PRIMJER 10. (Odgovor na ①; nekompaktnost ulaganja za $q = p^*$ i $\lambda = m - \frac{d}{p}$)

$\Omega \subseteq \mathbb{R}^d$, $j \geq 0$, $m \geq 1$

$p \in [1, \infty\rangle$

1) $mp < d$

Neka je $k \geq 1$ t.d. $d - mp < k \leq d$.

Radi jednostavnosti, promotrimo samo slučaj $k = d$.

$$q := p^* = \frac{dp}{d - mp}.$$

Želimo pokazati da ulaganje

$$W_0^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega_0)$$

nije kompaktno, pri čemu je $\Omega_0 \Subset \Omega$.

Neka su $a_m \in \Omega_0$ t.d. $K(a_m, r_m) \subseteq \Omega_0$

& $r_m > 0, r_m \leq 1$ & $K(a_m, r_m) \cap K(a_\ell, r_\ell) = \emptyset$, $m \neq \ell$.

Neka je $\varphi \in C_c^\infty(K(0,1))$, $\varphi \neq 0$.

$$\varphi_m(x) := r_m^{\,j+m-\frac{d}{p}}\, \varphi\left(\frac{x - a_m}{r_m}\right)$$

- $\varphi_m \in C_c^\infty(\Omega)$
- $\operatorname{supp} \varphi_m \subseteq K(a_m, r_m) \Rightarrow \varphi_m$ imaju disjunktne nosače
- $\partial^\alpha \varphi_m(x) = r_m^{\,j+m-\frac{d}{p}-|\alpha|} (\partial^\alpha \varphi)\left(\frac{x - a_m}{r_m}\right)$

$|\alpha| \leq j+m$:

$$\int_\Omega |\partial^\alpha \varphi_m(x)|^p dx = r_m^{(j+m-\frac{d}{p}-|\alpha|)p} \int_{K(a_m, r_m)} \left|(\partial^\alpha \varphi)\left(\frac{x - a_m}{r_m}\right)\right|^p dx = \left| \begin{array}{l} y = \frac{x - a_m}{r_m} \\ dy = r_m^{-d} dx \end{array} \right|$$

$$= r_m^{(j+m-|\alpha|)p} \int_{K(0,1)} |\partial^\alpha \varphi(y)|^p dy$$

$\boxed{\begin{array}{c} j+m-|\alpha| \geq 0 \\ \& \\ r_m \leq 1 \end{array}}$

$$\leq \|\partial^\alpha \varphi\|_{p,\Omega} \Rightarrow (\varphi_m) \text{ omeđen u } W^{j+m,p}(\Omega).$$

S druge strane, za $|\alpha| = j$ imamo

$$\int_{\Omega_0} |\partial^\alpha \varphi_m(x)|^q dx = \underbrace{r_m^{(j+m-\frac{d}{p}-|\alpha|)q+d}}_{} \int_{\Omega_0} |\partial^\alpha \varphi(y)|^q dy = \|\partial^\alpha \varphi\|_{q,\Omega_0}$$

$$= r_m^{\frac{mp-d}{p}\cdot\frac{dp}{d-mp}+d}$$

$$= r_m^0 = 1$$

$$\Rightarrow \quad \|\varphi_m\|_{j,q,\Omega_0} \geq |\varphi_m|_{j,q,\Omega_0} = \|\varphi\|_{j,q,\Omega_0} > 0$$

$\Rightarrow$ (φ_m) nije C-niz u $W^{j,q}(\Omega_0)$, naime

$B(a_m, r_m) \cap B(a_\ell, r_\ell) = \emptyset$

$$\|\varphi_m - \varphi_\ell\|^2_{j,q,\Omega_0} = \|\varphi_m - \varphi_\ell\|^2_{j,q,B(a_m,r_m)} + \|\varphi_m - \varphi_\ell\|^2_{j,q,B(a_\ell,r_\ell)}$$

$$= \|\varphi_m\|^2_{j,q,B(a_m,r_m)} + \|\varphi_\ell\|^2_{j,q,B(a_\ell,r_\ell)}$$

$$= \|\varphi_m\|^2_{j,q,\Omega_0} + \|\varphi_\ell\|^2_{j,q,\Omega_0}$$

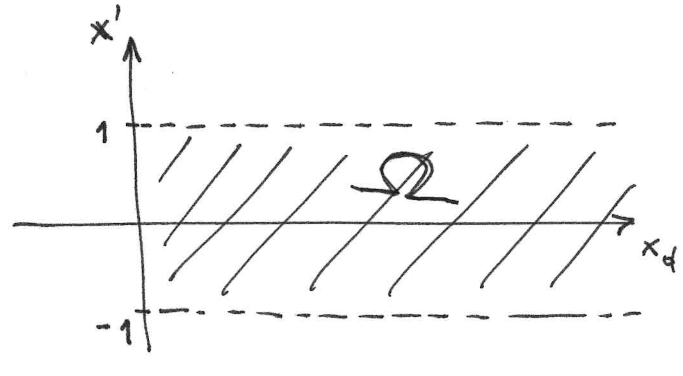
$$= 2\|\varphi\|^2_{j,q,\Omega_0} > 0 .$$

■

PRIMJER 11. (Odgovor (djelomični) na (2))

Pokažimo da ne možemo imati kompaktno ulaganje za

$$\Omega = \Omega_0 = \{(x', x_d) \in \mathbb{R}^d : |x'| < 1, \; x_d > 0\}$$



$K_m := K(\cancel{\ldots}\,(0, 1+2(m-1)), 1) \subseteq \Omega \quad \& \quad K_m \cap K_\ell = \emptyset .$

$m \geq 1$

Neka je $\varphi_1 \in C_c^\infty(K_1)$ t.d.

$$(\forall j \in \mathbb{N}_0)\ (\forall p \in [1, \infty\rangle) \quad \|\varphi_1\|_{j,p,K_1} =: A_{j,p} > 0 .$$

$$\varphi_m(x', x_d) := \varphi_1(x', x_d - 2(m-1)) \quad , \ m \geq 2$$

$$\|\varphi_m\|_{j,p,\Omega} = \|\varphi_m\|_{j,p,K_m} = \|\varphi_1\|_{j,p,K_1} = A_{j,p} > 0$$

$\Rightarrow (\varphi_m)$ je omeđen u $W^{j,p}(\Omega)$ za sve j,p.

Kao i u prošlom primjeru (φ_n) nije C-niz (disjunktni nosači).

Konstrukcija prošlog primjera se lako poopći na općeniti $\Omega \subseteq \mathbb{R}^d$ koji nije kvaziomeđen (v. [AF, 6.11]).

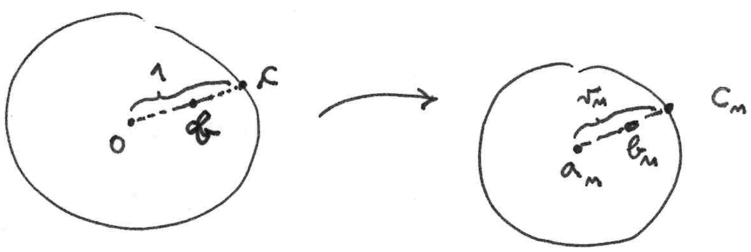
$$\left(\lim_{\substack{x \in \Omega \\ |x| \to \infty}} d(x, \partial\Omega) = 0 \ \ldots \text{kvaziomeđenost} \right)$$

Vratimo se na Primjer 10:

$(m-1)p \leq d < mp$

$$(\forall j \in \{0,1,\ldots,m\})\ (\exists b \in K(0,1))\ (\exists \beta \in \mathbb{N}_0^d,\ |\beta| = j) \quad |\partial^\beta \varphi(b)| = C > 0 .$$

(u suprotnom bi φ bila trivijalna f-ja)



$$\partial^{\beta}\varphi_m(b_m) = r_m^{\overbrace{j+m-\frac{d}{p}-|\beta|}^{=j}}\,(\partial^{\beta}\varphi)\Big(\underbrace{\frac{b_m-a_m}{r_m}}_{=b}\Big)$$

$$\Rightarrow\ |\partial^{\beta}\varphi_m(b_m)| = r_m^{m-\frac{d}{p}}\,C = r_m^{\lambda}C$$

$$\operatorname{supp}\varphi_m \subseteq K(a_m,r_m) \Rightarrow |\partial^{\beta}\varphi_m(c_m)| = 0$$

$$\Rightarrow \|\varphi_m\|_{C^{j,\lambda}(\overline{\Omega_0})} \geqslant \frac{|\partial^{\beta}\varphi_m(b_m)-\partial^{\beta}\varphi_m(c_m)|}{|b_m-c_m|^{\lambda}} = \frac{r^{\lambda}C}{|b_m-c_m|^{\lambda}}$$

$$\geqslant \frac{r_m^{\lambda}C}{r_m^{\lambda}} = C > 0$$

■

Dokaz Teorema 8. (Poincaréova nejednakost)

BSO pretp. da $\Omega \subseteq \{(x',x_d)\in\mathbb{R}^d : 0<x_d<c\}$.

$\varphi \in C_c^{\infty}(\Omega)$,

$$\varphi(x) = \int_0^{x_d} \partial_{x_d}\varphi(x',t)\,dt$$

$$\Rightarrow \|\varphi\|_{0,p,\Omega}^p = \int_{\mathbb{R}^{d-1}}\int_0^c |\varphi(x',x_d)|^p\,dx$$

$$\leqq \int_{\mathbb{R}^{d-1}}\int_0^c\Big(\int_0^{x_d}|\partial_{x_d}\varphi(x',t)|\,dt\Big)^p\,dx_d\,dx'$$

$$\leqslant \int_{\mathbb{R}^{d-1}}\int_0^c\Big(c\int_0^c|\partial_{x_d}\varphi(x',t)|\,\frac{dt}{c}\Big)^p\,dx_d\,dx'$$

Jensen

$$\leqslant \int_{\mathbb{R}^{d-1}}\int_0^c c^p\Big(\int_0^c|\partial_{x_d}\varphi(x',t)|^p\frac{dt}{c}\Big)\,dx_d\,dx'$$

$$\leqslant c^p|\varphi|_{1,p,\Omega}^p$$

ili Hölder:

$$\Big(\int_0^{x_d}1\cdot 1\,dt\Big)^p \leqslant \Big(\int_0^{x_d}|\partial_{x_d}\varphi|^p dt\Big)^{\frac{1}{p}\cdot p}\cdot\Big(\int_0^{x_d}dt\Big)^{\frac{1}{p'}\cdot p} = \Big(\int_0^{x_d}|\partial_{x_d}\varphi|^p dt\Big)\cdot x_d^{\overbrace{\frac{p}{p'}}^{p-1}}$$

■

Dokaz Korolara 9. $\varphi \in C_c^\infty(\Omega)$,

$$|\varphi|^p_{1,p,\Omega} \le \|\varphi\|^p_{1,p,\Omega} = \|\varphi\|^p_{0,p,\Omega} + |\varphi|^p_{1,p,\Omega}$$
$$\le (1+K^p)\,|\varphi|^p_{1,p,\Omega}$$

$\Rightarrow$ $|\cdot|_{1,p,\Omega}$ i $\|\cdot\|_{1,p,\Omega}$ su ekvivalentne norme na $W_0^{1,p}(\Omega)$. Iterativno dobivamo tvrdnju za $W_0^{m,p}(\Omega)$.

Poincaréova nejednakost se može za određene domene poopćiti i na $W^{m,p}(\Omega)$ (v. [Ev, §5.8.1], [Ma, §1.1.11]).

TEOREM 10. (Poincaréova nejednakost za $W^{m,p}$). $\Omega \subseteq \mathbb{R}^d$ omeđen i sa svojstvom konusa, $\Omega_0 \subset\subset \Omega$ otvoren, $m \in \mathbb{N}$, $p \in [1,\infty\rangle$.

$$\|u - \Pi_u\|_{m-1,p,\Omega} \le C|u|_{m,p,\Omega},$$

$$\Pi_u(x) := \sum_{|\alpha| \le m-1} \left(\int_\Omega u(x)\varphi_\alpha(x)\,dx \right) x^\alpha,$$

pri čemu $\varphi_\alpha \in C_c^\infty(\Omega_0)$ i C ne ovise o u.

KOROLAR 11. $\Omega \subseteq \mathbb{R}^d$ omeđen i sa svojstvom konusa. $\|\cdot\|_{m,p,\Omega}$ i $\|\cdot\|_{0,p,\Omega} + |\cdot|_{m,p,\Omega}$ su ekvivalentne norme na $W^{m,p}(\Omega)$. Štoviše, $\|\cdot\|_{0,p,\Omega}$ možemo zamijeniti s $\|\cdot\|_{0,p,\Omega_0}$ za $\Omega_0 \subset\subset \Omega$.

Za daljnja poopćenja vidi [Ma].

Zamislimo da rješavamo jednadžbu

$$-\Delta u = f, \quad u \ \Omega,$$

za $f \in L^2(\Omega)$. Jasno nam je da za $u \in H^2(\Omega) = W^{2,2}(\Omega)$ imamo $-\Delta u \in L^2(\Omega)$. Međutim, pitanje je postoji li $u \notin H^2(\Omega)$ t.d. $-\Delta u \in L^2(\Omega)$? Ako postoji, tada bismo htjeli promatrati i te f-je tako da obuhvatimo sve $f \in L^2(\Omega)$ za koje postoji rješenje gornje jednadžbe. Dakle, ima smisla promatrati prostor

općenito

$$\{u \in L^2(\Omega) : \Delta u \in L^2(\Omega)\} \neq H^2(\Omega)$$

uz normu $\|\cdot\|_{0,2,\Omega} + \|\Delta \cdot\|_{0,2,\Omega}$.

Gornji prostor se zove maksimalna domena u $L^2(\Omega)$ operatora Δ ili njegov prostor grafa.

Minimalna domena u $L^2(\Omega)$ operatora Δ je (obično) upotpunjenje prostora $(C_c^\infty(\Omega), \|\cdot\|_{0,2,\Omega} + \|\Delta \cdot\|_{0,2,\Omega})$.

Ova konstrukcija se može napraviti za proizvoljni diferencijalni operator, a ovdje ćemo kratko komentirati prostore grafa Friedrichsovih operatora (v. [K. BURAZIN: Prilozi teoriji Fried. i hiperboličkih sustava, PhD, 2008])