periodom $2l$:

$$\tilde{u}_0(2l+x) = -\tilde{u}_0(2l-(2l+x)) = -\tilde{u}_0(-x) = \tilde{u}_0(x)$$

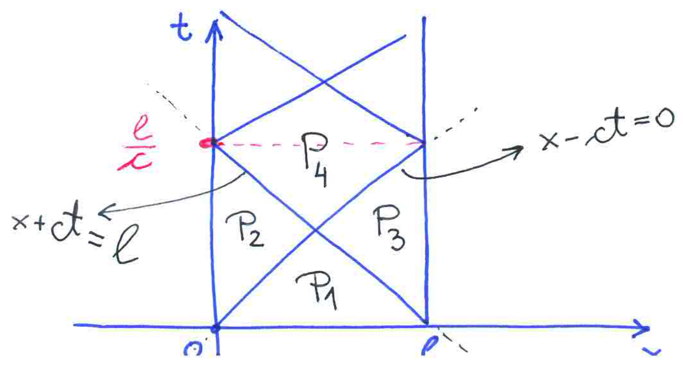
Kod rješavanja zadataka jednostavnije nam je gledati proširenja od u_0 i u_1, nego proširenja od F i G.

Rješenje u je sada dano formulom:

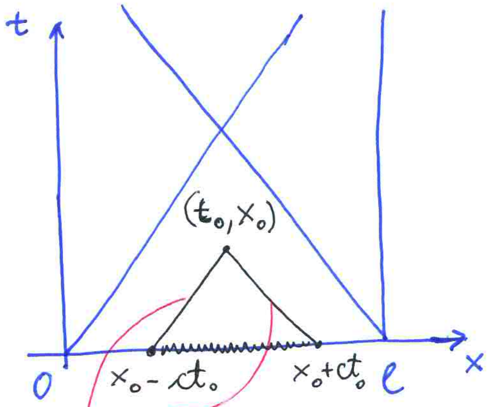
$$u(t,x) = \frac{\tilde{u}_0(x+ct) + \tilde{u}_0(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{u}_1(\xi)\,d\xi \qquad (6)$$

(usporedite da smo dobili istu stvar kao u slučaju s jednim rubnim uvjetom, samo što su proširenja $\tilde{u}_0$ i $\tilde{u}_1$ drugačija)

Sada ćemo geometrijski odrediti kako <u>točno</u> izgleda rješenje za pojedine točke (t,x) (zapisat ćemo u koristeći u_0 i u_1).



① $(t_0, x_0) \in P_1$



u ovom slučaju je

$$x_0 - ct_0 \geq 0$$
$$x_0 + ct_0 \leq \ell \quad ,$$

tj. $\quad x_0 - ct_0 \,,\, x_0 + ct_0 \in [0, \ell]$

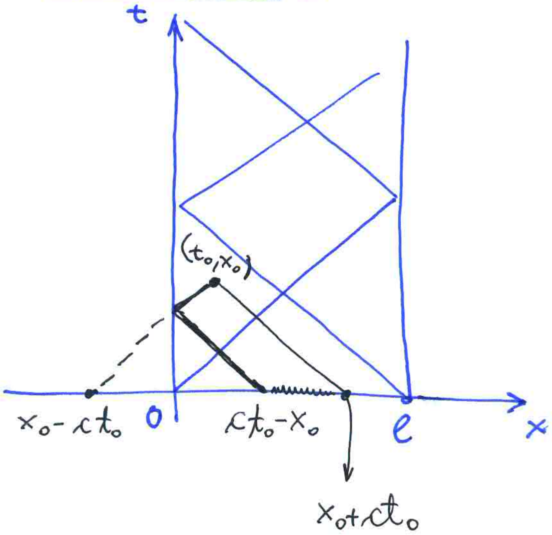
$$\Rightarrow \quad \tilde{u}_0(x_0 - ct_0) = u_0(x_0 - ct_0)$$
$$\tilde{u}_0(x_0 + ct_0) = u_0(x_0 + ct_0)$$
$$\tilde{u}_1(x_0 - ct_0) = u_1(x_0 - ct_0)$$
$$\tilde{u}_1(x_0 + ct_0) = u_1(x_0 + ct_0)$$

KARAKTERISTIKE KROZ TOČKU (t_0,x_0):

$$x - ct = x_0 - ct_0$$
$$x + ct = x_0 + ct_0$$

pa kad uvrstimo u (6) dobijemo formulu za u.

