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1) $\begin{cases} u_x + \sin x \, u_y = y & u \ \mathbb{R}^2 \\ u(0,y) = (y+1)^2 \end{cases}$

provjerimo ima li karakterističnih točaka: $\begin{bmatrix} 1 \\ \sin 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \neq 0 \Rightarrow$ NEMA

Rj: Koristimo metodu karakteristika.

$$\frac{dx}{d\tau} = 1 \Rightarrow x(\tau) = \tau + C_1$$

$$\frac{dy}{d\tau} = \sin x \Rightarrow dy = \sin(\tau + C_1)\, d\tau \Rightarrow y(\tau) = -\cos(\tau + C_1) + C_2$$

$$\frac{dz}{d\tau} = y \Rightarrow dz = (-\cos(\tau + C_1) + C_2)\, d\tau \Rightarrow z(\tau) = -\sin(\tau + C_1) + C_2\tau + C_3$$

Neka je $(0, y_0)$ proizvoljna točka na zadanoj krivulji.

$$X(0) = 0 \Rightarrow C_1 = 0$$

$$y(0) = y_0 \Rightarrow -\cos 0 + C_2 = y_0 \Rightarrow C_2 = y_0 + 1$$

$$\Rightarrow \boxed{\begin{aligned} x(\tau) &= \tau \\ y(\tau) &= -\cos\tau + y_0 + 1 \end{aligned}} \ (*)$$

$$z(0) = (y_0 + 1)^2 \Rightarrow C_3 = (y_0 + 1)^2 \Rightarrow \boxed{z(\tau) = -\sin\tau + (y_0 + 1)\tau + (y_0 + 1)^2}$$

Neka je $(x, y) \in \mathbb{R}^2$ proizvoljna. Iz $(*)$ slijedi

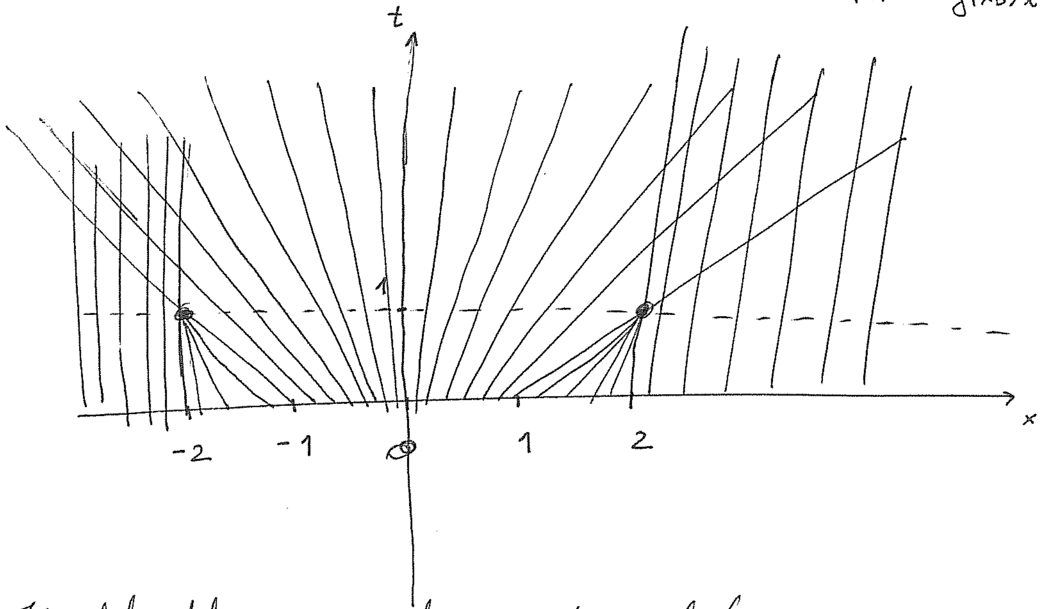
$$\boxed{\begin{aligned} \tau &= x \\ y_0 &= y + \cos x - 1 \end{aligned}}$$

$$\Rightarrow \boxed{u(x, y) = z(\tau; y_0) = -\sin x + (y + \cos x)x + (y + \cos x)^2}$$ rješenje.

