1) $$\begin{cases} u_t + c_1\partial_1 u + c_2\partial_2 u + f(t,x_1,x_2)u = 0 & u \quad \mathbb{R}^+\times\mathbb{R}^2 \\ u(0,\cdot,\cdot) = g \end{cases}$$

Rj:

1. NAČIN: Koristeći supstituciju

Cilj je uvesti nove varijable s

$$\xi_i = a_{i1}t + a_{i2}x_1 + a_{i3}x_2, \quad i=1,2,3$$

tako da

① gornje preslikavanje bude regularno, odnosno da matrica bude ~~sve~~ $[a_{ij}]_{ij}$ bude regularna.

② funkcija $v(\xi_1,\xi_2,\xi_3) := u(t,x_1,x_2)$ zadovoljava ODJ (odnosno da javljaju derivacije samo po jednoj varijabli).

Jedan mogući izbor je

$$\begin{aligned} \xi_1 &= c_1c_2t + c_2x_1 + c_1x_2 \\ \xi_2 &= x_1 - c_1t \\ \xi_3 &= x_2 - c_2t \end{aligned} \quad\Rightarrow\quad \begin{aligned} t &= \frac{1}{3c_1c_2}\xi_1 - \frac{1}{3c_1}\xi_2 - \frac{1}{3c_2}\xi_3 \\ x_1 &= \frac{1}{3c_2}\xi_1 + \frac{2}{3}\xi_2 - \frac{c_1}{3c_2}\xi_3 \\ x_2 &= \frac{1}{3c_1}\xi_1 - \frac{c_2}{3c_1}\xi_2 + \frac{2}{3}\xi_3 \end{aligned}$$

Naravno, gornje vrijedi uz pretpostavku $c_1, c_2 \neq 0$, međutim, konačno rješenje koje dobijemo vrijedit će i u tom slučaju.

Definirajmo

$$v(\xi_1,\xi_2,\xi_3) := u(t,x_1,x_2)$$
$$\tilde f(\xi_1,\xi_2,\xi_3) := f(t,x_1,x_2)$$

ovdje su t, x_1 i x_2 dani preko ξ_1, ξ_2 i ξ_3 kao gore

Imamo:

$$\begin{aligned} u_t &= v_{\xi_1}\frac{d\xi_1}{dt} + v_{\xi_2}\frac{d\xi_2}{dt} + v_{\xi_3}\frac{d\xi_3}{dt} \\ &= c_1c_2v_{\xi_1} - c_1v_{\xi_2} - c_2v_{\xi_3} \\ \partial_1 u &= c_2v_{\xi_1} + v_{\xi_2} \\ \partial_2 u &= c_1v_{\xi_1} + v_{\xi_3} \end{aligned}$$

$$\begin{aligned} \Rightarrow\quad 0 &= u_t + c_1\partial_1 u + c_2\partial_2 u + f(t,x_1,x_2)u \\ &= c_1c_2v_{\xi_1} - \cancel{c_1v_{\xi_2}} - \cancel{c_2v_{\xi_3}} + c_1c_2v_{\xi_1} + \cancel{c_1v_{\xi_2}} + c_1c_2v_{\xi_1} + \cancel{c_2v_{\xi_3}} + \tilde f(\xi_1,\xi_2,\xi_3)v \\ &= 3c_1c_2v_{\xi_1} + \tilde f v \end{aligned}$$

Dobili smo da funkcija v zadovoljava ODJ

$$\boxed{v_{\xi_1} + \frac{1}{3c_1c_2}\tilde{f}\,v = 0}$$

Odavde slijedi:

$$v(\xi_1,\xi_2,\xi_3) = v(\bar{\xi}_1,\xi_2,\xi_3)\, e^{\frac{-1}{3c_1c_2}\int_{\bar{\xi}_1}^{\xi_1}\tilde{f}(\eta,\xi_2,\xi_3)\,d\eta}.$$