② $(t_0, x_0) \in P_2$



u ovom slučaju je $x_0 + ct_0 \in [0, \ell]$, ali $x_0 - ct_0 \in [-\ell, 0]$ pa onda imamo:

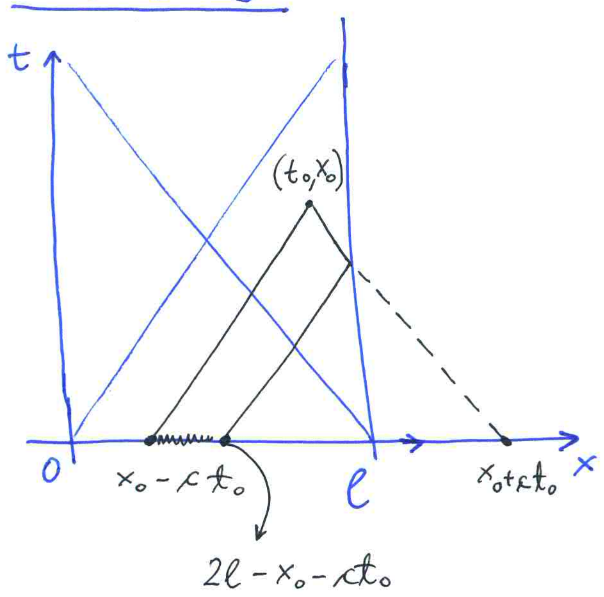
$$\tilde{u}_0(x_0 - ct_0) = -\tilde{u}_0(ct_0 - x_0)$$
$$= -u_0(\underset{\underset{[0,\ell]}{\cap}}{ct_0 - x_0})$$

Analogno napravimo i za u_1 pa dobivamo:

$$u(t_0,x_0) = \frac{u_0(x_0+ct_0) - u_0(ct_0 - x_0)}{2} + \frac{1}{2c}\int_{ct_0 - x_0}^{x_0+ct_0} u_1(\xi)\, d\xi$$

integral $\displaystyle\int_{x_0-ct_0}^{ct_0-x_0} \tilde{u}_1(\xi)\, d\xi = 0$ jer je $\tilde{u}_1$ neparna.

③ $(t_0, x_0) \in P_3$



$x_0 - ct_0 \in [0,\ell]$, ali

$x_0 + ct_0 > \ell$ & $x_0 + ct_0 < 2\ell$

pa onda imamo $2\ell - x_0 - ct_0 \in [0,\ell]$

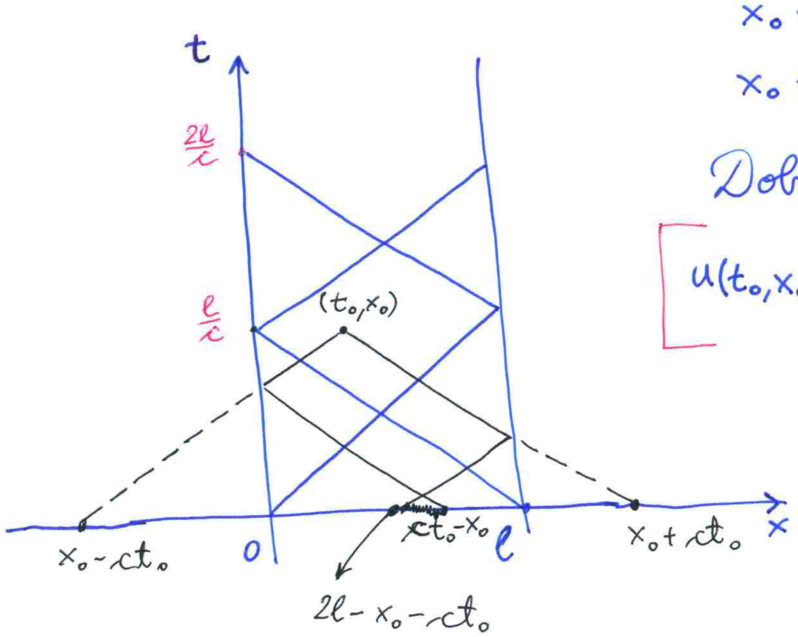
i :
$$\tilde{u}_0(x_0 + ct_0) = -\tilde{u}_0(2\ell - x_0 - ct_0)$$
$$= -u_0(2\ell - x_0 - ct_0)$$