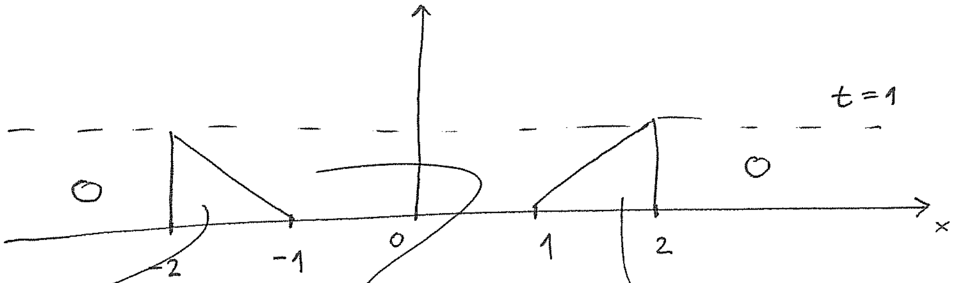
2) $\begin{cases} u_t + u u_x = 0 \\ u(0,\cdot) = g \end{cases}$, $g = \begin{cases} 0 & , \ x < -2 \\ -x-2 & , \ -2 \leq x < -1 \\ x & , \ -1 \leq x < 1 \\ -x+2 & , \ 1 \leq x < 2 \\ 0 & , \ x \geq 2 \end{cases}$

Rj:

Znamo da su karakteristike oblika $x(t) = g(x_0)t + x_0$.



Karakteristike se sijeku u dvije točke: $(1,-2)$ i $(1,2)$.
Prije određivanja krivulja skoka, odredimo rješenje do $t=1$.



$$x = (-x_0-2)t + x_0 = x_0(1-t) - 2t$$
$$\Rightarrow x_0 = \frac{x+2t}{1-t}$$
$$\Rightarrow u(t,x) = g(x_0) = -\frac{x+2t}{1-t} - 2 = \frac{-x-\cancel{2t}-2+\cancel{2t}}{1-t} = \frac{x+2}{t-1}$$

$$x = x_0 t + x_0$$
$$\Rightarrow x_0 = \frac{x}{t+1}$$
$$\Rightarrow u(t,x) = g(x_0) = \frac{x}{t+1}$$

$$x = (-x_0+2)t + x_0 = x_0(1-t) + 2t$$
$$\Rightarrow x_0 = \frac{x-2t}{1-t}$$
$$\Rightarrow u(t,x) = g(x_0) = -\frac{x-2t}{1-t} + 2 = \frac{-x+\cancel{2t}+2-\cancel{2t}}{1-t} = \frac{2-x}{1-t}$$

Odredimo sada krivulje šoka u točkama $(1,-2)$ i $(1,2)$.

R-H uvjet u $(1,-2)$

$$\left.\begin{array}{l} u_\ell = 0 \\ u_r = \frac{s}{t+1} \\ F_\ell = 0 \\ F_r = \left(\frac{s}{t+1}\right)^2 \frac{1}{2} \end{array}\right\} \Rightarrow \frac{\cancel{s}}{t+1}\dot{s} = \frac{1}{2}\left(\frac{s}{t+1}\right)^{\cancel{2}}$$

$$\frac{\dot{s}}{s} = \frac{1}{2}\frac{dt}{t+1} \Rightarrow \ln|s| = \ln\sqrt{t+1} + C$$

