1) $u_x - 2xu\,u_y = 0$ u $\mathbb{R}^2$

Rj: Koristit ćemo Lagrangeov postupak.

a)
$$\frac{dx}{1} = \frac{dy}{-2xu} = \frac{du}{0}$$

① $\frac{dx}{1} = \frac{du}{0} \Rightarrow du = 0 \Rightarrow u = C$

$$\Rightarrow \boxed{\varphi(x,y,u) := u}$$

② $\frac{dx}{1} = \frac{dy}{-2xu}$

$$-2Cx\,dx = dy \Rightarrow -Cx^2 = y + D$$

$$\Rightarrow -D = \boxed{ux^2 + y := \psi(x,y,u)}$$

Provjerimo da su φ i ψ linearno nezavisne:

$$\frac{\partial(\varphi,\psi)}{\partial(x,y,u)} = \begin{bmatrix} \varphi_x & \varphi_y & \varphi_u \\ \psi_x & \psi_y & \psi_u \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2xu & 1 & x^2 \end{bmatrix}$$

očito su ova dva stupca linearno nezavisna (za svaki x, y, u)

$\Rightarrow$ rješenje je dano jednadžbom

$$\boxed{F(u, ux^2 + y) = 0}$$

Ako je $F_\varphi \neq 0$, tada je gornji izraz ekvivalentan s

$$u = g(ux^2 + y)$$

za neku f-ju g.

b) $u = \frac{1}{x}$ na $y = 2x$

$$\Rightarrow \frac{1}{x} = g\left(\frac{1}{x}x^2 + 2x\right) = g(3x) \Rightarrow \boxed{g(x) = \frac{3}{x}}$$

$$\Rightarrow u = \frac{3}{ux^2 + y}$$

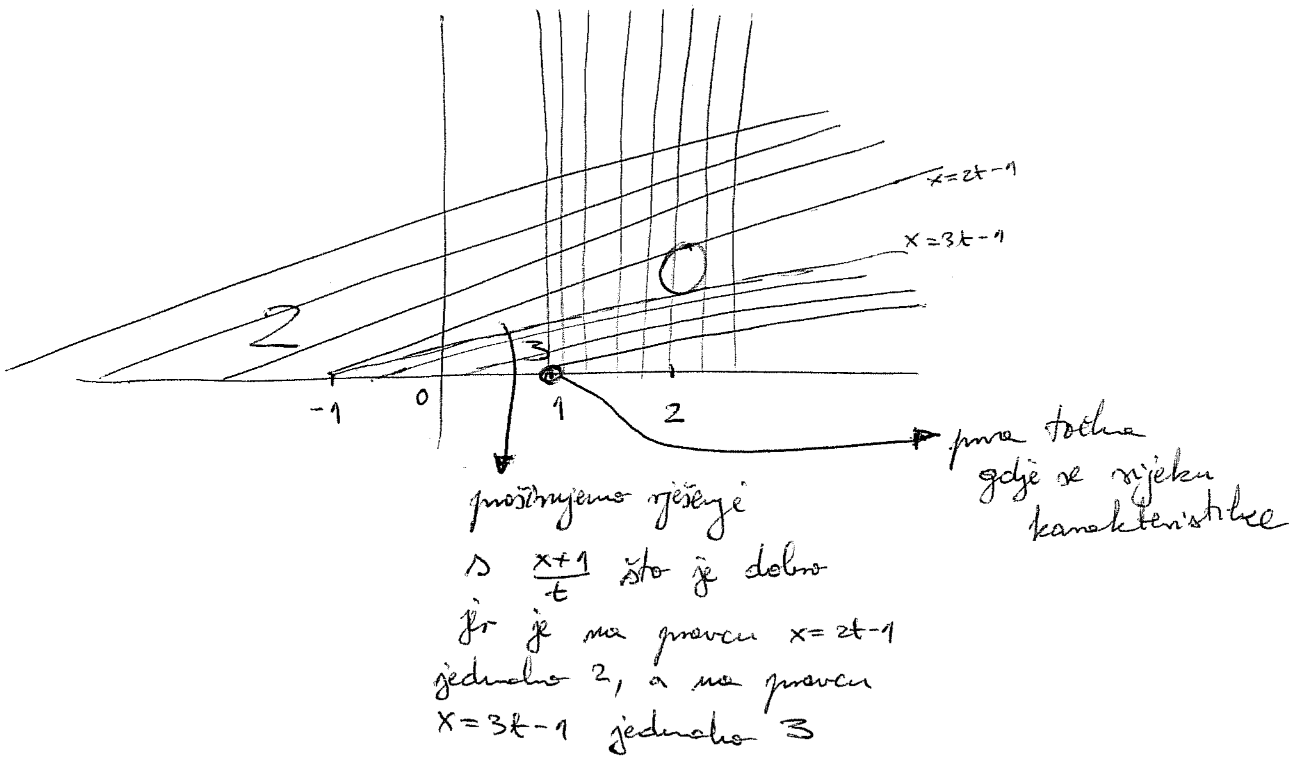
$$\Rightarrow x^2u^2 + yu - 3 = 0 \Rightarrow \boxed{u_{1,2} = \frac{-y \pm \sqrt{y^2 + 12x^2}}{2x^2}}$$

