PDJ 1

KOLOKVIJ – 2014/2015 (20.01.2015.)

1) a) $-xu_x + yu_y = xu^2$

① $y \quad x \quad 0$

$$\Rightarrow \varphi_x = y \Rightarrow \varphi = xy + C(y,u)$$
$$\varphi_y = x \Rightarrow x + C_y = x \Rightarrow C_y = 0 \Rightarrow C(y,u) = D(u)$$
$$\varphi_u = 0 \Rightarrow D' = 0 \Rightarrow D = \text{const.}$$
$$\Rightarrow \boxed{\varphi(x,y,u) = xy}$$

② $1 \quad 0 \quad \frac{1}{u^2}$

$$\Rightarrow \psi_x = 1 \Rightarrow \psi = x + C(y,u)$$
$$\psi_y = 0 \Rightarrow C(y,u) = D(u)$$
$$\psi_u = \frac{1}{u^2} \Rightarrow D' = \frac{1}{u^2} \Rightarrow D = -\frac{1}{u} + \text{const.}$$
$$\Rightarrow \boxed{\psi(x,y,u) = x - \frac{1}{u}}$$

Provjera nezavisnosti:

$$\frac{\partial(\varphi,\psi)}{\partial(x,y,u)} = \begin{bmatrix} y & x & 0 \\ 1 & 0 & \frac{1}{u^2} \end{bmatrix} \Rightarrow \left|\frac{\partial(\varphi,\psi)}{\partial(x,y)}\right| = x = 0 \quad \text{samo za } x=0$$

$$\left|\frac{\partial(\varphi,\psi)}{\partial(x,u)}\right| = \frac{y}{u^2} = 0 \quad \text{samo za } y=0$$

$\Rightarrow$ van ishodišta je gornja matrica punog ranga.

$\Rightarrow$ rješenje je dano s $F(xy, x-\frac{1}{u}) = 0$

$\Rightarrow \exists g$ t.d. $x - \frac{1}{u} = g(xy)$

$$\Rightarrow \boxed{u = \frac{1}{x - g(xy)}}$$

1) b) $u(x,y) = \dfrac{1}{x - g(xy)}$

$$u(x,1) = e^{-x} \Rightarrow \frac{1}{x-g(x)} = e^{-x} \Rightarrow x - g(x) = e^{x}$$

$$\Rightarrow \boxed{g(x) = x - e^{x}}$$

$$\Rightarrow \boxed{u(x,y) = \frac{1}{x - xy - e^{xy}}}$$

PROVJERA:

$$u_x = -\frac{1}{(x-xy-e^{xy})^2}\left(1 - y - e^{xy}y\right)$$

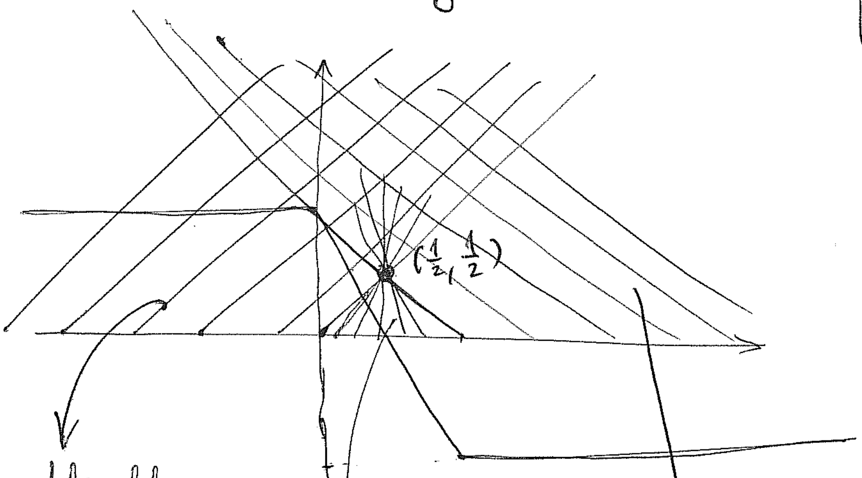
$$u_y = -\frac{1}{(x-xy-e^{xy})^2}\left(-x - e^{xy}x\right)$$

$$\Rightarrow -xu_x + yu_y = -\frac{1}{(x-xy-e^{xy})^2}\left(-x + \cancel{xy} + \cancel{xye^{xy}} - \cancel{xy} - \cancel{xye^{xy}}\right)$$