$$\Rightarrow s(t) = C\sqrt{t+1}$$

$$s(1) = -2 \Rightarrow \sqrt{2}C = -2 \Rightarrow C = -\sqrt{2}$$

$$\Rightarrow \boxed{s_1(t) = -\sqrt{2t+2}}$$

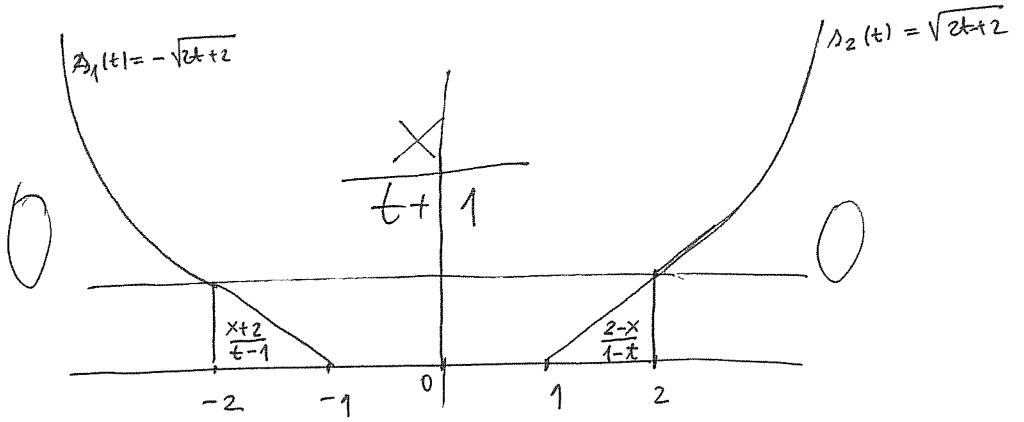
RH uvjet u $(1,2)$

$$\left.\begin{array}{l} u_\ell = \frac{s}{t+1} \\ u_r = 0 \\ F_\ell = \frac{1}{2}\frac{s^2}{(t+1)^2} \\ F_r = 0 \end{array}\right\} \Rightarrow \dot{s} = \frac{1}{2}\frac{s}{t+1} \Rightarrow (\text{analogno}) \Rightarrow s(t) = C\sqrt{t+1}$$

$$s(1) = 2 \Rightarrow \sqrt{2}C = 2 \Rightarrow C = \sqrt{2}$$

$$\Rightarrow \boxed{s_2(t) = \sqrt{2t+2}}$$

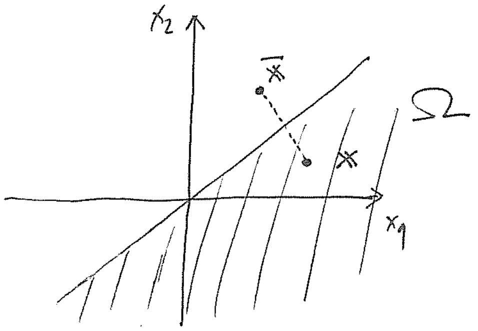
Time smo dobili



Uočite da je rješenje neparno po x. To je posljedica toga što je g neparna f-ja pa je stoga moguće i rješenje neparno.

3.)

a) $\Omega = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 > x_2 \}$



$x = (x_1, x_2)$

$\bar{x} = (x_2, x_1)$

Greenova f-ja je dana s

$$G(x, y) = \Phi(|x - y|) - \Phi(|\bar{x} - y|)$$

$$= -\frac{1}{4\pi} \ln\left((x_1-y_1)^2 + (x_2-y_2)^2\right)$$

$$+ \frac{1}{4\pi} \ln\left((x_2-y_1)^2 + (x_1-y_2)^2\right)$$

b) $\begin{cases} \Delta u = 0 \\ u(x,x) = g(x) \end{cases}$.

$$\Rightarrow \quad u(x_1, x_2) = -\int_{\partial\Omega} \nabla_{\vec{n}_y} G(x, y)\, g(y)\, dS_y$$

Vanjska normala na $\partial\Omega$ je vektor $\frac{1}{\sqrt{2}}(-1, 1)$ pa imamo:

$$\nabla_{\vec{n}_y} G(x,y) = \nabla_y G(x,y) \cdot \vec{n}_y = \frac{1}{\sqrt{2}} \left(-\partial_{y_1} G(x,y) + \partial_{y_2} G(x,y) \right)$$

- $\partial_{y_1} G(x,y) = -\frac{1}{4\pi} \frac{-2(x_1-y_1)}{(x_1-y_1)^2 + (x_2-y_2)^2} + \frac{1}{4\pi} \frac{-2(x_2-y_1)}{(x_2-y_1)^2 + (x_1-y_2)^2}$

$$\Rightarrow \partial_{y_1} G(x_1, x_2; \underbrace{y, y}_{\in\, \partial\Omega}) = \frac{1}{2\pi} \frac{x_1 - y - x_2 + y}{(x_1-y)^2 + (x_2-y)^2} = \frac{x_1 - x_2}{2\pi} \frac{1}{(x_1-y)^2 + (x_2-y)^2}$$