Ako je $F_\varphi = 0$ tada se trivijalno dobije da bi u ovom slučaju bilo $F \equiv 0$, odnosno to nije moguće jer onda nemamo nikakvu informaciju o rješenju.

2) $\begin{cases} u_t + u u_x = 0 \\ u(0,\cdot) = g \end{cases}$

$$g(x) = \begin{cases} 2 & , x < -1 \\ 3 & , -1 \le x < 1 \\ 0 & , x \ge 1 \end{cases}$$

Karakteristike su oblika
$x(t) = g(x_0)t + x_0$.



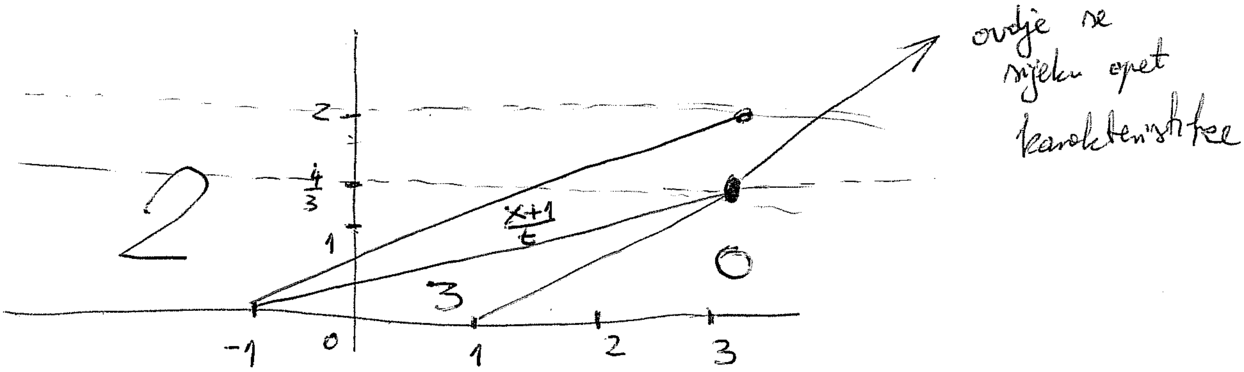
R-H uvjeti u $t=0, x=1$

$$\left.\begin{array}{l} \left.\begin{array}{l} u_\ell = 3 \\ u_r = 0 \end{array}\right\} [u] = 3 \\ \left.\begin{array}{l} F(u_\ell) = \frac{9}{2} \\ F(u_r) = 0 \end{array}\right\} [F] = \frac{9}{2} \end{array}\right\} \Rightarrow 3\dot{s} = \frac{9}{2} \Rightarrow s(t) = \frac{3}{2}t + C$$