$$= \frac{x}{(x-xy-e^{xy})^2}$$

$$= xu^2 \checkmark$$

2) $\begin{cases} u_t + u u_x = 0 \\ u(0,\cdot) = g \end{cases}$, $g(x) = \begin{cases} 1 & , \ x < 0 \\ 1-2x & , \ 0 \le x < 1 \\ -1 & , \ x \geq 1 \end{cases}$



karakteristike su oblika
$x(t) = g(x_0)t + x_0 = t + x_0$

karakteristike su oblika
$x(t) = (1-2x_0)t + x_0$

karakteristike su oblika
$x(t) = -t + x_0$

Pogledajmo presjek s prvom lijevom i prvom desnom karakteristikom:

$\begin{cases} x(t) = t \\ x(t) = (1-2x_0)t + x_0 \end{cases} \Rightarrow t = (1-2x_0)t + x_0$

$2x_0 t = x_0 \ /: x_0 \neq 0 \quad (x_0 \in \langle 0,1 \rangle$

$2t = 1$

$\boxed{t = \frac{1}{2}}$

$\Rightarrow$ sijeku se u točki $(\frac{1}{2}, \frac{1}{2})$

$\begin{cases} x(t) = -t + 1 \\ x(t) = (1-2x_0)t + x_0 \end{cases} \Rightarrow -t + 1 = (1-2x_0)t + x_0$

$2(x_0 - 1)t = x_0 - 1 \ /: (x_0 - 1) \neq 0$
$(x_0 \in \langle 0,1 \rangle)$

$2t = 1$

$t = \frac{1}{2}$

$\Rightarrow$ sve karakteristike iz područja $x_0 \in \langle 0,1 \rangle$ prolaze točkom $(\frac{1}{2}, \frac{1}{2})$.

U točki $(\frac{1}{2}, \frac{1}{2})$ se krivulje sijeku za prvi t pa izvodimo krivulju šoka:

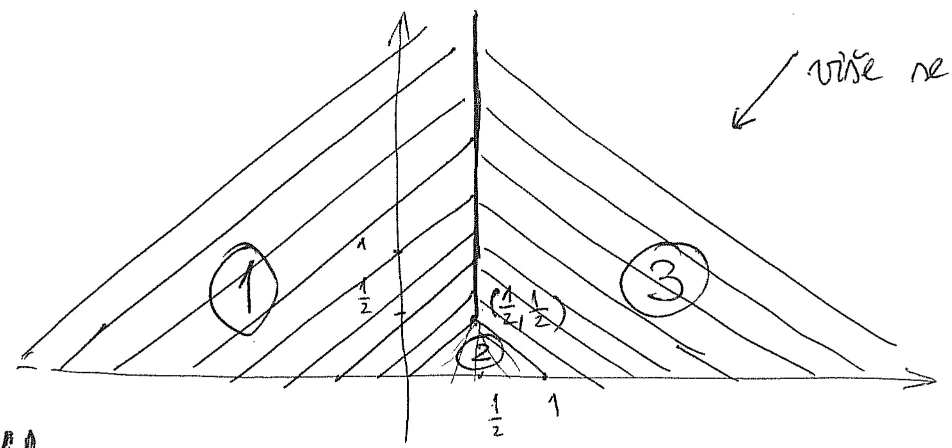
R-H uvjet u točki $\underline{t = \frac{1}{2}}, \underline{x = \frac{1}{2}}$

$$\left.\begin{array}{l} \left.\begin{array}{l} u_l = 1 \\ u_r = -1 \end{array}\right\} \Rightarrow [u] = 2 \\ \\ \left.\begin{array}{l} F_l = \frac{1}{2} \\ F_r = \frac{1}{2} \end{array}\right\} \Rightarrow [F] = 0 \end{array}\right\} \Rightarrow \dot{s} = 0 \Rightarrow s(t) = C \ \ldots \text{ konstanta}$$

$$s\left(\tfrac{1}{2}\right) = \frac{1}{2} \Rightarrow C = \frac{1}{2}$$

$$\Rightarrow \boxed{x(t) = s(t) = \frac{1}{2}} \ \ldots \text{ krivulja šoka}$$

Sada opet crtamo:



U ① je rješenje jednako 1.
U ③ je rješenje —"— -1.
U ② moramo malo računati:

$$x(t) = (1-2x_0)t + x_0 \Rightarrow x = t + (1-2t)x_0$$

$$z(t) = g(x_0) \qquad \Rightarrow \boxed{x_0 = \frac{x-t}{1-2t}}$$

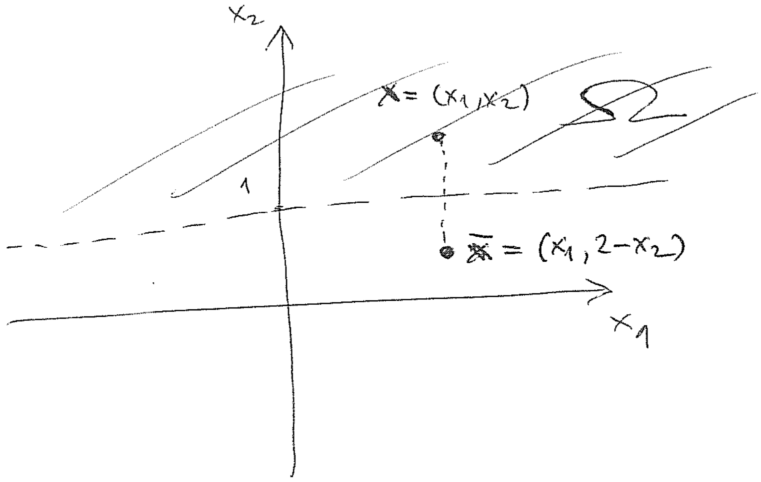
$$\Rightarrow u(t,x) = z(t) = g\left(\frac{x-t}{1-2t}\right) = 1 - 2\left(\frac{x-t}{1-2t}\right)$$

$$= \frac{1 - \cancel{2t} - 2x + \cancel{2t}}{1-2t} = \frac{1-2x}{1-2t}$$

$$\Rightarrow u(t,x) = \begin{cases} 1, & x < t \\ \frac{1-2x}{1-2t}, & t \leq x \leq -t+1 \\ -1, & x > -t+1 \end{cases}, \quad t < \frac{1}{2}$$

$$u(t,x) = \begin{cases} 1, & x < \frac{1}{2} \\ -1, & x \geq \frac{1}{2} \end{cases}, \quad t \geq \frac{1}{2}$$

3.) a)



$$\Rightarrow \quad G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \Phi(\bar{\mathbf{x}}-\mathbf{y})$$

$$= -\frac{1}{4\pi} \ln\left((x_1-y_1)^2 + (x_2-y_2)^2\right) + \frac{1}{4\pi} \ln\left((x_1-y_1)^2 + (x_2+y_2-2)^2\right)$$

b) Rješenje je dano s

$$u(x_1,x_2) = -\int_{-\infty}^{+\infty} \frac{\partial G}{\partial \nu_{\mathbf{y}}}(x_1,x_2;y_1,1)\, g(y_1)\, dy_1$$

$$-\frac{\partial G}{\partial \nu_{\mathbf{y}}}(\mathbf{x};\mathbf{y}) = \frac{\partial G}{\partial y_2}(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi} \frac{2(y_2-x_2)}{(x_1-y_1)^2+(x_2-y_2)^2} + \frac{1}{4\pi} \frac{2(x_2+y_2-2)}{(x_1-y_1)^2+(x_2+y_2-2)^2}$$

$$\Rightarrow \quad -\frac{\partial G}{\partial \nu_{\mathbf{y}}}(\mathbf{x};y_1,1) = \frac{1}{\pi} \frac{x_2-1}{(x_1-y_1)^2+(x_2-1)^2}$$

$$\Rightarrow \quad u(x_1,x_2) = \frac{x_2-1}{\pi} \int_{-\infty}^{+\infty} \frac{g(y_1)}{(x_1-y_1)^2+(x_2-1)^2}\, dy_1$$

c)
$$g(x) = \begin{cases} |x|-1, & |x|<1 \\ 0, & \text{inače} \end{cases}$$

$$\Rightarrow \quad u(x_1,x_2) = \frac{x_2-1}{\pi} \underbrace{\int_{-1}^{0} \frac{-y_1-1}{(x_1-y_1)^2+(x_2-1)^2}\, dy_1}_{I_1} + \frac{x_2-1}{\pi} \underbrace{\int_{0}^{1} \frac{y_1-1}{(x_1-y_1)^2+(x_2-1)^2}\, dy_1}_{I_2}$$