- $\partial_{y_2} G(x,y) = -\frac{1}{4\pi} \frac{-2(x_2-y_2)}{(x_1-y_1)^2 + (x_2-y_2)^2} + \frac{1}{4\pi} \frac{-2(x_1-y_2)}{(x_2-y_1)^2 + (x_1-y_2)^2}$

$$\Rightarrow \partial_{y_1} G(x_1, x_2, y, y) = \frac{1}{2\pi} \frac{x_2 - y - x_1 + y}{(x_1-y)^2 + (x_2-y)^2} = \frac{x_2 - x_1}{2\pi} \frac{1}{(x_1-y)^2 + (x_2-y)^2}$$

$$\Rightarrow \nabla_{\vec{n}_y} G(x, y)\Big|_{\partial\Omega} = \frac{x_2 - x_1}{\sqrt{2}\pi} \frac{1}{(x_1-y)^2 + (x_2-y)^2}$$

$$\Rightarrow u(x_1, x_2) = \frac{x_1 - x_2}{\sqrt{2}\pi} \int_{-\infty}^{+\infty} \frac{g(y)\ \sqrt{2}}{(x_1-y)^2 + (x_2-y)^2}\, dy$$

ovaj član dolazi jer imamo integral po krivulji parametriziranoj s $\gamma(y) = (y, y)$ pa je $|\gamma'(y)| = |(1,1)| = \sqrt{2}$

c)

$$\begin{aligned}(x_1-y)^2+(x_2-y)^2 &= x_1^2-2x_1y+y^2+x_2^2-2x_2y+y^2\\
&= 2y^2-2(x_1+x_2)y+x_1^2+x_2^2\\
&= 2\left(y^2-2\,\frac{x_1+x_2}{2}\,y+\frac{(x_1+x_2)^2}{4}\right)-\frac{(x_1+x_2)^2}{2}+x_1^2+x_2^2\\
&= 2\left(y-\frac{x_1+x_2}{2}\right)^2+\frac{(x_1-x_2)^2}{2} = 2\left(\left(y-\frac{x_1+x_2}{2}\right)^2+\frac{(x_1-x_2)^2}{4}\right)\end{aligned}$$

$$\Rightarrow\quad u(x_1,x_2)=\frac{x_1-x_2}{2\cancel{\sqrt{2}}\,\pi}\underbrace{\int_{-\infty}^{+\infty}\frac{\cancel{\sqrt{2}}\,g(y)}{\left(y-\frac{x_1+x_2}{2}\right)^2+\frac{(x_1-x_2)^2}{4}}\,dy}_{=:I}=\frac{x_1-x_2}{2\pi}\,I$$

$$g(x)=\begin{cases}A-|x|, & |x|\le A\\ 0, & |x|>A\end{cases}\qquad (A>0)$$

$$\begin{aligned}I &= \int_{-\infty}^{+\infty}\frac{g(y)}{\left(y-\frac{x_1+x_2}{2}\right)^2+\left(\frac{x_1-x_2}{2}\right)^2}\,dy = \int_{-\infty}^{+\infty}\frac{g\left(y+\frac{x_1+x_2}{2}\right)}{y^2+\left(\frac{x_1-x_2}{2}\right)^2}\,dy\\
&= \int_{-A-\frac{x_1+x_2}{2}}^{A-\frac{x_1+x_2}{2}}\frac{A-\left|y+\frac{x_1+x_2}{2}\right|}{y^2+\left(\frac{x_1-x_2}{2}\right)^2}\,dy\\
&= \int_{-A-\frac{x_1+x_2}{2}}^{-\frac{x_1+x_2}{2}}\frac{A+y+\frac{x_1+x_2}{2}}{y^2+\left(\frac{x_1-x_2}{2}\right)^2}\,dy+\int_{-\frac{x_1+x_2}{2}}^{A-\frac{x_1+x_2}{2}}\frac{A-y-\frac{x_1+x_2}{2}}{y^2+\left(\frac{x_1-x_2}{2}\right)^2}\,dy\\
&= \frac12\ln\left(y^2+\left(\frac{x_1-x_2}{2}\right)^2\right)\Bigg|_{-A-\frac{x_1+x_2}{2}}^{-\frac{x_1+x_2}{2}}+\left(A+\frac{x_1+x_2}{2}\right)\frac{2}{x_1-x_2}\operatorname{arctg}\frac{2y}{x_1-x_2}\Bigg|_{-A-\frac{x_1+x_2}{2}}^{-\frac{x_1+x_2}{2}}\\
&\quad -\frac12\ln\left(y^2+\left(\frac{x_1-x_2}{2}\right)^2\right)\Bigg|_{-\frac{x_1+x_2}{2}}^{A-\frac{x_1+x_2}{2}}+\left(A-\frac{x_1+x_2}{2}\right)\frac{2}{x_1-x_2}\operatorname{arctg}\frac{2y}{x_1-x_2}\Bigg|_{-\frac{x_1+x_2}{2}}^{A-\frac{x_1+x_2}{2}}\\
&= \frac12\ln\frac{x_1^2+x_2^2}{2}-\frac12\ln\left(A^2+A(x_1+x_2)+\frac{x_1^2+x_2^2}{2}\right)-\left(A+\frac{x_1+x_2}{2}\right)\frac{2}{x_1-x_2}\operatorname{arctg}\frac{x_1+x_2}{x_1-x_2}\\
&\quad +\left(A+\frac{x_1+x_2}{2}\right)\frac{2}{x_1-x_2}\operatorname{arctg}\frac{2A+x_1+x_2}{x_1-x_2}-\frac12\ln\left(A^2-A(x_1+x_2)+\frac{x_1^2+x_2^2}{2}\right)+\frac12\ln\frac{x_1^2+x_2^2}{2}\\
&\quad -\left(A-\frac{x_1+x_2}{2}\right)\frac{2}{x_1-x_2}\operatorname{arctg}\left(\frac{x_1+x_2-2A}{x_1-x_2}\right)+\left(A-\frac{x_1+x_2}{2}\right)\frac{2}{x_1-x_2}\operatorname{arctg}\frac{x_1+x_2}{x_1-x_2}\end{aligned}$$