- $v(\xi_1,\xi_2,\xi_3) = u(t,x_1,x_2)$
- $v(\bar{\xi}_1,\xi_2,\xi_3) = u\Big(\underbrace{\frac{1}{3c_1c_2}\bar{\xi}_1 - \frac{1}{3c_1}\xi_2 - \frac{1}{3c_2}\xi_3}, \frac{1}{3c_2}\bar{\xi}_1 + \frac{2}{3}\xi_2 - \frac{c_1}{3c_2}\xi_3, \frac{1}{3c_1}\bar{\xi}_1 - \frac{c_2}{3c_1}\xi_2 + \frac{2}{3}\xi_3\Big)$

želimo odabrati takav $\bar{\xi}_1$
t.d. ovdje bude 0 pa
da možemo iskoristiti
početni uvjet

$\Rightarrow \boxed{\bar{\xi}_1 = c_2\xi_2 + c_1\xi_3}$, a od prije znamo $\boxed{\begin{aligned}\xi_2 &= x_1 - c_1t\\ \xi_3 &= x_2 - c_2t\end{aligned}}$

- $\frac{1}{3c_2}\bar{\xi}_1 + \frac{2}{3}\xi_2 - \frac{c_1}{3c_2}\xi_3 = \frac{1}{3}\xi_2 + \cancel{\frac{c_1}{3c_2}\xi_3} + \frac{2}{3}\xi_2 - \cancel{\frac{c_1}{3c_2}\xi_3} = \xi_2 = x_1 - c_1t$
- $\frac{1}{3c_2}\bar{\xi}_1 - \frac{c_2}{3c_1}\xi_2 + \frac{2}{3}\xi_3 = \ldots = \xi_3 = x_2 - c_2t$

$\Rightarrow v(\bar{\xi}_1,\xi_2,\xi_3) = u(0, x_1-c_1t, x_2-c_2t) = g(x_1-c_1t, x_2-c_2t)$

- $\int_{\bar{\xi}_1}^{\xi_1}\tilde{f}(\eta,\xi_2,\xi_3)\,d\eta = \int_{\bar{\xi}_1}^{\xi_1} f\Big(\frac{1}{3c_1c_2}\eta - \frac{1}{3c_1}\xi_2 - \frac{1}{3c_2}\xi_3, \frac{1}{3c_2}\eta + \frac{2}{3}\xi_2 - \frac{c_1}{3c_2}\xi_3, \frac{1}{3c_1}\eta - \frac{c_2}{3c_1}\xi_2 + \frac{2}{3}\xi_3\Big)d\eta$

$$\left[\begin{aligned}
& s = \frac{1}{3c_1c_2}\eta - \frac{1}{3c_1}\xi_2 - \frac{1}{3c_2}\xi_3 \Rightarrow ds = \frac{1}{3c_1c_2}d\eta\\
& \eta = \bar{\xi}_1 \to s = \frac{1}{3c_1c_2}\bar{\xi}_1 - \frac{1}{3c_1}\xi_2 - \frac{1}{3c_2}\xi_3 = 0\\
& \eta = \xi_1 \to s = \frac{1}{3c_1c_2}\xi_1 - \frac{1}{3c_1}\xi_2 - \frac{1}{3c_2}\xi_3 = t\\
& \text{------}\\
& \eta = 3c_1c_2 s + c_2\xi_2 + c_1\xi_3\\
& \Rightarrow \frac{1}{3c_2}\eta + \frac{2}{3}\xi_2 - \frac{c_1}{3c_2}\xi_3 = c_1 s + \xi_2 = x_1 + c_1(s-t)\\
& \quad\ \frac{1}{3c_1}\eta - \frac{c_2}{3c_1}\xi_2 + \frac{2}{3}\xi_3 = c_2 s + \xi_3 = x_2 + c_2(s-t)
\end{aligned}\right.$$

$$= 3c_1c_2\int_0^t f(s, x_1+c_1(s-t), x_2+c_2(s-t))\,ds$$

$$\Rightarrow \boxed{u(t,x_1,x_2) = g(x_1-c_1t, x_2-c_2t)\, e^{-\int_0^t f(s, x_1+c_1(s-t), x_2+c_2(s-t))\,ds}}$$

$\hookrightarrow$ nije teško provjeriti da je ovo rješenje i u slučaju kad je $c_1 = 0$ ili $c_2 = 0$

