① ZAMJENA VARIJABLE U PLOŠNOM INTEGRALU

$$\int_{S(0,1)} u(x+rw)\, dS_w = \int_A u(x+r\varphi(y))\sqrt{\det \nabla\varphi(y)^T \nabla\varphi(y)}\, dy$$

$\tilde{\varphi}(y) := \psi(\varphi(y))$, $\psi(z) = x + rz$

$\varphi, \tilde{\varphi} : A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^{m-1}$, $\psi : \mathbb{R}^m \to \mathbb{R}^m$

$$\underset{m\times(m-1)}{\nabla\tilde{\varphi}(y)} = \underset{m\times m}{\nabla\psi(\varphi(y))}\ \underset{m\times(m-1)}{\nabla\varphi(y)}$$

$$\Rightarrow \nabla\tilde{\varphi}(y)^T \nabla\tilde{\varphi}(y) = \nabla\varphi(y)^T \nabla\psi(\varphi(y))^T \nabla\psi(\varphi(y))\nabla\varphi(y)$$

$$\left\{ \nabla\psi(\varphi(y))^T = \nabla\psi(\varphi(y)) = rI \right.$$

$$= r^2 \nabla\varphi(y)^T \nabla\varphi(y)$$

$$\Rightarrow \det(\nabla\tilde{\varphi}(y)^T \nabla\tilde{\varphi}(y)) = r^{2(m-1)} \det(\nabla\varphi(y)^T \nabla\varphi(y)) \quad /\sqrt{\ }$$

$$\Rightarrow \sqrt{\det \nabla\tilde{\varphi}(y)^T \nabla\tilde{\varphi}(y)} = r^{m-1}\sqrt{\det \nabla\varphi(y)^T \nabla\varphi(y)}$$

$$\Rightarrow \sqrt{\det \nabla\varphi(y)^T \nabla\varphi(y)} = \frac{1}{r^{m-1}}\sqrt{\det \nabla\tilde{\varphi}(y)^T \nabla\tilde{\varphi}(y)}$$

$$= \int_A u(\tilde{\varphi}(y)) \frac{1}{r^{m-1}}\sqrt{\det \nabla\tilde{\varphi}(y)^T \nabla\tilde{\varphi}(y)}\, dy$$

$$= \frac{1}{r^{m-1}} \int_{\tilde{\varphi}} u$$

$$= \frac{1}{r^{m-1}} \int_{S(x,r)} u(y)\, dS_y$$ jer je $\tilde{\varphi}$ parametrizacija od $S(x,r)$

② POLARNE KOORDINATE

TVRDNJA: $$\int_{K[0,r]} f(x)\,dx = \int_0^r \int_{S(0,s)} f(y)\,dS_y\,ds$$

Dz.

Neka je $\varphi : A \subseteq \mathbb{R}^{n-1} \to \mathbb{R}^n$ parametrizacija jedinične sfere $S(0,1)$.

Definirajmo $\psi : [0,r]\times A \to \mathbb{R}^n$
$$\psi(s,x') = s\varphi(x')$$

Očito je $\psi([0,r]\times A) = K[0,r]$ i injekcija pa se radi o dobroj zamjeni varijable.

$$\nabla\psi(s,x') = \begin{bmatrix} \varphi_1(x') & s\varphi_{1,x'_1}(x') & \cdots & s\varphi_{1,x'_{n-1}}(x') \\ \varphi_2(x') & s\varphi_{2,x'_1}(x') & \cdots & s\varphi_{2,x'_{n-1}}(x') \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_n(x') & s\varphi_{n,x'_1}(x') & \cdots & s\varphi_{n,x'_{n-1}}(x') \end{bmatrix}_{n\times n}$$

$$|\det \nabla\psi| = \sqrt{\det \nabla\psi^T \nabla\psi}$$

$$\nabla\psi(s,x')^T \nabla\psi(s,x') = \left[\begin{array}{c|ccc} |\varphi(x')|^2 & s\sum\varphi_i\varphi_{i,x'_1} & \cdots & s\sum\varphi_i\varphi_{i,x'_{n-1}} \\ \hline s\sum\varphi_i\varphi_{i,x'_1} & & & \\ \vdots & & s^2\,\nabla\varphi(x')^T\nabla\varphi(x') & \\ s\sum\varphi_i\varphi_{i,x'_{n-1}} & & & \end{array}\right]$$

Znamo $|\varphi(x')| = 1$ (jer je $\varphi(A) = S(0,1)$)
$$\Rightarrow |\varphi(x')|^2 = 1 \quad / \, \partial x'_i$$
$$2\sum_j \varphi_j\varphi_{j,x'_i} = 0$$

$$\Rightarrow \nabla\psi(s,x')^T\nabla\psi(s,x') = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & s^2\,\nabla\varphi(x')^T\nabla\varphi(x') & \\ 0 & & & \end{bmatrix}$$

$$\Rightarrow |\det\nabla\psi| = \sqrt{\det \nabla(s\varphi)^T\,\nabla(s\varphi)}$$, $s\varphi$ je parametrizacija $S(0,s)$

Konačno,
$$\int_{K[0,r]} f(x)\,dx = \int_0^r\int_A f(s\varphi(x'))\underbrace{}_{|\det\nabla\psi|}\,dx'\,ds = \int_0^r \underbrace{\int_A f(s\varphi(x'))\sqrt{\det\nabla(s\varphi)^T\nabla(s\varphi)}\,dx'}\,ds$$
$$= \int_{s\varphi} f = \int_{S(0,s)} f(y)\,dS_y$$