$$\Rightarrow I = \ln\frac{x_1^2+x_2^2}{2} - \frac{1}{2}\ln\left(\left(A^2+\frac{x_1^2+x_2^2}{2}\right)^2 - A^2(x_1+x_2)^2\right)$$
$$- \frac{2(x_1+x_2)}{x_1-x_2}\,\mathrm{arctg}\,\frac{x_1+x_2}{x_1-x_2} + \left(A+\frac{x_1+x_2}{2}\right)\frac{2}{x_1-x_2}\,\mathrm{arctg}\,\frac{2A+x_1+x_2}{x_1-x_2}$$
$$- \left(A-\frac{x_1+x_2}{2}\right)\frac{2}{x_1-x_2}\,\mathrm{arctg}\,\frac{x_1+x_2-2A}{x_1-x_2}$$

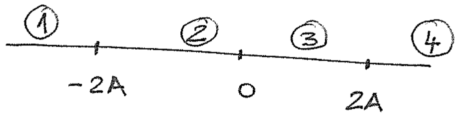
$$\Rightarrow u(x_1,x_2) = \frac{x_1-x_2}{2\pi}\left(\ln\frac{x_1^2+x_2^2}{2} - \frac{1}{2}\ln\left(\left(A^2+\frac{x_1^2+x_2^2}{2}\right)^2 - A^2(x_1+x_2)^2\right)\right)$$
$$+ \frac{x_1+x_2}{\pi}\left(\frac{1}{2}\,\mathrm{arctg}\,\frac{2A+x_1+x_2}{x_1-x_2} + \frac{1}{2}\,\mathrm{arctg}\,\frac{x_1+x_2-2A}{x_1-x_2} - \mathrm{arctg}\left(\frac{x_1+x_2}{x_1-x_2}\right)\right)$$
$$+ \frac{A}{\pi}\left(\mathrm{arctg}\,\frac{2A+x_1+x_2}{x_1-x_2} - \mathrm{arctg}\,\frac{x_1+x_2-2A}{x_1-x_2}\right)$$

Očito za $A=0$ dobivamo $u \equiv 0$ što bismo i trebali dobiti.

PROVJERA RUBNOG UVJETA:

Za $x_1 = x_2$ prvi član s ln je nula.

Daljnju analizu dijelimo ovisno o ~~predznaku~~ vrijednosti x_1+x_2.



① $x_1+x_2 < -2A$ (odnosno $x_1 < -A$)

$$\mathrm{arctg}\,\frac{2A+x_1+x_2}{x_1-x_2},\ \mathrm{arctg}\,\frac{x_1+x_2-2A}{x_1-x_2},\ \mathrm{arctg}\,\frac{x_1+x_2}{x_1-x_2} \longrightarrow -\frac{\pi}{2}$$

$$\Rightarrow u(x_1,x_2) \longrightarrow \frac{x_1+x_2}{\pi}\underbrace{\left(\frac{1}{2}\left(-\frac{\pi}{2}\right)+\frac{1}{2}\left(-\frac{\pi}{2}\right)-\left(-\frac{\pi}{2}\right)\right)}_{=0} + \frac{A}{\pi}\underbrace{\left(-\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)}_{=0}$$

Analogno za ④ $= 0$
samo su svi limesi $\frac{\pi}{2}$, a ne $-\frac{\pi}{2}$.