3) c) (nastavak)

$$I_1 = \int_{-1}^{0} \frac{-y_1-1}{(x_1-y_1)^2+(x_2-1)^2}\,dy_1$$

$$= -\frac{1}{2}\int_{-1}^{0} \frac{2(y_1-x_1)}{(x_1-y_1)^2+(x_2-1)^2}\,dy_1 - \frac{1}{2}\int_{-1}^{0} \frac{x_1+2}{(x_1-y_1)^2+(x_2-1)^2}\,dy_1 \quad \Big\downarrow z = y_1 - x_1$$

$$= -\frac{1}{2}\ln\left((x_1-y_1)^2+(x_2-1)^2\right)\Big|_{-1}^{0} - \frac{x_1+2}{2}\int_{-1-x_1}^{-x_1} \frac{dz}{z^2+(x_2-1)^2}$$

$$= -\frac{1}{2}\ln\left(x_1^2+(x_2-1)^2\right) + \frac{1}{2}\ln\left((x_1+1)^2+(x_2-1)^2\right)$$

$$- \frac{x_1+2}{2}\,\frac{1}{x_2-1}\,\mathrm{arctg}\left(\frac{y_1-x_1}{x_2-1}\right)\Big|_{-1}^{0}$$

$$= \ln\sqrt{\frac{(x_1+1)^2+(x_2-1)^2}{x_1^2+(x_2-1)^2}} - \frac{x_1+2}{2(x_2-1)}\,\mathrm{arctg}\left(\frac{x_1}{1-x_2}\right) +$$

$$+ \frac{x_1+2}{2(x_2-1)}\,\mathrm{arctg}\left(\frac{x_1+1}{1-x_2}\right)$$

$$I_2 = \int_0^1 \frac{y_1-1}{(x_1-y_1)^2+(x_2-1)^2}\,dy_1 = \frac{1}{2}\int_0^1 \frac{2(y_1-x_1)}{(x_1-y_1)^2+(x_2-1)^2}\,dy_1 + \frac{x_1-2}{2}\int_0^1 \frac{dy_1}{(y_1-x_1)^2+(x_2-1)^2}$$

$$= \frac{1}{2}\ln\left((x_1-y_1)^2+(x_2-1)^2\right)\Big|_0^1 + \frac{x_1-2}{2}\,\frac{1}{x_2-1}\,\mathrm{arctg}\left(\frac{y_1-x_1}{x_2-1}\right)\Big|_0^1$$

$$= \frac{1}{2}\ln\left((x_1-1)^2+(x_2-1)^2\right) - \frac{1}{2}\ln\left((x_1)^2+(x_2-1)^2\right) + \frac{x_1-2}{2(x_2-1)}\,\mathrm{arctg}\left(\frac{x_1-1}{1-x_2}\right)$$

$$- \frac{x_1-2}{2(x_2-1)}\,\mathrm{arctg}\left(\frac{x_1}{1-x_2}\right)$$

3, c) (nastavak)

$$\Rightarrow I_1+I_2 = \ln\sqrt{\frac{(x_1+1)^2+(x_2-1)^2}{x_1^2+(x_2-1)^2}} + \ln\sqrt{\frac{(x_1-1)^2+(x_2-1)^2}{x_1^2+(x_2-1)^2}}$$

$$-\frac{x_1}{x_2-1}\operatorname{arctg}\left(\frac{x_1}{1-x_2}\right)+\frac{x_1+2}{2(x_2-1)}\operatorname{arctg}\left(\frac{x_1+1}{1-x_2}\right)$$

$$+\frac{x_1-2}{2(x_2-1)}\operatorname{arctg}\left(\frac{x_1-1}{1-x_2}\right)$$

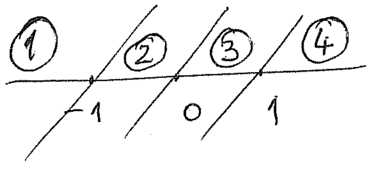
$$\Rightarrow u(x_1,x_2)=\frac{x_2-1}{\pi}\ln\frac{\sqrt{\left((x_1+1)^2+(x_2-1)^2\right)\left((x_1-1)^2+(x_2-1)^2\right)}}{x_1^2+(x_2-1)^2}$$

$$-\frac{x_1}{\pi}\operatorname{arctg}\left(\frac{x_1}{1-x_2}\right)+\frac{x_1+2}{2\pi}\operatorname{arctg}\left(\frac{x_1+1}{1-x_2}\right)$$

$$+\frac{x_1-2}{2\pi}\operatorname{arctg}\left(\frac{x_1-1}{1-x_2}\right)$$

PROVJERA RUBNOG UVJETA:

- izraz s ln je nula za $x_2=1$ jer imamo član $\frac{x_2-1}{\pi}$
- $1-x_2$ se javlja u nazivniku arctg pa moramo raditi detaljniju analizu



① $x_1,\ x_1+1,\ x_1-1<0$

$\Rightarrow$ svi arctg-si teže k $-\frac{\pi}{2}$ kad $x_2\to 1$

$\Rightarrow$ na limesu imamo

$$u(x_1,\cancel{x_2}) = -\frac{x_1}{\pi}\left(-\frac{\pi}{2}\right)+\frac{x_1+2}{2\pi}\left(-\frac{\pi}{2}\right)+\frac{x_1-2}{2\pi}\left(-\frac{\pi}{2}\right)=\frac{x_1}{2}-\frac{x_1+2}{4}-\frac{x_1-2}{4}=0$$

④ $x_1,x_1+1,x_1-1>0$

$\Rightarrow$ svi arctg-si teže k $\frac{\pi}{2}$ kad $x_2\to 1$

pa analogno kao u ① imamo $u(x_1,1)=0$

(2) $x_1+1>0$

$x_1, x_1-1<0$

$$\Rightarrow u(x_1,1) = -\frac{x_1}{\pi}\left(-\frac{\pi}{2}\right) + \frac{x_1+2}{2\pi}\frac{\pi}{2} + \frac{x_1-2}{2\pi}\left(\frac{\pi}{2}\right)$$

$$= \frac{x_1}{2} + \frac{x_1+2}{4} - \frac{x_1-2}{4} = \frac{2x_1+\cancel{x_1}+2-\cancel{x_1}+2}{4} = \frac{2x_1+4}{4}$$

(3) $x_1+1>0$

$x_1>0$

$x_1-1<0$

$$u(x_1,1) = -\frac{x_1}{\pi}\frac{\pi}{2} + \frac{x_1+2}{2\pi}\frac{\pi}{2} + \frac{x_1-2}{2\pi}\left(-\frac{\pi}{2}\right)$$

$$= \frac{-2x_1+\cancel{x_1}+2-x_1+2}{4} = \frac{-2x_1+4}{4}$$

4) Uvedimo $v_1(t,x) = u(t,x) - Mt$, gdje je $M := \|f\|_{L^\infty(Cl\,\Omega_T)}$ i u rj.

$$\begin{cases} u_t - \Delta u = f & u\ \Omega_T \\ u = g & na\ \Gamma_T \end{cases}.$$

Tada imamo:
$$\partial_t v_1 - \Delta v_1 = u_t - \Delta u - M = f - M \leq 0$$

$\Rightarrow$ (Uputa)
$$\max_{Cl\,\Omega_T} v_1 = \max_{\Gamma_T} v_1$$

Sada imamo:

$$\max_{Cl\,\Omega_T} u = \max_{Cl\,\Omega_T}(v_1 + Mt) \leq \max_{Cl\,\Omega_T} v_1 + \max_{Cl\,\Omega_T} Mt$$
$$= \max_{\Gamma_T} v_1 + MT$$
$$= \max_{\Gamma_T}(u - Mt) + MT$$
$$\leq \max_{\Gamma_T} u + MT$$
$$\leq \max_{\Gamma_T} |u| + MT$$
$$= \max_{\Gamma_T}|g| + T \max_{Cl\,\Omega_T}|f|$$

Analogno drugu ocjenu dobivamo s $v_2(t,x) = -u(t,x) - Mt$ jer
$$\partial_t v_2 - \Delta v_2 = -(u_t - \Delta u) - M = -f - M \leq 0$$

$$\Rightarrow \max_{Cl\,\Omega_T}(-u) = \max_{Cl\,\Omega_T}(v_2 + Mt) \leq \max_{\Gamma_T}(-u - Mt) + MT$$
$$\leq \max_{\Gamma_T}|g| + T\max_{Cl\,\Omega_T}|f|$$

$$\Rightarrow \max_{Cl\,\Omega_T}|u| = \max\left\{\max_{Cl\,\Omega_T} u, \max_{Cl\,\Omega_T}(-u)\right\} \leq \max_{\Gamma_T}|g| + T\max_{Cl\,\Omega_T}|f|.$$

4) drugi način

Probajmo s: $w_1(t,x) = u(t,x) + \frac{M}{2d}|x|^2$

$$\Rightarrow \quad \partial_t w_1 = u_t$$
$$\Delta w_1 = \Delta u + \frac{M}{2d} 2d = \Delta u + M$$
$$\Rightarrow \quad \partial_t w_1 - \Delta w_1 = u_t - \Delta u - M = f - M \leq 0$$

$$\Rightarrow \quad \max_{Cl\Omega_T} u = \max_{Cl\Omega_T} \underbrace{\left(w_1 - \frac{M}{2d}|x|^2\right)}_{\leq w_1}$$
$$\leq \max_{\Gamma_T} \left(u + \frac{M}{2d}|x|^2\right)$$
$$\leq \max_{\Gamma_T} |u| + \underbrace{\left(\max_{\Gamma_T} |x|^2\right)}_{= \max_{Cl\Omega} |x|^2 =: C} \frac{M}{2d}$$

$\rightsquigarrow$ konstanta koja ovisi o domeni

$$\leq \max_{\Gamma_T} |g| + \frac{C}{2d} \max_{Cl\Omega_T} |f|$$

Analogno s $w_2(t,x) = -u(t,x) + \frac{M}{2d}|x|^2$

$$\Rightarrow \quad \partial_t w_2 - \Delta w_2 = -u_t + \Delta u - M = -(u_t - \Delta u) - M = -f - M \leq 0$$

$$\Rightarrow \quad \max_{Cl\Omega_T}(-u) = \max_{Cl\Omega_T}\left(w_2 - \frac{M}{2d}|x|^2\right)$$
$$\leq \max_{\Gamma_T}\left(-u + \frac{M}{2d}|x|^2\right)$$
$$\leq \max_{\Gamma_T}|g| + \frac{C}{2d}\max_{Cl\Omega_T}|f|$$

$$\Rightarrow \quad \max_{Cl\Omega_T} \leq \max_{\Gamma_T}|g| + \frac{C}{2d}\max_{Cl\Omega_T}|f|$$

5) $\begin{cases} u_{tt} - c^2 u_{xx} + a u_t + b u = 0 \\ u(0,\cdot) = g \\ u_t(0,\cdot) = h \end{cases}$ $\left(b = \frac{a^2}{4} \right)$

Uvodimo supstituciju $v(t,x) = u(t,x)\, e^{\alpha t}$

$$v_t = u_t e^{\alpha t} + \alpha u e^{\alpha t}$$
$$v_{tt} = u_{tt} e^{\alpha t} + 2\alpha u_t e^{\alpha t} + \alpha^2 u e^{\alpha t}$$
$$v_{xx} = u_{xx} e^{\alpha t}$$

$$\Rightarrow \quad v_{tt} - c^2 v_{xx} = \left(u_{tt} - c^2 u_{xx} + 2\alpha u_t + \alpha^2 u \right) e^{\alpha t}$$

Stavimo $2\alpha = a \Rightarrow \boxed{\alpha = \frac{a}{2}}$

$$\Rightarrow \quad \alpha^2 = \frac{a^2}{4} = b$$

$$\Rightarrow \quad v_{tt} - c^2 v_{xx} = \underbrace{\left(u_{tt} - c^2 u_{xx} + a u_t + b u \right)}_{=0} e^{\frac{at}{2}}$$

$$\Rightarrow \quad \begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(0,\cdot) = g \\ v_t(0,\cdot) = h + \frac{a}{2} g \end{cases}$$

$$\Rightarrow \quad v(t,x) = \frac{1}{2}\left(g(x+ct) + g(x-ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \left(h(\xi) + \frac{a}{2} g(\xi) \right) d\xi$$

$$\Rightarrow \quad \boxed{\begin{aligned} u(t,x) &= e^{-\frac{at}{2}} v(t,x) \\ &= \frac{e^{-\frac{at}{2}}}{2} \left(g(x+ct) + g(x-ct) \right) + \frac{e^{-\frac{at}{2}}}{2c} \int_{x-ct}^{x+ct} \left(h(\xi) + \frac{a}{2} g(\xi) \right) d\xi \end{aligned}}$$