Također je
$$\int_{2\ell - x_0 - ct_0}^{x_0 + ct_0} \tilde{u}_1(\xi) = 0 \quad \text{jer je}$$

$\tilde{u}_1$ neparna s obzirom na ℓ.

$$u(t_0, x_0) = \frac{-u_0(2\ell - x_0 - ct_0) + u_0(x_0 - ct_0)}{2} + \frac{1}{2c}\int_{x_0 - ct_0}^{2\ell - x_0 - ct_0} u_1(\xi)\, d\xi .$$

④ $(t_0, x_0) \in P_4$



$x_0 - ct_0 \in [-\ell, 0] \Rightarrow ct_0 - x_0 \in [0,\ell]$

$x_0 + ct_0 \in [\ell, 2\ell] \Rightarrow 2\ell - x_0 - ct_0 \in [0,\ell]$

Dobivamo :

$$u(t_0, x_0) = \frac{-u_0(2\ell - x_0 - ct_0) - u_0(ct_0 - x_0)}{2} + \frac{1}{2c}\int_{ct_0 - x_0}^{2\ell - x_0 - ct_0} u_1(\xi)\, d\xi$$

Postavlja se pitanje možemo li na temelju prvih nekoliko četverokuta P_i zaključiti što se događa sa svima? Pozitivan odgovor bismo dobili u slučaju da je $\tilde{u}$ periodička s obzirom na t.

($\tilde{u}$ je proširenje od u, tj. $u(t,x) = \tilde{u}(t,x)$ za $t \in \mathbb{R}^+, x \in [0,\ell]$)

$$\tilde{u}\left(t + \frac{2\ell}{c}, x\right) = \frac{\tilde{u}_0(x+ct+2\ell) + \tilde{u}_0(x-ct-2\ell)}{2} + \frac{1}{2c}\int_{x-ct-2\ell}^{x+ct+2\ell} \tilde{u}_1(\xi)\,d\xi$$

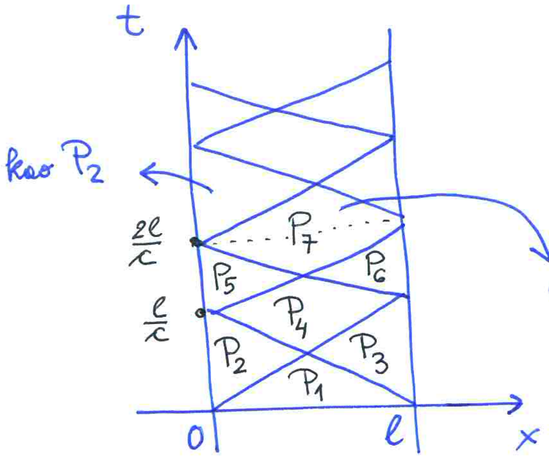
$$= \frac{-\tilde{u}_0(-x-ct) - \tilde{u}_0(2\ell-x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{u}_1(\xi)\,d\xi$$