2. NAČIN : metoda karakteristika

$$S = \{ (0, x_1, x_2) \in \mathbb{R}^3 \}$$

$$\frac{dt}{d\tau} = 1 \quad \Rightarrow \quad t(\tau) = \tau + C_1$$

$$\frac{dx_1}{d\tau} = c_1 \quad \Rightarrow \quad x_1(\tau) = c_1\tau + C_2$$

$$\frac{dx_2}{d\tau} = c_2 \quad \Rightarrow \quad x_2(\tau) = c_2\tau + C_3$$

$$\frac{dz}{d\tau} = -f(t, x_1, x_2)\, z \rightsquigarrow \frac{dz}{z} = -f(\tau + C_1, c_1\tau + C_2, c_2\tau + C_3)\, d\tau \;\Big/ \int_0^\tau$$

$$\Rightarrow z(\tau) = z(0)\, e^{-\int_0^\tau f(s + C_1, c_1 s + C_2, c_2 s + C_3)\, ds}$$

$(0, x_1^0, x_2^0) \in S \rightsquigarrow$

$$\left.\begin{array}{l} t(0) = 0 \Rightarrow C_1 = 0 \\ x_1(0) = x_1^0 \Rightarrow C_2 = x_1^0 \\ x_2(0) = x_2^0 \Rightarrow C_3 = x_2^0 \\ z(0) = g(x_1^0, x_2^0) \end{array}\right\} \Rightarrow \begin{array}{l} t(\tau) = \tau \\ x_1(\tau) = c_1\tau + x_1^0 \\ x_2(\tau) = c_2\tau + x_2^0 \\ z(\tau) = g(x_1^0, x_2^0)\, e^{-\int_0^\tau f(s,\, c_1 s + x_1^0,\, c_2 s + x_2^0)\, ds} \end{array}$$

$$\left.\begin{array}{l} \tau = t \\ x_1^0 = x_1 - c_1 t \\ x_2^0 = x_2 - c_2 t \end{array}\right\} \Rightarrow u(t, x_1, x_2) = z(\tau; x_1^0, x_2^0)$$

$$= g(x_1 - c_1 t, x_2 - c_2 t)\, e^{-\int_0^t f(s,\, x_1 + c_1(s-t),\, x_2 + c_2(s-t))\, ds}$$

2) a)

$$\begin{cases} xu_x - 2yu_y - u_z = u^2 & u\ \mathbb{R}^3 \\ u(x,y,0) = x^2+xy+y^2 \end{cases}$$

$\rightsquigarrow$ $S \ldots \{(x,y,0) : x,y \in \mathbb{R}\}$ normala je $\nu = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\begin{bmatrix}0\\0\\1\end{bmatrix} \cdot \begin{bmatrix}x\\-2y\\-1\end{bmatrix} = -1 \neq 0 \Rightarrow$ nema karakterističnih točaka

Rj. METODA KARAKTERISTIKA

$$\frac{dx}{d\tau} = x \Rightarrow \frac{dx}{x} = d\tau \Rightarrow \ln|x| = \tau + C \Rightarrow \cancel{x(\tau)} = C_1 e^{\tau}$$

$$\frac{dy}{d\tau} = -2y \Rightarrow \ldots \Rightarrow y(\tau) = C_2 e^{-2\tau}$$

$$\frac{dz}{d\tau} = -1 \Rightarrow z(\tau) = -\tau + C_3$$

rješenje $\rightarrow$ $$\frac{dv}{d\tau} = v^2 \Rightarrow \frac{dv}{v^2} = d\tau \Rightarrow -\frac{1}{v} = \tau + C_4 \Rightarrow v = \frac{-1}{\tau + C_4}$$

Iz rubnog uvjeta dobivamo: $(x_0, y_0, 0) \in S \ldots$ ploha na kojoj je zadano rj:

$$\left.\begin{aligned} x_0 &= x(0) = C_1 \\ y_0 &= y(0) = C_2 \\ 0 &= z(0) = C_3 \\ x_0^2 + x_0y_0 + y_0^2 &= v(0) = -\frac{1}{C_4} \Rightarrow C_4 = -\frac{1}{x_0^2+x_0y_0+y_0^2} \end{aligned}\right\} \Rightarrow \begin{aligned} x(\tau) &= x_0 e^{\tau} \\ y(\tau) &= y_0 e^{-2\tau} \\ z(\tau) &= -\tau \\ v(\tau) &= \frac{-x_0^2 - x_0y_0 - y_0^2}{(x_0^2+x_0y_0+y_0^2)\tau - 1} \end{aligned}$$