$s(0) = 1 \Rightarrow C = 1$

$\Rightarrow s_1(t) = \frac{3}{2}t + 1$

Pravci $x = \frac{3}{2}t + 1$ i $x = 3t - 1$ se sijeku u točki $(t,x) = (\frac{4}{3}, 3)$ pa imamo



R-H uvjeti u $t=\frac{4}{3}$, $x=3$

$$\left.\begin{array}{l} u_l = \frac{s+1}{t} \\ u_r = 0 \end{array}\right\} \Rightarrow [u] = \frac{s+1}{t}$$

$$\left.\begin{array}{l} F(u_l) = \frac{1}{2}\left(\frac{s+1}{t}\right)^2 \\ F(u_r) = 0 \end{array}\right\} [F] = \frac{1}{2}\left(\frac{s+1}{t}\right)^2$$

$$\Rightarrow \dot{s} = \frac{s+1}{2t}$$

$$\frac{ds}{s+1} = \frac{dt}{2t} \Rightarrow \ln|s+1| = \tfrac{1}{2}\ln|t| + C$$

$$\Rightarrow s+1 = C\sqrt{t}$$

$$\Rightarrow s(t) = C\sqrt{t} - 1$$

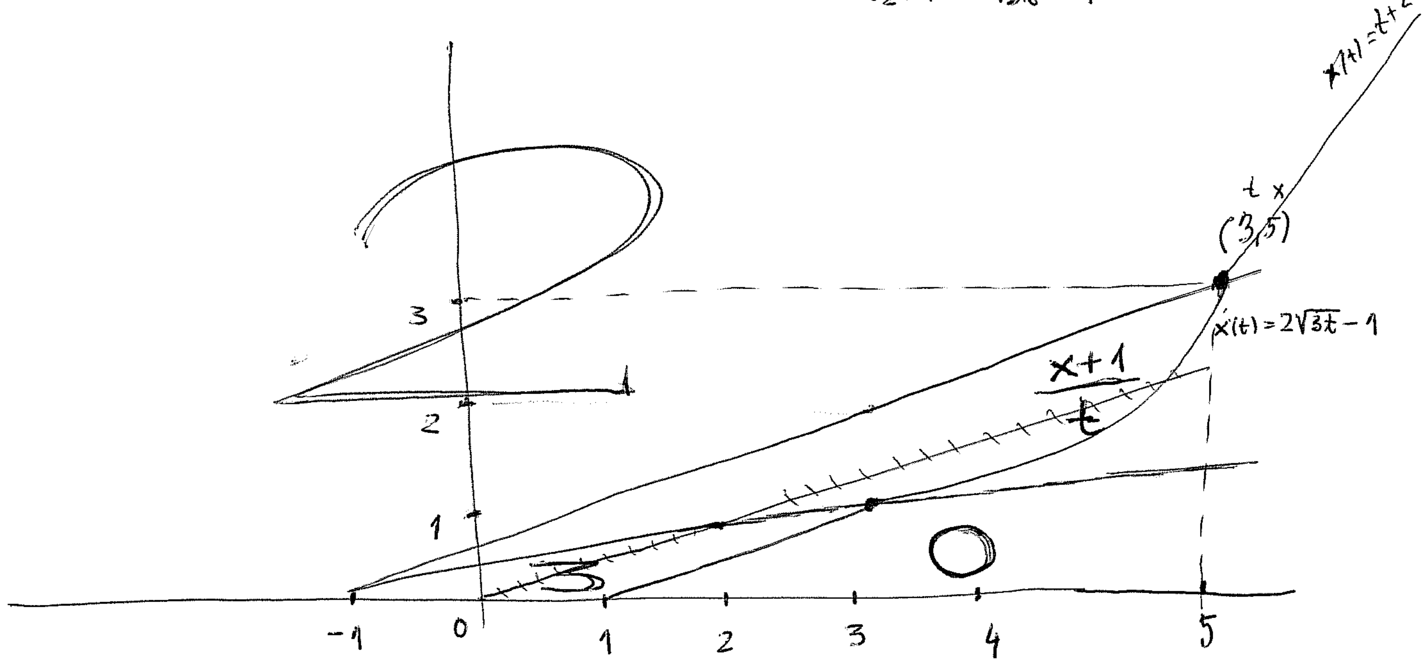
$$s(\tfrac{4}{3}) = 3 \Rightarrow \sqrt{\tfrac{4}{3}}\,C - 1 = 3$$