(preostala dva integrala su nula jer je $\tilde{u}_1$ neparna f-ja perioda 2ℓ, a duljina intervala po kojem integriramo je jednaka periodu 2ℓ)

$$= \frac{\tilde{u}_0(x+ct) + \tilde{u}_0(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{u}_1(\xi)\,d\xi$$

$$= \tilde{u}(t,x)$$

$\Rightarrow$ rješenje je periodičko s obzirom na t, a period je $\frac{2\ell}{c}$ (možda to nije temeljni period, ali nam to nije bitno)



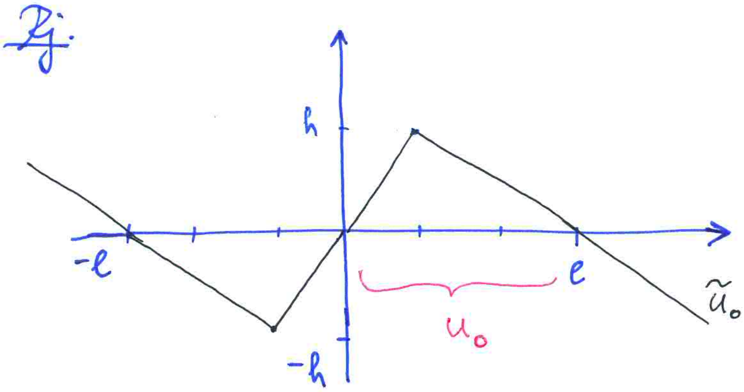
Zaključujemo da formule na $P_1, \dots, P_6$ određuju cijelo rješenje.

ovdje je opet formula ista kao na P_1 (na cijelom P_7 : POKAŽITE!)

Kod rješavanja zadataka je cilj problem dovesti u $t \in [0, \frac{2l}{c}]$, gdje smo izveli formule rješenja. Formule nije potrebno pamtiti napamet jer se lako izvode.

ZAD. Žica duljine l učvršćena je na krajevima $x=0$ i $x=l$. U času $t=0$ pomaknuta je u točki $x=\frac{l}{3}$ na visinu h iz ravnotežnog položaja i potom ispuštena. Naći oblik žice za $t \in [0, \frac{l}{3c}]$.

Rj.



– u početnom trenutku je brzina 0 (krećemo od stacionarnog položaja)

$$u_0(x) = \begin{cases} \frac{3h}{l}x & , x \in [0, \frac{l}{3}] \\ -\frac{3h}{2l}x + \frac{3h}{2} & , x \in [\frac{l}{3}, l] \end{cases}$$

u_0 je neprekidna i po dijelovima klase C^2 (u stvarnosti se ne dogodi šiljak, ali mi tako aproksimiramo i tako rješavamo zadatak) pa možemo provjeriti UVJETE KOMPATIBILNOSTI koji su trivijalno zadovoljeni.

Budući da gledamo rješenje samo za $t \in [0, \frac{l}{3c}]$, nalazimo se samo u tri slučaja (P_1, P_2, P_3).

Nađimo kako izgleda u u pojedinim slučajevima:

(P_1) $u(t,x) = \frac{1}{2}\left(u_0(x+ct) + u_0(x-ct)\right)$

Neka je t fiksan.

a) $x - ct \leq \frac{l}{3} \iff x \leq \frac{l}{3} + ct$

$x + ct \leq \frac{l}{3} \iff x \leq \frac{l}{3} - ct$

$x - ct \geq 0 \iff x \geq ct$

$x + ct \leq l \iff x \leq l - ct$

$ct = \frac{l}{3} - ct$

$\Rightarrow \quad t = \frac{l}{6c}$

$\boxed{t < \frac{l}{6c}}$ $\quad x \in [ct, \frac{l}{3} - ct]$

$$u(t,x) = \frac{1}{2}\left(\frac{3h}{l}(x+ct) + \frac{3h}{l}(x-ct)\right) = \frac{3h}{l}x$$