Izrazimo τ, x_0 i y_0 preko x, y i z:

$$\left.\begin{aligned} &\boxed{\tau = -z} \\ &\boxed{y_0 = ye^{2\tau} = ye^{-2z}} \\ &\boxed{x_0 = xe^{-\tau} = xe^{z}} \end{aligned}\right\} \Rightarrow \begin{aligned} u(x,y,z) &= v(\tau; x_0, y_0) \\ &= \frac{-x^2e^{2z} - xye^{-z} - y^2e^{-4z}}{-(+x^2e^{2z} + xye^{-z} + y^2e^{-4z})z - 1} \\ &= \frac{x^2e^{2z} + xye^{-z} + y^2e^{-4z}}{1 + (x^2e^{2z} + xye^{z} + y^2e^{-4z})z} \end{aligned}$$

2) b)

$$\begin{cases} ux\,u_x - yu\,u_y = y^2 - x^2 & u\ \mathbb{R}^2 \\ u(x,x) = f(x) \end{cases}$$

Rj:

$S \ldots \{(x,x) : x \in \mathbb{R}\} \Rightarrow \nu = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} ux \\ -\cancel{y}u \end{bmatrix} = ux + \cancel{y}u = 2ux\cancel{(x+y)}$ ⇝ karakteristična točka je samo ishodište

na S gledamo pa je $y = x$ ~~pravca $y=-x$~~

Nakon što podijelimo jednadžbu s u dobivamo:

to možemo jer ako je $u \equiv 0$ tada slijedi da je nužno $y^2 - x^2 = 0$, odnosno to je moguće samo na pravcima $y = -x$ i $y = x$. Iz toga slijedi da je u na okolini S (ne računajući sam skup S) sigurno različito od 0.

$\frac{dx}{d\tau} = x \Rightarrow x(\tau) = c_1 e^{\tau}$

$\frac{dy}{d\tau} = -y \Rightarrow y(\tau) = c_2 e^{-\tau}$

$\frac{dv}{d\tau} = \frac{y^2 - x^2}{v} \Rightarrow v\,dv = (c_2^2 e^{-2\tau} - c_1^2 e^{2\tau})\, d\tau$

$\frac{v^2}{2} = -\frac{c_2^2}{2} e^{-2\tau} - \frac{c_1^2}{2} e^{2\tau} + c_3$

$v^2(\tau) = -c_2^2 e^{-2\tau} - c_1^2 e^{2\tau} + c_3$ ⇝ očito neće imati smisla za sve τ

$(x_0, x_0) \in S \Rightarrow$

$x_0 = x(0) = c_1$

$x_0 = y(0) = c_2$

$f(x_0) = v(0) \Rightarrow f^2(x_0) = v^2(0) = -c_2^2 - c_1^2 + c_3$

$f^2(x_0) = -2x_0^2 + c_3 \Rightarrow c_3 = f^2(x_0) + 2x_0^2$

$$\Rightarrow \left.\begin{aligned} x(\tau) &= x_0 e^{\tau} \\ y(\tau) &= x_0 e^{-\tau} \\ v^2(\tau) &= -x_0^2 (e^{-2\tau} + e^{2\tau}) + f^2(x_0) + 2x_0^2 \end{aligned}\right\} \Rightarrow x_0^2 = xy$$

budući da je xy nužno ≥ 0, rješenje možemo definirati samo na I i III kvadrantu

$e^{\tau} = \frac{x}{x_0} \Rightarrow e^{2\tau} = \frac{x^2}{x_0^2} = \frac{x^{\cancel{2}}}{\cancel{x}y} = \frac{x}{y} \Rightarrow e^{-2\tau} = \frac{y}{x}$

$x_0^2 = xy \Rightarrow \boxed{x_0 = \operatorname{sign}(x)\sqrt{xy}}$ ⇝ x_0, x i y su istog predznaka