② ~~$-2A \le x_1+x_2 < 0$ ($-A \le x_1 < 0$)~~ $(x_1,x_2) \to (x,x)$, $\underline{\underline{-A \le x < 0}}$

$$\lim_{(x_1,x_2)\to(x,x)} u(x_1,x_2) = \frac{2x}{\pi}\left(\frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{2} + \frac{\pi}{2}\right) + \frac{A}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \underline{\underline{x+A}}$$

4) $$\begin{cases} u_{tt} - \Delta u = t^2 x_1^2 \\ u(0,x_1,x_2,x_3) = x_2 \\ u_t(0,x_1,x_2,x_3) = x_3^2 \end{cases}$$

Rj: Rješavamo homogenu zadaću

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(0,x) = x_2 \\ u_t(0,x) = x_3^2 \end{cases}$$

Kirch.

$$\Rightarrow \quad u(t,x) = \fint_{S(x,t)} t y_3^2 + y_2 + y_2 - x_2 \, dS_y$$

$$= \fint_{S(0,t)} t(y_3+x_3)^2 + y_2 + x_2 + y_2 \, dS_y$$

$$= t \fint_{S(0,t)} y_3^2 + 2y_3x_3 + x_3^2 \, dS_y + 2\underbrace{\fint_{S(0,t)} y_2 \, dS_y}_{\substack{=0 \\ \text{(neparna f-ja} \\ \text{na simetričnoj} \\ \text{domeni)}}} + x_2 \underbrace{\fint_{S(0,t)} dS_y}_{=1}$$

$$= t \fint_{S(0,t)} y_3^2 \, dS_y + 2x_3 t \underbrace{\fint_{S(0,t)} y_3 \, dS_y}_{=0} + t x_3^2 \underbrace{\fint_{S(0,t)} dS_y}_{=1} + x_2$$

$$= t \fint_{S(0,t)} y_3^2 \, dS_y + t x_3^2 + x_2$$

Izračunajmo $\fint_{S(0,t)} y_3^2 \, dS_y$.

$$\fint_{S(0,t)} y_3^2 \, dS_y = \frac{1}{4\pi t^2} \int_{S(0,t)} y_3^2 \, dS_y = \left\{ \begin{array}{ll} y_1 = t\cos\varphi\cos\psi, & \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ y_2 = t\cos\varphi\sin\psi & \\ y_3 = t\sin\varphi, & \psi \in [0, 2\pi] \\ \text{Jacobijan: } t^2\cos\varphi & \end{array} \right\}$$

$$= \frac{1}{4\pi t^2} t^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos\varphi \, t^2 \sin^2\varphi \, d\psi \, d\varphi$$

$$= \frac{2\pi t^2}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\varphi \cos\varphi \, d\varphi = \left\{ \begin{array}{l} v = \sin\varphi \\ dv = \cos\varphi \, d\varphi \end{array} \right\} = \frac{t^2}{2} \int_{-1}^{1} v^2 \, dv$$

$$= \frac{t^2}{6} v^3 \Big|_{-1}^{1} = \frac{t^2}{3}$$

$\Rightarrow$ rješenje homogene je $\underline{u_1(t,x) = \frac{t^3}{3} + t x_3^2 + x_2}$.

Sada promatramo nehomogenu zadaću s homogenim početnim uvjetima koju rješavamo koristeći Duhamelovo načelo.

$$\begin{cases} \tilde{v}_{tt} - \Delta \tilde{v} = 0 \\ \tilde{v}(0,\cdot) = 0 \\ \tilde{v}_t(0,x) = s^2 x_1^2 \end{cases}$$

Kirch.

$$\Rightarrow \tilde{v}(t,x) = \fint_{S(x,t)} t s^2 y_1^2 \, dS_y = s^2 t \fint_{S(0,t)} (y_1 + x_1)^2 \, dS_y$$

$$= \underbrace{s^2 t \fint_{S(0,t)} y_1^2 \, dS_y}_{= \frac{t^2}{3}} + \underbrace{2 s^2 t x_1 \fint_{S(0,t)} y_1 \, dS_y}_{=0} + \underbrace{s^2 t x_1^2 \fint_{S(0,t)} dS_y}_{=1}$$