$x \in [\frac{l}{3} - ct, \frac{l}{3} + ct]$

$$\begin{aligned} u(t,x) &= \frac{1}{2}\left(\frac{3h}{l}(x+ct) - \frac{3h}{2l}(x-ct) + \frac{3h}{2}\right) \\ &= \frac{1}{2}\left(\frac{3h}{2l}x + \frac{9h}{2l}ct + \frac{3h}{2}\right) \\ &= \frac{3h}{4l}x + \frac{9h}{4l}ct + \frac{3h}{4} \end{aligned}$$

$x \in [\frac{l}{3} + ct, l - ct]$

$$\begin{aligned} u(t,x) &= \frac{1}{2}\left(-\frac{3h}{2l}(x+ct) + \frac{3h}{2} - \frac{3h}{2l}(x-ct) + \frac{3h}{2}\right) \\ &= \frac{1}{2}\left(-\frac{3h}{l}x + \frac{3h}{1}\right) \\ &= -\frac{3h}{2l}x + \frac{3h}{2} \end{aligned}$$

Za $x \in [0, ct]$ nalazimo se u P_2, a za $x \in [\ell - ct, \ell]$ nalazimo se u P_3.

Sada treba dalje ta dva slučaja rastaviti na podslučajeve da znamo koju formulu uvrstiti za u_0 i onda bismo dobili formulu za $u(t, \cdot)$, $t \in [0, \frac{\ell}{6c}]$.

Ako je $t \geq \frac{\ell}{6c}$, onda je $\frac{\ell}{3} - ct \leq ct$ pa se mijenja poredak slučajeva i zato se treba sve iz početka raditi.

...

Na kolokviju će biti zadane konkretne vrijednosti za t tako da će biti lakše.

ZAD. (realni :))

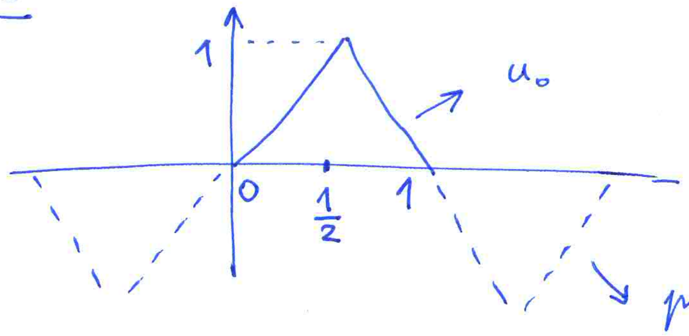
$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = \begin{cases} 2x & , x \in [0, \frac{1}{2}] \\ 2 - 2x & , x \in [\frac{1}{2}, 1] \end{cases} \\ u_t(0,x) = 0 \end{cases}$$

skicirati rješenje u trenucima $t = 1, t = \frac{3}{2}, t = 2, t = \frac{7}{2}$.

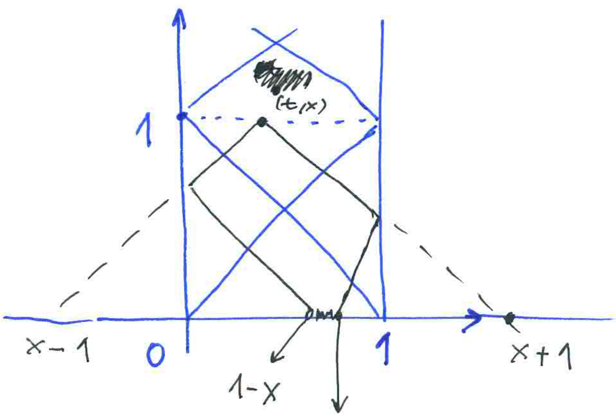
Rj:

$\left.\begin{matrix} c = 1 \\ \ell = 1 \end{matrix}\right\} \Rightarrow$ period od u je 2