$\Rightarrow u^2(x,y) = v^2(\tau; x_0) = -xy\left(\frac{y}{x} + \frac{x}{y}\right) + f^2(\operatorname{sign}(x)\sqrt{xy}) + 2xy$

$= \underline{-(x-y)^2 + f^2(\operatorname{sign}(x)\sqrt{xy})}$ ⇝ ovo mora biti ≥ 0 pa skup na kojem možemo definirati rješenje ovisi o f

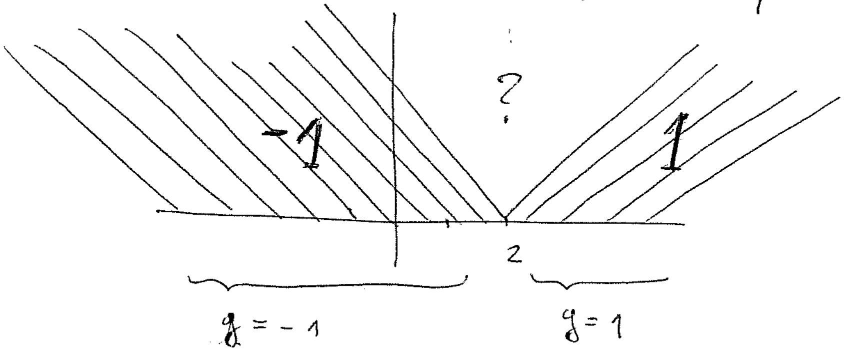
NAP. Alternativno se mogla uvesti na početku supstitucija $v = \frac{u^2}{2}$ jer je tad:

$$\left.\begin{aligned} v_x &= u u_x \\ v_y &= u u_y \end{aligned}\right\} \Rightarrow \left.\begin{aligned} &\boxed{x v_x - y v_y = y^2 - x^2} \\ &v(x,x) = \frac{u^2(x,x)}{2} = \frac{f^2(x)}{2} \end{aligned}\right\}$$

riješimo po v i samo na kraju vratimo supstituciju

3) $\begin{cases} u_t + u u_x = 0 \\ u(0,\cdot) = g \end{cases}$, $g(x) = \begin{cases} -1 , & x < 2 \\ 1 , & x \geq 2 \end{cases}$

Rj: Karakteristike su oblika $x(t) = g(x_0)t + x_0$, pri čemu $x_0 \in \mathbb{R}$.



Odredimo rješenje u području gdje nemamo karakteristika.

① R-H uvjet u točki $t=0, x=2$

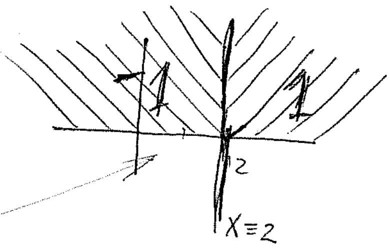
$$\left.\begin{array}{l} u_\ell = -1 \\ u_r = 1 \\ F_\ell = 1 \\ F_r = 1 \end{array}\right\} \quad -2\dot{s} = 0 \Rightarrow s(t) = C$$

$$s(0) = 2 \Rightarrow \boxed{C = 2}$$

$\Rightarrow x(t) = s(t) = 2$ je krivulja šoka

Time smo dobili :

$$u(t,x) = \begin{cases} -1 , & x < 2 \\ 1 , & x \geq 2 \end{cases} , \quad t \in \mathbb{R}_0^+$$



Međutim, ovo nije entropijsko rješenje jer je $u_\ell < u_r$
(također, uočavamo da karakteristike izlaze iz krivulje šoka što nije dobro)