$$\Rightarrow C = 4 \cdot \sqrt{\tfrac{3}{4}} = \cancel{3}\, 2\sqrt{3}$$

~~Time smo konačno dobili:~~

$$\Rightarrow \boxed{\cancel{s_2(t) = 3t - 1}}$$

$$s_2(t) = 2\sqrt{3t} - 1$$

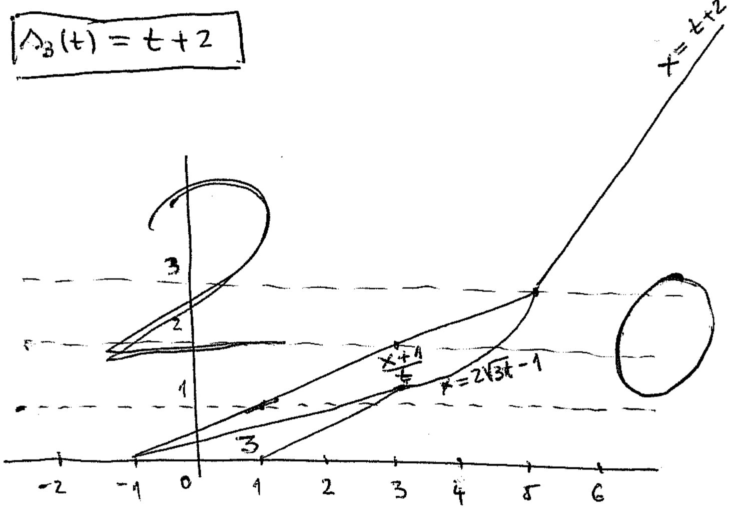


R-H uvjeti u $t=3$, $x=5$

$$\left.\begin{array}{l} u_l = 2 \\ u_r = 0 \\ F_l = 2 \\ F_r = 0 \end{array}\right\} \quad \dot{s} = 1 \Rightarrow s(t) = t + C$$

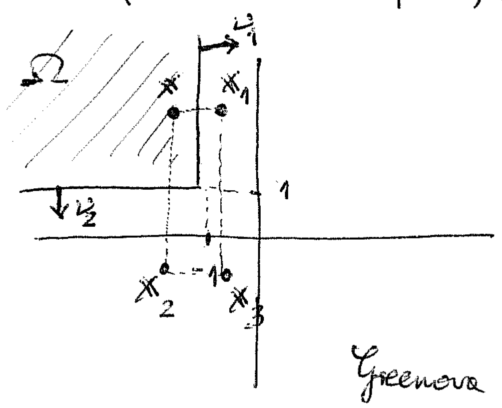
$$s(3) = 5 \Rightarrow 3 + C = 5 \Rightarrow \boxed{C = 2}$$

$$\Rightarrow \boxed{s_3(t) = t + 2}$$



3)

a) $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : \ x_1 < -1, \ x_2 > 1\}$



$\mathbb{x} = (x_1, x_2)$
$\mathbb{x}_1 = (-2-x_1, x_2)$
$\mathbb{x}_2 = (x_1, 2-x_2)$
$\mathbb{x}_3 = (-2-x_1, 2-x_2)$