$t = 0$



$t = 1$

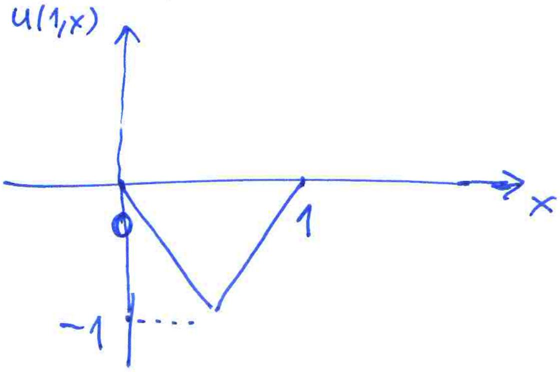


$$u(1,x) = \frac{1}{2}\left(\tilde{u}_0(x+1) + \tilde{u}_0(x-1)\right)$$

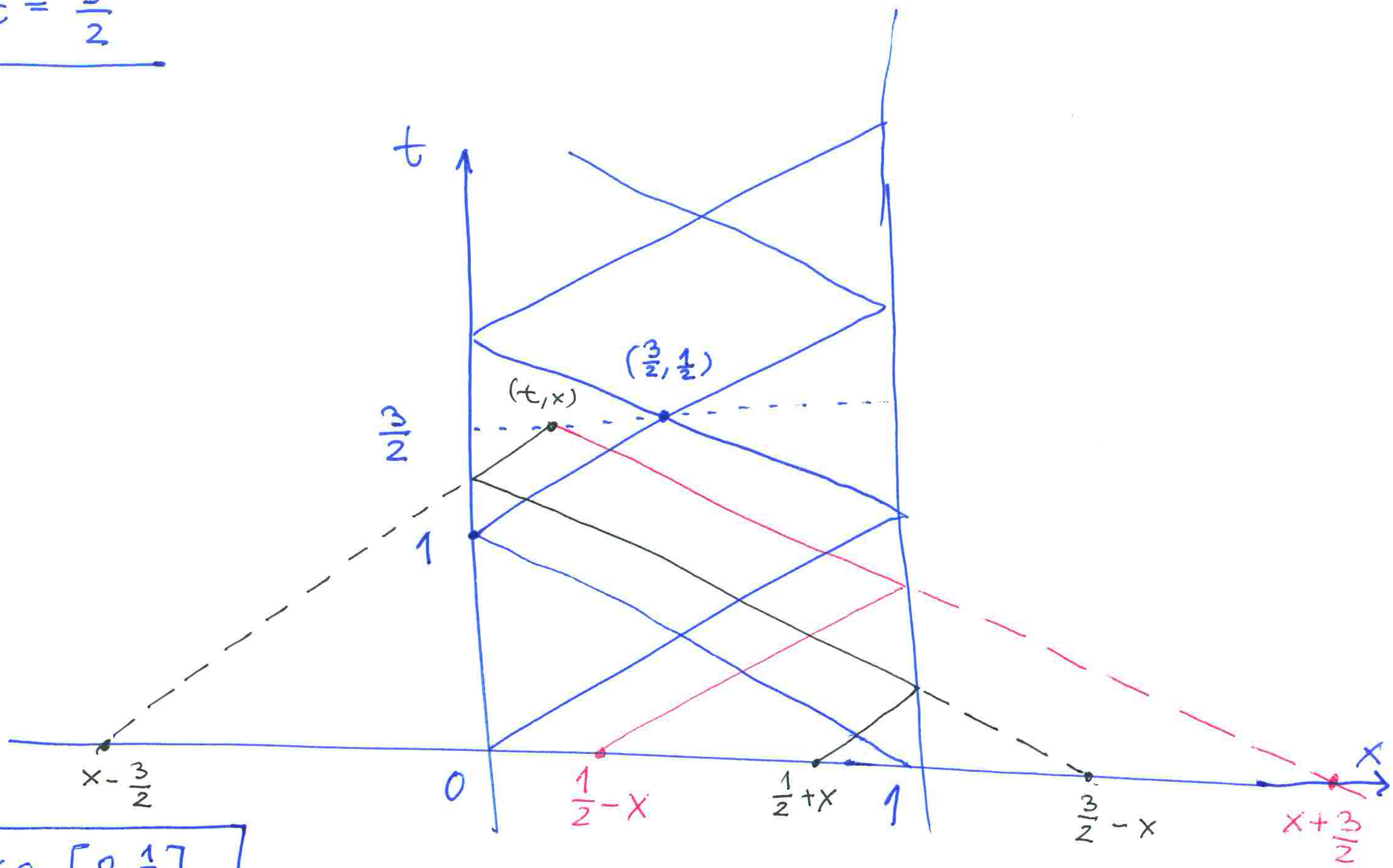
$$= \frac{1}{2}\left(-u_0(1-x) - u_0(1-x)\right)$$