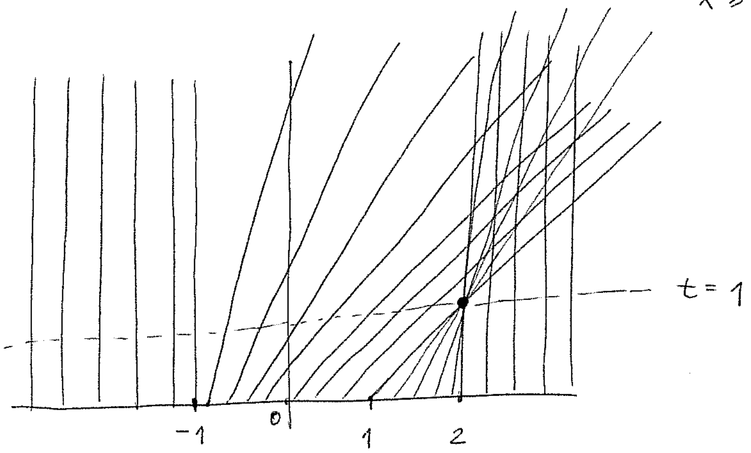
② ENTROPIJSKO RJEŠENJE

Proširimo neprekinuto rješenje. Želimo da na pravcu $t = -x+2$ bude vrijednost -1, dok na pravcu $t = x-2$ bude vrijednost 1. To postižemo t.d. definiramo

$$u(t,x) = \begin{cases} -1 , & t < -x+2 \\ \frac{x-2}{t} , & -x+2 \leq t \leq x-2 \\ 1 , & t > x-2 \end{cases}$$

4) a) $\begin{cases} u_t + u u_x = 0 \\ u(0,\cdot) = g \end{cases}$, $g(x) = \begin{cases} 0 & , \; x < -1 \\ x+1 & , \; -1 \le x < 0 \\ 1 & , \; 0 \le x < 1 \\ 2-x & , \; 1 \le x < 2 \\ 0 & , \; x \ge 2 \end{cases}$

Rj:



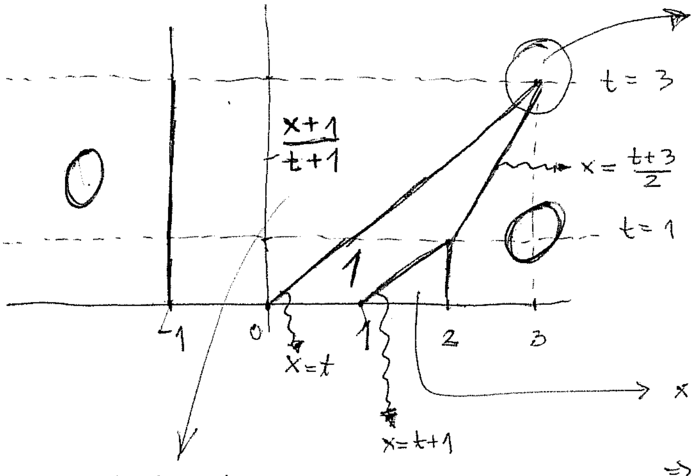
Trebamo izračunati krivulju šoka u točki $t=1$, $x=2$.

$$\left.\begin{aligned} u_l &= 1 \\ u_r &= 0 \\ F_l &= \tfrac{1}{2} \\ F_r &= 0 \end{aligned}\right\} \Rightarrow \dot{s} = \frac{1}{2} \Rightarrow s(t) = \frac{1}{2}t + C$$

$$s(1) = 2 \Rightarrow \frac{1}{2} + C = 2 \Rightarrow \boxed{C = \frac{3}{2}}$$

Krivulja šoka je $\boxed{x(t) = s(t) = \frac{t+3}{2}}$.

Skicirajmo sada rješenje:



u ovoj točki moramo ponovno računati krivulju šoka jer se mijenja vrijednost slijeva

$$x = (x_0+1)t + x_0 = x_0(t+1) + t$$

$$\Rightarrow x_0 = \frac{x-t}{t+1}$$

$$\Rightarrow u(t,x) = g(x_0) = \frac{x-t}{t+1} + 1 = \frac{x+1}{t+1}$$

$$x = (2-x_0)t + x_0 = x_0(1-t) + 2t$$

$$\Rightarrow x_0 = \frac{x-2t}{1-t}$$

$$\Rightarrow u(t,x) = 2 - \frac{x-2t}{1-t} = \boxed{\frac{2-x}{1-t}}$$

Krivulja šoka u točki $t=3$, $x=3$.

$$\left.\begin{aligned} u_l &= \frac{s(t)+1}{t+1} \\ u_r &= 0 \\ F_l &= \frac{1}{2}\left(\frac{s(t)+1}{t+1}\right)^2 \\ F_r &= 0 \end{aligned}\right\} \Rightarrow \quad \dot{s} = \frac{s+1}{2t+2}$$

$$\frac{ds}{s+1} = \frac{dt}{2t+2} \quad / \int \quad \Rightarrow \quad \ln|s+1| = \ln\sqrt{2t+2} + C$$

