Laplaceova jednadžba

- **Elementarno rješenje**

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\pi}\ln|\mathbf{x}| & , \quad d=2 \\ \frac{1}{d(d-2)\omega_d}\frac{1}{|\mathbf{x}|^{d-2}} & , \quad d \geqslant 3 \end{cases}$$

$\omega_d \ldots$ volumen jedinične kugle u $\mathbf{R}^d$

- **Nehomogena (Poissonova) jednadžba**

$$\Delta u = f \quad \text{u} \quad \mathbf{R}^d$$

$$u(\mathbf{x}) = \int_{\mathbf{R}^d} \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} = \int_{\mathbf{R}^d} \Phi(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\, d\mathbf{y}$$

- **Harmonijske funkcije**

u harmonijska: $\Delta u = 0$

$$u \in \mathrm{C}^2(\Omega) \text{ harmonijska} \iff u(\mathbf{x}) = \fint_{\mathrm{S}(\mathbf{x},r)} u\, ds = \fint_{\mathrm{K}[\mathbf{x},r]} u\, d\mathbf{y} \ ,$$

- **Greenova funkcija**

$\Omega \subseteq \mathbf{R}^d$ otvoren, $\Gamma = \partial\Omega$

$$\begin{cases} -\Delta u = f \quad \text{u} \quad \Omega \\ u_{|\Gamma} = g \end{cases}$$

$$u(\mathbf{x}) = -\int_\Gamma g(\mathbf{y}) \nabla_{\vec{n}_\mathbf{y}} G(\mathbf{x},\mathbf{y})\, dS_\mathbf{y} + \int_\Omega G(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}$$

$G(\mathbf{x},\mathbf{y}) = \Phi(|\mathbf{x}-\mathbf{y}|) - w(\mathbf{x},\mathbf{y})$, gdje $w(\mathbf{x},\cdot)$ zadovoljava:

$$\begin{cases} \Delta w(\mathbf{x},\cdot) = 0 \\ w(\mathbf{x},\cdot)_{|\Gamma} = \Phi(|\mathbf{x}-\cdot|) \end{cases}$$

- **Greenova fukcija za kuglu i poluprostor**

Poluprostor:

$\mathbf{R}^d_+ = \{(\mathbf{x}', x_d) \in \mathbf{R}^d \ : \ x_d > 0\}$,

$$G(\mathbf{x},\mathbf{y}) = \Phi(|\mathbf{x}-\mathbf{y}|) - \Phi(|\bar{\mathbf{x}}-\mathbf{y}|) \ , \quad \mathbf{x},\mathbf{y} \in \mathbf{R}^d_+ \ , \ \mathbf{x} \neq \mathbf{y} \ , \ \bar{\mathbf{x}} = \overline{(\mathbf{x}', x_d)} = (\mathbf{x}', -x_d)$$

Kugla:

$$G(\mathbf{x},\mathbf{y}) = \Phi(|\mathbf{x}-\mathbf{y}|) - \Phi\Big(\frac{|\mathbf{x}|}{R}|\tilde{\mathbf{x}}-\mathbf{y}|\Big) \ , \quad \mathbf{x},\mathbf{y} \in \mathrm{K}(0,R) \ , \ \mathbf{x} \neq \mathbf{y} \ , \ \tilde{\mathbf{x}} = \frac{R^2}{|\mathbf{x}|^2}\mathbf{x}$$

Jednadžba provođenja

$$\begin{cases} u_t - k\triangle u = f & \text{u} \quad \mathbf{R}^+ \times \mathbf{R}^d \ (k>0) \\ u(0,\cdot) = g \end{cases}$$

$$u(t,\mathbf{x}) = \int_{\mathbf{R}^d} \Phi(t,\mathbf{x}-\mathbf{y})g(\mathbf{y})d\mathbf{y} + \int_0^t \int_{\mathbf{R}^d} \Phi(t-s,\mathbf{x}-\mathbf{y})f(s,\mathbf{y})\,d\mathbf{y}ds\ ,$$

$$\Phi(t,\mathbf{x}) = \begin{cases} \frac{1}{(4\pi kt)^{d/2}} e^{-\frac{|\mathbf{x}|^2}{4kt}} & ,\quad t>0 \\ 0 & ,\quad \text{inače} \end{cases}$$

Svojstva:

$$\int_{\mathbf{R}^d} \Phi(t,\mathbf{x})\,d\mathbf{x} = 1\ , \quad t>0\,,$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\cos(bx)dx = \sqrt{\frac{\pi}{a}}\, e^{-\frac{b^2}{4a}} \quad , \quad a>0,\ b\in\mathbf{R}\,.$$

Valna jednadžba

$$\begin{cases} u_{tt} - \triangle u = 0 & \text{u} \quad \mathbf{R}^+ \times \mathbf{R}^d \\ u(0,\cdot) = g \\ u_t(0,\cdot) = h \end{cases}$$

- **d=1**

  D'Alambertova formula:

$$u(t,x) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$$

- **d=2**

  Poissonova formula:

$$u(t,\mathbf{x}) = \frac{1}{2}\fint_{\mathrm{K}(\mathbf{x},t)} \frac{tg(\mathbf{y}) + t^2 h(\mathbf{y}) + t\nabla g(\mathbf{y})\cdot(\mathbf{y}-\mathbf{x})}{\sqrt{t^2 - |\mathbf{y}-\mathbf{x}|^2}}\,d\mathbf{y}$$

- **d=3**

  Krichhoffova formula:

$$u(t,\mathbf{x}) = \fint_{\mathrm{S}(\mathbf{x},t)} th(\mathbf{y}) + g(\mathbf{y}) + \nabla g(\mathbf{y})\cdot(\mathbf{y}-\mathbf{x})\,dS_{\mathbf{y}}$$

- **Nehomogena zadaća**

$$\begin{cases} u_{tt} - \triangle u = f & \text{u} \quad \mathbf{R}^+ \times \mathbf{R}^d \\ u(0,\cdot) = 0 \\ u_t(0,\cdot) = 0 \end{cases}$$

  Duhamelovo načelo:

$$u(t,\mathbf{x}) = \int_0^t v(t,\mathbf{x};s)\,ds\,,$$

pri čemu

$$\begin{cases} v_{tt}(\cdot,\cdot;s) - \triangle v(\cdot,\cdot;s) = 0 & \text{u} \quad \mathbf{R}^+ \times \mathbf{R}^d \\ v(s,\cdot;s) = 0 \\ v_t(s,\cdot;s) = f(s,\cdot) \end{cases}$$