(računali smo taj integral kod homogene j.) (neparna f-ja na simetričnoj domeni)

$$= \frac{t^3}{3} s^2 + t s^2 x_1^2$$

Nama zapravo treba v koji zadovoljava

$$\begin{cases} v_{tt} - \Delta v = 0 \\ v(s,\cdot\,; s) = 0 \\ v_t(s,\cdot\,; s) = s^2 x_1^2 \end{cases},$$

pa je (samo translacija u varijabli t) $v(t,x;s) = \tilde{v}(t-s,x)$

$$= \frac{(t-s)^3}{3} s^2 + (t-s) s^2 x_1^2 .$$

Konačno imamo

$$u_2(t,x) = \int_0^t \frac{(t-s)^3}{3} s^2 + (t-s) s^2 x_1^2 \, ds$$

$$\underline{= \dots = \frac{t^6}{180} + \frac{t^4 x_1^2}{12}},$$

pa je konačno rješenje :

$$\boxed{\begin{aligned} u(t,x) &= u_1(t,x) + u_2(t,x) \\ &= \frac{t^3}{3} + t x_3^2 + x_2 + \frac{t^6}{180} + \frac{t^4 x_1^2}{12} \end{aligned}}$$

5) $u \in C^2(\langle 0,T] \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$

$$\begin{cases} u_t - \Delta u = 0 & u \ \langle 0,T\rangle \times \mathbb{R}^d \\ u(0,\cdot) = u_0 \end{cases},$$

$$\lim_{R \to +\infty} \sup_{[0,T] \times K[0,R]^c} |u(t,x)| = 0 .$$

Treba pokazati: $\displaystyle \sup_{[0,T] \times \mathbb{R}^d} u(t,x) = \max\{0, \sup_{\mathbb{R}^d} u_0\}$.

Rj: Koristimo princip maksimuma za jednadžbu provođenja.

$\boxed{\geq}$ Očito vrijedi

$$\sup_{[0,T] \times \mathbb{R}^d} u(t,x) \geq \sup_{\mathbb{R}^d} u(0,x) = \sup_{\mathbb{R}^d} u_0 .$$

Pokažimo još da je $\displaystyle \sup_{[0,T] \times \mathbb{R}^d} u(t,x) \geq 0$.

Pretpostavimo suprotno, tada postoji $\varepsilon > 0$ t.d.

$$\sup_{[0,T] \times \mathbb{R}^d} u(t,x) < -\varepsilon .$$

Postoji dovoljno veliki $R > 0$ t.d. $\displaystyle \sup_{[0,T] \times K[0,R]^c} |u(t,x)| < \frac{\varepsilon}{2}$

$$\Rightarrow \underbrace{\sup_{[0,T] \times \mathbb{R}^d} u(t,x)}_{-\varepsilon >} \geq \sup_{[0,T] \times K[0,R]^c} u(t,x) > -\frac{\varepsilon}{2} \quad \text{↯}$$

$\boxed{\leq}$ Neka je $\varepsilon > 0$ po volji odabran. Tada postoji $R > 0$ t.d. $\displaystyle \max_{[0,T] \times S(0,R)} |u(t,x)| < \varepsilon$. Po principu maksimuma za sloj $[0,T] \times K[0,R]$ imamo

ovo vrijedi i za sve $R' > R$

$$\max_{[0,T] \times K[0,R]} u(t,x) = \max_{([0,T] \times S(0,R)) \cup (\{0\} \times K[0,R])} u(t,x) = \max\left\{ \max_{[0,T] \times S(0,R)} u(t,x), \max_{K[0,R]} u_0 \right\}$$

$$\leq \max\{\varepsilon, \max_{\mathbb{R}^d} u_0\}$$

$$\Rightarrow \sup_{[0,T] \times \mathbb{R}^d} u(t,x) = \sup_{R} \max_{[0,T] \times K[0,R]} u(t,x) \leq \max\{\varepsilon, \max_{\mathbb{R}^d} u_0\}$$

Iz proizvoljnosti $\varepsilon > 0$ slijedi

$$\sup_{[0,T]\times\mathbb{R}^d} u(t,x) = \max\{0, \sup_{\mathbb{R}^d} u_0\}.$$