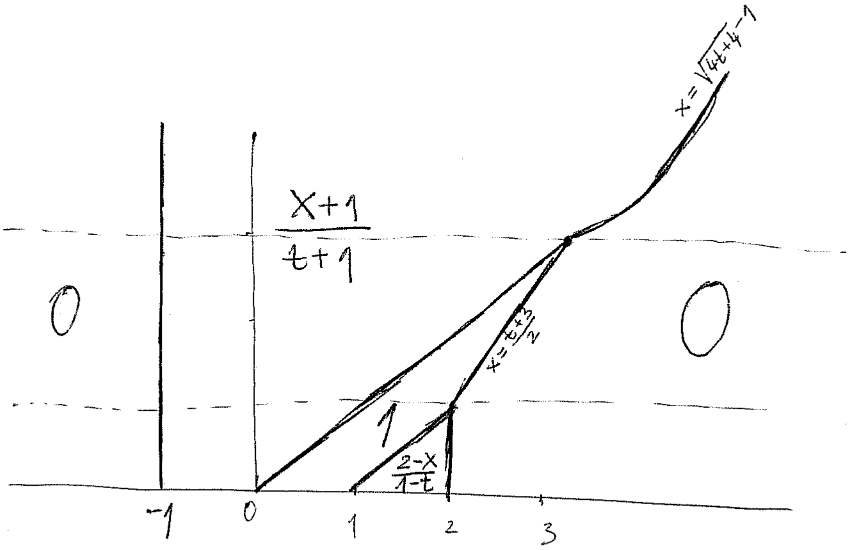
$$\Rightarrow \quad s+1 = C\sqrt{2t+2}$$

$$s(t) = C\sqrt{2t+2} - 1$$

$$s(3) = 3 \Rightarrow \sqrt{8}\,C - 1 = 3 \Rightarrow C = \frac{4}{2\sqrt{2}} = \sqrt{2}$$

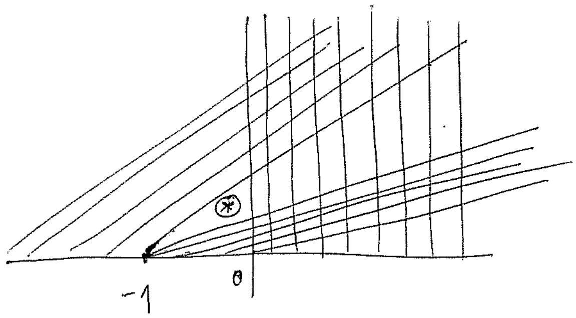
$$\Rightarrow \boxed{s(t) = \sqrt{4t+4} - 1} =: x(t)$$

Sad možemo nacrtati skicu rješenja:



4) b) $\begin{cases} u_t + u u_x = 0 \\ u(0,\cdot) = g \end{cases}$, $g(x) = \begin{cases} 1 , & x < -1 \\ 2 , & -1 \leq x < 0 \\ 0 , & x \geq 0 \end{cases}$

Rj:



(1) Odredimo krivulju šoka u točki $t=0, x=0$.

$$\left.\begin{array}{l} u_l = 2 \\ u_r = 0 \\ F_l = 2 \\ F_r = 0 \end{array}\right\} \Rightarrow 2\dot{s} = 2 \Rightarrow s(t) = t + C$$

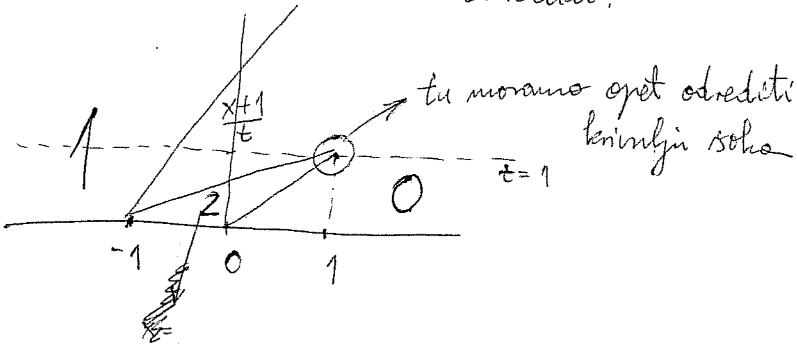
$$s(0) = 0 \Rightarrow C = 0$$

$\boxed{s_1(t) = t}$ ... prva krivulja šoka

(2) Odredimo rješenje na području (*).