$$= -u_0(1-x)$$

$$u(1,x) = \begin{cases} -2x & , x \in [0, \frac{1}{2}] \\ -2 + 2x & , x \in [\frac{1}{2}, 1] \end{cases}$$



$\underline{t = \frac{3}{2}}$



$\boxed{x \in [0,\frac{1}{2}]}$

$$u(\tfrac{3}{2},x) = \frac{1}{2}\left(\tilde{u}_0(x+\tfrac{3}{2}) + \tilde{u}_0(x-\tfrac{3}{2})\right)$$
$$= \frac{1}{2}\left(-u_0(\tfrac{1}{2}-x) + u_0(\tfrac{1}{2}+x)\right)$$

AKO IMAMO NEPARAN BROJ REFLEKSIJA BIT ĆE PREDZNAK <u>MINUS</u>, A ZA PARAN BROJ <u>PLUS</u>.

$$x \in [0,\tfrac{1}{2}]$$
$$\Rightarrow -x \in [-\tfrac{1}{2},0]$$
$$\Rightarrow \tfrac{1}{2}-x \in [0,\tfrac{1}{2}]$$

$$x \in [0,\tfrac{1}{2}]$$
$$\Rightarrow \tfrac{1}{2}+x \in [\tfrac{1}{2},1]$$

$$u(\tfrac{3}{2},x) = \frac{1}{2}\left(-2(\tfrac{1}{2}-x) + 2 - 2(\tfrac{1}{2}+x)\right)$$
$$= -\tfrac{1}{2}+x+1-\tfrac{1}{2}-x$$
$$= 0$$

Analogno $\boxed{x \in [\tfrac{1}{2}, 1]}$

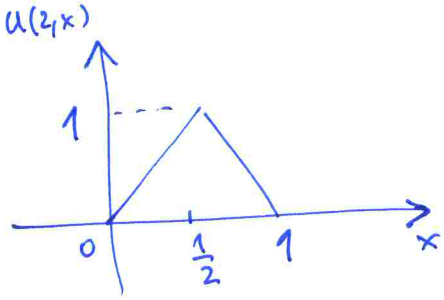
$$u(\tfrac{3}{2},x) = \tfrac{1}{2}\left(u_0(x-\tfrac{1}{2}) - u_0(\tfrac{3}{2}-x)\right)$$
$$= \tfrac{1}{2}\left(2(x-\tfrac{1}{2}) - 2 + 2(\tfrac{3}{2}-x)\right)$$
$$= x - \tfrac{1}{2} - 1 + \tfrac{3}{2} - x$$
$$= 0$$

$$\Rightarrow \quad \boxed{u(\tfrac{3}{2},x) = 0 \quad , \quad x \in [0,1]}$$

$\underline{t=2}$

$$u(2,x) \underset{\uparrow}{=} u(0,x) = u_0(x) = \begin{cases} 2x & , x \in [0,\tfrac{1}{2}] \\ 2-2x & , x \in [\tfrac{1}{2},0] \end{cases}$$

PERIODIČNOST



$\underline{t=\tfrac{7}{2}}$

$$u(\tfrac{7}{2},x) = u(\tfrac{3}{2},x) \quad \checkmark$$