Greenova funkcija je dana s

$$G(\mathbb{x},\mathbb{y}) = \Phi(|\mathbb{x}-\mathbb{y}|) - \Phi(|\mathbb{x}_1-\mathbb{y}|) - \Phi(|\mathbb{x}_2-\mathbb{y}|) + \Phi(|\mathbb{x}_3-\mathbb{y}|)$$

b)
$$\begin{cases} \Delta u = 0 \\ u(-1,x_2) = g(x_2-1) \\ u(x_1,1) = g(-x_1-1) \end{cases}$$

pri čemu
$$g(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases}$$

Normala na $\{x_1=-1\}$ je $\underbrace{(1,0)=e_1}_{\nu_1}$, dok je na $\{x_2=1\}$ jednaka $\underbrace{(0,-1)=e_2}_{\nu_2}$.

$$\Rightarrow \ -\frac{\partial G}{\partial \nu_1}(\mathbb{x},\mathbb{y}) = -\frac{\partial G}{\partial y_1}(\mathbb{x},\mathbb{y})$$
$$-\frac{\partial G}{\partial \nu_2}(\mathbb{x},\mathbb{y}) = \frac{\partial G}{\partial y_2}(\mathbb{x},\mathbb{y})$$

$$G(x_1,x_2,y_1,y_2) = -\frac{1}{4\pi}\ln\left((x_1-y_1)^2+(x_2-y_2)^2\right) + \frac{1}{4\pi}\ln\left((-2-x_1-y_1)^2+(x_2-y_2)^2\right)$$
$$+\frac{1}{4\pi}\ln\left((x_1-y_1)^2+(2-x_2-y_2)^2\right) - \frac{1}{4\pi}\ln\left((2+x_1+y_1)^2+(x_2+y_2-2)^2\right)$$

$$\frac{\partial G}{\partial y_1}(\mathbb{x},\mathbb{y}) = -\frac{1}{4\pi}\frac{-2(x_1-y_1)}{(x_1-y_1)^2+(x_2-y_2)^2} + \frac{1}{4\pi}\frac{2(x_1+y_1+2)}{(x_1+y_1+2)^2+(x_2-y_2)^2}$$
$$+\frac{1}{4\pi}\frac{-2(x_1-y_1)}{(x_1-y_1)^2+(x_2+y_2-2)^2} - \frac{1}{4\pi}\frac{2(x_1+y_1+2)}{(x_1+y_1+2)^2+(x_2+y_2-2)^2}$$

$$\Rightarrow \ -\frac{\partial G}{\partial \nu_1}(x_1,x_2,-1,y_2) = -\frac{1}{\pi}\frac{x_1+1}{(x_1+1)^2+(x_2-y_2)^2} + \frac{1}{\pi}\frac{x_1+1}{(x_1+1)^2+(x_2+y_2-2)^2}$$

$$\frac{\partial G}{\partial y_2}(x,y) = -\frac{1}{4\pi}\,\frac{-2(x_2-y_2)}{(x_1-y_1)^2+(x_2-y_2)^2} + \frac{1}{4\pi}\,\frac{-2(x_2-y_2)}{(x_1+y_1+2)^2+(x_2-y_2)^2}$$

$$+\frac{1}{4\pi}\,\frac{2(x_2+y_2-2)}{(x_1-y_1)^2+(x_2+y_2-2)^2} - \frac{1}{4\pi}\,\frac{2(x_2+y_2-2)}{(x_1+y_1+2)^2+(x_2+y_2-2)^2}$$

$$\Rightarrow\ -\frac{\partial G}{\partial \nu_2}(x_1,x_2,y_1,1) = \frac{1}{\pi}\,\frac{x_2-1}{(x_1-y_1)^2+(x_2-1)^2} - \frac{1}{\pi}\,\frac{x_2-1}{(x_1+y_1+2)^2+(x_2-1)^2}$$

$$u(x_1,x_2) = -\int_1^{+\infty}\frac{\partial G}{\partial \nu_1}(x_1,x_2,-1,y_2)\,g(y_2-1)\,dy_2 - \int_{-\infty}^{-1}\frac{\partial G}{\partial \nu_2}(x_1,x_2,y_1,1)\,g(y_1+1)\,dy_1$$

$$= -\frac{x_1+1}{\pi}\int_1^2\frac{dy_2}{(x_1+1)^2+(x_2-y_2)^2} + \frac{x_1+1}{\pi}\int_1^2\frac{dy_2}{(x_1+1)^2+(x_2+y_2-2)^2}$$

$$+\frac{x_2-1}{\pi}\int_{-2}^{-1}\frac{dy_1}{(x_1-y_1)^2+(x_2-1)^2} - \frac{x_2-1}{\pi}\int_{-2}^{1}\frac{dy_1}{(x_1+y_1+2)^2+(x_2-1)^2}$$

$\sim\cdot\sim$

Svi integrali su oblika:

$$\int_a^b\frac{dx}{c^2+(x-d)^2} = \left\{\begin{matrix} y = x-d \\ dy = dx\end{matrix}\right\} = \int_{a-d}^{b-d}\frac{dy}{c^2+y^2}$$

$$= \frac{1}{c}\,\mathrm{arctg}\,\frac{y}{c}\Bigg|_{a-d}^{b-d}$$

bez $|\cdot|$

$\sim\cdot\sim$

$$u(x_1,x_2) = -\frac{\cancel{x_1+1}}{\pi}\,\frac{1}{\cancel{x_1+1}}\,\mathrm{arctg}\,\frac{y_2-x_2}{x_1+1}\Bigg|_1^2 + \frac{\cancel{x_1+1}}{\pi}\,\frac{1}{\cancel{x_1+1}}\,\mathrm{arctg}\,\frac{y_2+x_2-2}{x_1+1}\Bigg|_1^2$$

$$+\frac{\cancel{x_2-1}}{\pi}\,\frac{1}{\cancel{x_2-1}}\,\mathrm{arctg}\,\frac{y_1-x_1}{x_2-1}\Bigg|_{-2}^{-1} - \frac{\cancel{x_2-1}}{\pi}\,\frac{1}{\cancel{x_2-1}}\,\mathrm{arctg}\,\frac{y_1+x_1+2}{x_2-1}\Bigg|_{-2}^{-1}$$

$$= \frac{1}{\pi}\left(-\mathrm{arctg}\left(\cancel{\frac{2-x_2}{x_1+1}}\right) + \mathrm{arctg}\left(\frac{1-x_2}{x_1+1}\right) + \mathrm{arctg}\left(\frac{x_2}{x_1+1}\right) - \mathrm{arctg}\left(\cancel{\frac{x_2-1}{x_1+1}}\right)\right)$$

$$+\frac{1}{\pi}\left(\mathrm{arctg}\left(\frac{-1-x_1}{x_2-1}\right) - \mathrm{arctg}\left(\frac{-2-x_1}{x_2-1}\right) - \mathrm{arctg}\left(\frac{x_1+1}{x_2-1}\right) + \mathrm{arctg}\left(\frac{x_1}{x_2-1}\right)\right)$$

$$= \frac{1}{\pi}\left(\mathrm{arctg}\left(\frac{x_2-2}{x_1+1}\right) - 2\,\mathrm{arctg}\left(\frac{x_2-1}{x_1+1}\right) + \mathrm{arctg}\left(\frac{x_2}{x_1+1}\right) + 2\,\mathrm{arctg}\left(\frac{-1-x_1}{x_2-1}\right)\right.$$

$$\left. - \mathrm{arctg}\left(\frac{-2-x_1}{x_2-1}\right) + \mathrm{arctg}\left(\frac{x_1}{x_2-1}\right)\right)$$

4) $\begin{cases} u_t - \Delta u = \sin t \sin x_1 \sin x_2 \\ u(0,\cdot) = 1 \end{cases}$

Ř.:
$$u(t,x_1,x_2) = \underbrace{\int_{\mathbb{R}^2} \Phi(t, \mathbb{x}-\mathbb{y})\, d\mathbb{y}}_{=1} + \int_0^t \int_{\mathbb{R}^2} \Phi(t-s, \mathbb{x}-\mathbb{y}) \sin s \sin y_1 \sin y_2 \, d\mathbb{y}\, ds$$

$$= 1 + \int_0^t \frac{\sin s}{4\pi(t-s)} \int_{\mathbb{R}^2} e^{-\frac{|\mathbb{x}-\mathbb{y}|^2}{4(t-s)}} \sin y_1 \sin y_2 \, d\mathbb{y}\, ds$$

$$= 1 + \int_0^t \frac{\sin s}{4\pi(t-s)} \left(\int_{\mathbb{R}} e^{-\frac{(x_1-y_1)^2}{4(t-s)}} \sin y_1 \, dy_1 \right) \left(\int_{\mathbb{R}} e^{-\frac{(x_2-y_2)^2}{4(t-s)}} \sin y_2 \, dy_2 \right) ds$$

$$\cancel{= 1 + \int_0^t \frac{\sin s}{4\pi(t-s)} \int_{\mathbb{R}} e^{-\frac{(y-x)^2}{4(t-s)}} \sin y}$$

$\sim\cdot\sim$

$$\int_{\mathbb{R}} e^{-\frac{(y-x)^2}{4(t-s)}} \sin y \, dy = \left\{ \begin{matrix} z = y - x \\ y = z + x \end{matrix} \right\} = \int_{\mathbb{R}} e^{-\frac{z^2}{4(t-s)}} \sin(z+x)\, dz$$

$$= \cos x \underbrace{\int_{\mathbb{R}} e^{-\frac{z^2}{4(t-s)}} \sin z \, dz}_{=0} + \sin x \int_{\mathbb{R}} e^{-\frac{z^2}{4(t-s)}} \cos z \, dz$$

$$= \sin x \sqrt{4\pi(t-s)}\, e^{-(t-s)}$$

$\sim\cdot\sim$

$$\Rightarrow u(t,x_1,x_2) = 1 + \int_0^t \frac{\sin s}{\cancel{4\pi(t-s)}} \sin x_1 \, \cancel{4\pi(t-s)} \sin x_2 \, e^{-2(t-s)}\, ds$$

$$= 1 + \sin x_1 \sin x_2 \int_0^t e^{-2(t-s)} \sin s \, ds$$

$$= 1 + e^{-2t} \sin x_1 \sin x_2 \underbrace{\int_0^t e^{2s} \sin s \, ds}_{I} = \boxed{1 + \tfrac{1}{5} \sin x_1 \sin x_2 \left(2\sin t - \cos t + e^{-2t}\right)}$$

$$I \overset{\text{P.I.}}{=} \tfrac{1}{2} e^{2s} \sin s \Big|_0^t - \tfrac{1}{2} \int_0^t e^{2s} \cos s \, ds \overset{\text{P.I.}}{=} \tfrac{1}{2} e^{2t} \sin t - \tfrac{1}{4} e^{2s} \cos s \Big|_0^t - \tfrac{1}{4} \int_0^t e^{2s} \sin s \, ds$$

$$= \tfrac{1}{2} e^{2t} \sin t - \tfrac{1}{4} e^{2t} \cos t + \tfrac{1}{4} - \tfrac{1}{4} I$$

$$\Rightarrow \tfrac{5}{4} I = \tfrac{1}{4} e^{2t} (2\sin t - \cos t) + \tfrac{1}{4} \Rightarrow I = \underline{\tfrac{1}{5} e^{2t} (2\sin t - \cos t) + \tfrac{1}{5}}$$