Stavimo $\frac{x+1}{t}$. Tada je na pravcu $x = +t - 1$ vrijednost jednaka $\frac{+t-1+1}{t} = +1$, dok je na pravcu $x = +2t - 1$ jednaka $\frac{+2t-1+1}{t} = +2$, što smo i htjeli.

Skicirajmo što smo do sad odredili.



(3)
$$\left.\begin{array}{l} u_l = \frac{s+1}{t} \\ u_r = 0 \\ F_l = \frac{1}{2}\left(\frac{s+1}{t}\right)^2 \\ F_r = 0 \end{array}\right\} \Rightarrow \dot{s} = \frac{s+1}{2t} \Rightarrow \frac{ds}{s+1} = \frac{dt}{2t} \Big/ \int$$

$$\Rightarrow \ln|s+1| = \ln\sqrt{2t} + C$$

$$\Rightarrow s+1 = C\sqrt{2t}$$

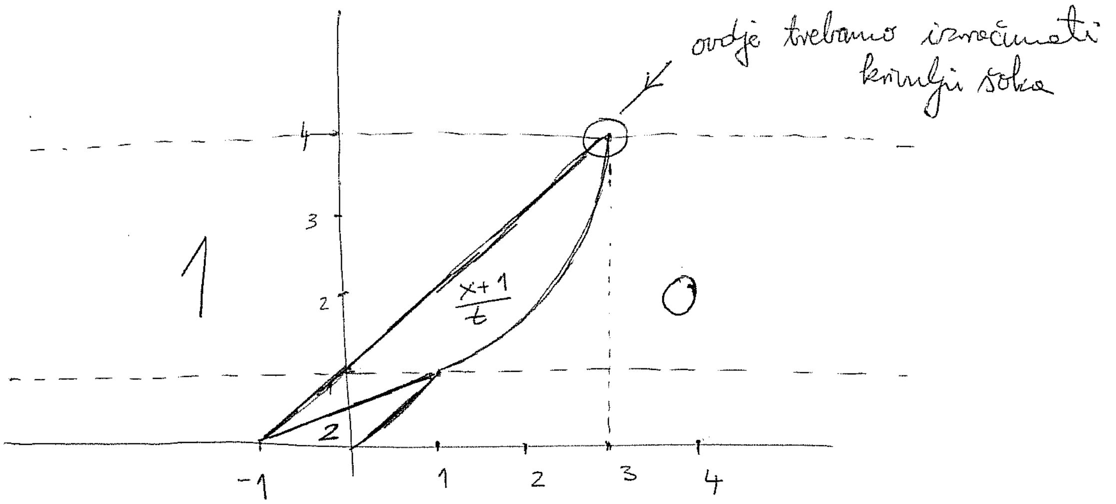
$$s(t) = C\sqrt{2t} - 1$$

$$s(1) = 1 \Rightarrow \sqrt{2}C - 1 = 1$$

$$\Rightarrow \boxed{C = \sqrt{2}}$$

$\Rightarrow \boxed{s_2(t) = \sqrt{4t} - 1}$ ... druga krivulja šoka

Sada imamo:



④ krivulja šoka u točki $t=4, x=3$.

$$\left.\begin{array}{l} u_l=1 \\ u_r=0 \\ F_l=\frac{1}{2} \\ F_r=0 \end{array}\right\} \Rightarrow \dot{s}=\frac{1}{2} \Rightarrow s(t)=\frac{1}{2}t+C$$

$$\left.\begin{array}{l} s(t)=\frac{1}{2}t+C \\ 3=s(4)=2+C \Rightarrow C=1 \end{array}\right\} \Rightarrow \boxed{s_3(t)=\frac{1}{2}t+1} \text{ ... treća krivulja šoka}$$

Konačno je rješenje dano s: