OPERATORI PROŠIRENJA I TRAGA

1) Za $u \in W^{m,p}(\Omega)$ pitamo se postoji li njeno proširenje $v_u \in W^{m,p}(\mathbb{R}^d)$ t.d. $v_u = u$ s.s. u Ω.
Štoviše, htjeli bismo da je preslikavanje

$$W^{m,p}(\Omega) \ni u \xmapsto{E} v_u \in W^{m,p}(\mathbb{R}^d)$$

linearno i neprekidno.
Postojanje operatora E bi olakšalo mnoge tehničke poteškoće izazvane domenom Ω.
Na primjer, ukoliko vrijedi $W^{m,p}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ postojanjem operatora E izravno slijedi $W^{m,p}(\Omega) \hookrightarrow L^2(\Omega)$. Naime,

$$\|u\|_{0,2,\Omega} = \|Eu\|_{0,2,\Omega} \leq \|Eu\|_{0,2,\mathbb{R}^d} \underbrace{\leq}_{C_1} \|Eu\|_{m,p,\mathbb{R}^d} \leq C_1 C_2 \|u\|_{m,p,\Omega} .$$

Dakle, Soboljevljeva ulaganja bi bilo dovoljno pokazati samo za $\Omega = \mathbb{R}^d$ (ovaj pristup je npr. korišten u knjizi „Evans").
Mi nismo koristili taj pristup jer slabe pretpostavke na Ω koje smo imali (uvjet konusa) nažalost ne osiguravaju postojanje operatora E.

2) U teoriji parcijalnih diferencijalnih jednadžbi za dobru postavljenost osim diferencijalne relacije trebamo zadati i neki uvjet o ponašanju f-je na rubu domene, npr.

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g \end{cases}.$$

Ukoliko u nije neprekidna, tada oznaka $u|_{\partial\Omega}$ nema smisla i trebamo joj dati značenje.

To ćemo napraviti sljedećom konstrukcijom:

$(p \neq \infty)$ $\quad u \in W^{m,p}(\Omega) \Rightarrow Eu \in W^{m,p}(\mathbb{R}^d) \Rightarrow \exists (v_n)$ u $C_c^\infty(\mathbb{R}^d)$, $v_n \xrightarrow{W^{m,p}} Eu$

$(v_n|_{\partial\Omega})$ je dobro definiran niz

i pokazat ćemo da je konvergentan

i da limes ne ovisi o izboru E, (v_n).

OPERATOR PROŠIRENJA

DEFINICIJA 1. $\Omega \subseteq \mathbb{R}^d$, $m \in \mathbb{N}_0$, $p \in [1, \infty]$.

Linearan operator $E : W^{m,p}(\Omega) \longrightarrow W^{m,p}(\mathbb{R}^d)$ nazivamo jednostavan (m,p)-operator proširenja za Ω ako
$(\exists K = K(m,p) > 0)(\forall u \in W^{m,p}(\Omega))$

(i) $Eu(x) = u(x)$ s.s. u Ω;

(ii) $\|Eu\|_{m,p,\mathbb{R}^d} \leq K \|u\|_{m,p,\Omega}$.

Kažemo da je E jaki m-operator proširenja za Ω ako je linearno preslikavanje s prostora f-ja definiranih s.s. u Ω u prostor f-ja def. s.s. u $\mathbb{R}^d$, te
$(\forall p \in [1,\infty\rangle)(\forall k \in \{0,1,\dots,m\})$ $E|_{W^{k,p}(\Omega)}$ je jednostavan (k,p)-operator proširenja za Ω.

E je potpuni operator proširenja za Ω ako je E za svaki $m \in \mathbb{N}_0$ jaki m-operator proširenja za Ω.

Postoje tri metode kojima se pokazuje postojanje operatora proširenja:

a) metoda refleksija (Whitney (1934), Hestenes (1941), Seeley (1964))

b) integralna usrednjenja (Stein)

c) Calderón-Zygmundova teorija singularnih integrala (Calderón (1961))

Koristimo pristup a), a kasnije ćemo samo iskazati rezultate preostala dva pristupa.

TEOREM 2. (jaki m-operator proširenja za poluprostor)

$\Omega = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_d > 0\}$, $m \in \mathbb{N}_0$.

Postoji jaki m-operator proširenja E za Ω.

Nadalje, za svaki multiindeks $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq m$, postoji jaki $(m-|\alpha|)$-operator proširenja E_α t.d.

$$\partial^\alpha E u = E_\alpha \partial^\alpha u \, .$$

Dokaz. Za u na $\mathbb{R}^d_+$ definiramo Eu i $E_\alpha u$ (s.s.) na $\mathbb{R}^d$:

$$Eu(x) = \begin{cases} u(x) & : x_d > 0 \\ \sum_{j=1}^{m+1} \lambda_j u(x', -jx_d) & : x_d < 0 \end{cases}$$

$$E_\alpha u(x) = \begin{cases} u(x) & : x_d > 0 \\ \sum_{j=1}^{m+1} (-j)^{\alpha_d} \lambda_j u(x', -jx_d) & : x_d < 0 \end{cases} ,$$

pri čemu su $\lambda_1, \dots, \lambda_{m+1}$ jedinstvena rj. $(m+1) \times (m+1)$ sustava

$$(*) \quad \sum_{j=1}^{m+1} (-j)^k \lambda_j = 1 \, , \quad k = 0, \dots, m$$

matrica sustava je Vandermondova matrica

$$u \in C^m(\overline{\mathbb{R}^d_+}) \Rightarrow Eu \in C^m(\mathbb{R}^d) \quad \&$$

$$\partial^\alpha Eu = E_\alpha \partial^\alpha u \quad , \quad |\alpha| \leq m .$$

$$\|\partial^\alpha(Eu)\|_{p,\mathbb{R}^d} \overset{2^{1/p}}{\leq} \left(\|\partial^\alpha(Eu)\|_{p,\mathbb{R}^d_+} + \|\partial^\alpha(Eu)\|_{p,\mathbb{R}^d_-} \right)$$

- $$\|\partial^\alpha(Eu)\|_{p,\mathbb{R}^d_-} = \|E_\alpha \partial^\alpha u\|_{p,\mathbb{R}^d_-}$$

$$= \Big\| \sum_{j=1}^{m+1} (-j)^{\alpha_d} \lambda_j (\partial^\alpha u)(x', -jx_d) \Big\|_{p,\mathbb{R}^d_-}$$

$$\leq \sum_{j=1}^{m+1} |\lambda_j| j^{\alpha_d} \left(\int_{\mathbb{R}^{d-1}} \int_{-\infty}^{0} |(\partial^\alpha u)(x', -jx_d)|^p dx' dx_d \right)^{1/p}$$

$y = -jx_d$

$dy = -j dx_d$

$\Rightarrow dx_d = -\frac{dy}{j}$

$\left(\begin{array}{l} x_d \to -\infty, \ y \to +\infty \\ x_d = 0, \ y = 0 \end{array} \right)$

$$= \sum_{j=1}^{m+1} |\lambda_j| j^{\alpha_d} j^{1/p} \|\partial^\alpha u\|_{p,\mathbb{R}^d_+}$$

$$\Rightarrow \|\partial^\alpha(Eu)\|_{p,\mathbb{R}^d} \leq \underbrace{2^{1/p}\left(1 + \sum_{j=1}^{m+1} j^{\alpha_d + \frac{1}{p}} |\lambda_j|\right)}_{\substack{\text{ovisi o } m, p, \alpha, \\ \text{ali \underline{ne} o } u}} \|\partial^\alpha u\|_{p,\mathbb{R}^d_+} .$$

Iz $\partial^\beta E_\alpha u = E_{\alpha+\beta} \partial^\beta u$ analogno slijedi tvrdnja za E_α.

■

Kad bismo mogli poopćiti (*) za $m=\infty$, tada bismo analognom konstrukcijom mogli dobiti potpuni operator proširenja za $\mathbb{R}^d_+$.
Za to nam treba sljedeća lema.

LEMA 3. (Poopćenje (*) za $m=\infty$)
Postoji niz realnih brojeva (a_k) t.d. $(\forall m \in \mathbb{N}_0)$

$$(**) \qquad \sum_{k=0}^{\infty} 2^{mk} a_k = (-1)^m ,$$

$$(***) \qquad \sum_{k=0}^{\infty} 2^{mk} |a_k| < \infty .$$

Dokaz. Za $N \in \mathbb{N}$, neka $a_{0,N}, a_{1,N}, \ldots, a_{N,N}$ čine jedinstveno rj. sustava

$$\sum_{k=0}^{N} (2^k)^m a_{k,N} = (-1)^m , \quad m = 0,1,\ldots,N .$$

Štoviše, koristeći Cramerovo pravilo i determinantu Vandermondeove matrice

$$V(x_0,x_1,\ldots,x_N) := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_N \\ x_0^2 & x_1^2 & \cdots & x_N^2 \\ \vdots & & & \\ x_0^N & x_1^N & \cdots & x_N^N \end{vmatrix} = \prod_{\substack{i,j=0 \\ i<j}}^{N} (x_j - x_i) ,$$

imamo

uz konvenciju $\prod_{i=l}^{m} P_i = 1$ ako $m<l$

$$a_{k,N} = \frac{V(1,2,\ldots,2^{k-1},-1,2^{k+1},\ldots,2^N)}{V(1,2,\ldots,2^N)}$$

$$= \frac{V(1,2,\ldots,2^{k-1},2^{k+1},\ldots,2^N) \prod_{i=0}^{k-1}(-1-2^i) \prod_{j=k+1}^{N}(2^j+1)}{V(1,2,\ldots,2^{k-1},2^{k+1},\ldots,2^N) \prod_{i=0}^{k-1}(2^k-2^i) \prod_{j=k+1}^{N}(2^j-2^k)}$$

$$= \underbrace{\prod_{i=0}^{k-1} \frac{1+2^i}{2^i-2^k}}_{A_k} \underbrace{\prod_{j=k+1}^{N} \frac{1+2^j}{2^j-2^k}}_{B_{k,N}}$$

$$= A_k B_{k,N}$$

• $\underset{(k\geq 1)}{|A_k|} = \prod_{i=0}^{k-1} \frac{1+2^i}{2^k-2^i} \leq \prod_{i=0}^{k-1} \frac{2^{i+1}}{2^k-2^{k-1}} = \prod_{i=0}^{k-1} \frac{2^{i+1}}{2^{k-1}} = \frac{2^{\sum_{i=1}^{k} i}}{2^{k(k-1)}} = 2^{\frac{k^2}{2}+\frac{k}{2}-k^2+k}$

$= 2^{-\frac{1}{2}(k^2-3k)}$

za $k=0$ je $A_k=1$, dok je $2^{-\frac{1}{2}(0^2-3\cdot 0)}=1$, pa i za $k=0$ vrijedi ocjena

• $\underset{\substack{>0 \\ (k\leq N-1)}}{\underbrace{\ln B_{k,N}}} = \sum_{j=k+1}^{N} \ln\left(\frac{1+2^j}{2^j-2^k}\right) = \sum_{j=k+1}^{N} \ln\left(1+\underbrace{\frac{1+2^k}{2^j-2^k}}_{>0}\right)$

$\leq \sum_{j=k+1}^{N} \frac{1+2^k}{2^j-2^k} \leq (1+2^k)\sum_{j=k+1}^{N}\frac{1}{2^{j-1}}$

$\ln(1+x)<x$ za $x>0$

$\leq \underbrace{\frac{1+2^k}{2^k}}_{\leq 2} \underbrace{\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j}_{=2} \leq 4$

$\Rightarrow \quad B_{k,N} \leq e^4$

za $k=N$ je $B_{N,N}=1\leq e^4$ ✓

Jer je $\frac{1+2^j}{2^j-2^k}\geq 1$ za $j\geq k+1$, za čvrst $k\in\mathbb{N}$ imamo da je niz $(B_{k,N})_N$ rastući, a kako je i omeđen odozgo, to je konvergentan

$$B_k := \lim_{N\to\infty} B_{k,N} \quad (\leq e^4).$$

Definiramo $a_k := A_k B_k$. Pokažimo (***):

$$\sum_{k=0}^{\infty} 2^{mk}|a_k| = \sum_{k=0}^{\infty} 2^{mk}|A_k||B_k| \leq e^4 \underbrace{\sum_{k=0}^{\infty} 2^{-\frac{1}{2}(k^2-(3+2m)k)}}$$

ovo je konvergentan red (npr. D'Alembertov kriterij:

$$\frac{2^{-\frac{1}{2}((k+1)^2-(3+2m)(k+1))}}{2^{-\frac{1}{2}(k^2-(3+2m)k)}} = \left(\frac{1}{2}\right)^{k-1-m} \xrightarrow[(k\to\infty)]{} 0$$

$$\Rightarrow \sum_{k=0}^{\infty} 2^{mk}|a_k| < \infty .$$

Pokažimo (**): Za $\varepsilon>0$ neka je $N_0\in\mathbb{N}$ t.d.

$$\sum_{k=N_0+1}^{\infty} 2^{mk}|A_k| < \frac{\varepsilon}{2e^4} \quad \text{(ostatak konvergentnog reda)}.$$

Neka je $N\geq N_0$ proizvoljan i izaberimo $M_N\geq N_0$ t.d.

$$(\forall k\in\{0,1,\ldots,N\}) \quad |B_k - B_{k,M_N}| \leq \frac{\varepsilon}{2\sum_{k=0}^{\infty} 2^{mk}|A_k|}.$$

Bez smanjenja općenitosti možemo pretpostaviti: $M_N > N$.

Naime, ukoliko je $M_N \leq N$, tada je $B_{N,M_N} = 1$, dok je $B_N > 1$ (vidi se iz definicije niza $B_{N,M}$), pa je za dovoljno malen ε sigurno $M_N > N$.

$$\left| \sum_{k=0}^{N} 2^{mk} a_k - (-1)^m \right| \leq \left| \sum_{k=0}^{N} 2^{mk} (a_k - a_{k,M_N}) \right| + \left| \sum_{k=0}^{N} 2^{mk} a_{k,M_N} - (-1)^m \right|$$

$$\leq \sum_{k=0}^{N} 2^{mk} |A_k| \, |B_k - B_{k,M_N}| + \left| \sum_{k=N+1}^{M_N} 2^{mk} a_{k,M_N} \right|$$

$$+ \underbrace{\left| \sum_{k=0}^{M_N} 2^{mk} a_{k,M_N} - (-1)^m \right|}_{=0}$$

$$\leq \underbrace{\frac{\varepsilon}{2 \sum_{k=0}^{\infty} 2^{mk} |A_k|} \sum_{k=0}^{\infty} 2^{mk} |A_k|}_{< \frac{\varepsilon}{2}} + e^4 \underbrace{\sum_{k=N_0+1}^{\infty} 2^{mk} |A_k|}_{< \frac{\varepsilon}{2}} < \varepsilon$$

$\blacksquare$

TEOREM 4. (potpuni operator proširenja za poluprostor)

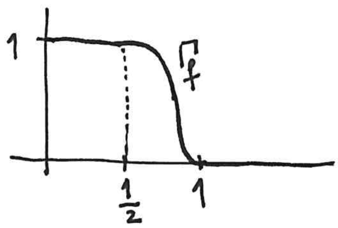
Postoji potpuni operator proširenja za $\mathbb{R}^d_+$.

$\mathbb{R}^d_+$ zadovoljava uvjet segmenta

Dokaz. Kako je $\{ \phi|_{\mathbb{R}^d_+} : \phi \in C_c^\infty(\mathbb{R}^d) \}$ gusto u $W^{m,p}(\mathbb{R}^d_+)$, $m \in \mathbb{N}_0$, $p \in [1, \infty\rangle$, promatramo samo te f-je, a onda se jednostavno po gustoći tvrdnje proširi na $W^{m,p}(\mathbb{R}^d_+)$.

Neka je $f \in C^\infty([0,\infty\rangle)$ t.d.

- $f \equiv 1$ na $[0, \frac{1}{2}]$
- $\operatorname{supp} f \subseteq [0, 1\rangle$



Za $\phi \in C_c^\infty(\mathbb{R}^d)$ definirajmo

$$E\phi(x) = E\phi(x', x_d) = \begin{cases} \phi(x) & : x_d \geq 0 \\ \sum_{k=0}^{\infty} a_k f(-2^k x_d)\, \phi(x', -2^k x_d) & : x_d < 0 \end{cases},$$

pri čemu je (a_k) niz iz prethodne leme.

Očito je $E\phi|_{\mathbb{R}^d_+} = \phi|_{\mathbb{R}^d_+} \in C^\infty(\overline{\mathbb{R}^d_+})$.

S druge strane, za $x \in \mathbb{R}^d_-$ je $E\phi(x)$ dobro definirano jer

$$x_d < 0 \Rightarrow (\exists k_0 \in \mathbb{N}) \ -2^{k_0} x_d \geq 1 \Rightarrow \sum_{k=0}^{\infty} a_k f(-2^k x_d)\, \phi(x', -2^k x_d) \underbrace{=}_{\text{konačna suma}} \sum_{k=0}^{k_0-1} a_k f(-2^k x_d)\, \phi(x', -2^k x_d),$$

pa je onda i $E\phi|_{\mathbb{R}^d_-} \in C^\infty(\mathbb{R}^d_-)$,

te $E\phi$ ima kompaktan nosač (jer ϕ ima kompaktan nosač).

Pokažimo da je zapravo $E\phi|_{\mathbb{R}^d_-} \in C^\infty(\overline{\mathbb{R}^d_-})$, te

$$(\forall \alpha \in \mathbb{N}_0^d)(\forall x' \in \mathbb{R}^{d-1}) \quad \lim_{x_d \to 0-} \partial^\alpha (E\phi)(x', x_d) = \partial^\alpha \phi(x', 0) = \lim_{x_d \to 0+} \partial^\alpha (E\phi)(x', x_d),$$

čime bismo dobili $E\phi \in C^\infty(\mathbb{R}^d)$.

Neka je $\alpha = (\alpha', \alpha_d) \in \mathbb{N}_0^d$, $x = (x', x_d) \in \mathbb{R}^d_-$, tada

$$\partial^\alpha E\phi(x) = \sum_{k=0}^{\infty} \underbrace{a_k \sum_{j=0}^{\alpha_d} \binom{\alpha_d}{j} (-2^k)^{\alpha_d} f^{(\alpha_d - j)}(-2^k x_d)\, (\partial^{\alpha'} \partial^j \phi)(x', -2^k x_d)}_{=: \psi_k(x)},$$

gdje je kao i ranije za svaki $x \in \mathbb{R}^d_-$ gornja suma konačna.

Međutim, pokažimo da vrijedi i jače svojstvo, odnosno da red $\sum_{k=0}^{\infty} \psi_k(x)$ apsolutno konvergira jednoliko po $x \in \mathbb{R}^d_-$.

Naime, iz

$$|\psi_k(x)| \leq 2^{\alpha_d} \|f\|_{C^{\alpha_d}([1,\infty))} \|\phi\|_{C^{|\alpha|}(\mathbb{R}^d_+)}\, 2^{k\alpha_d} |a_k|$$

i $(***)$ slijedi tvrdnja.

Iz toga slijedi

$$\lim_{x_d\to 0-} \partial^\alpha(E\phi)(x',x_d) = \sum_{k=0}^{\infty} \lim_{x_d\to 0-} \psi_k(x',x_d) .$$

Kako je

$$\lim_{x_d\to 0-}\psi_k(x',x_d) = a_k \sum_{j=0}^{\alpha_d}\binom{\alpha_d}{j}(-2^k)^{\alpha_d}\underbrace{f^{(\alpha_d-j)}(0+)}_{=\delta_{\alpha_d,j}}(\partial^{(\alpha',j)}\phi)(x',0+)$$

$$= a_k(-2^k)^{\alpha_d}(\partial^\alpha\phi)(x',0+)$$

$$= (-1)^{\alpha_d}(\partial^\alpha\phi)(x',0+)\, 2^{k\alpha_d}a_k$$

$$\Rightarrow \lim_{x_d\to 0-}\partial^\alpha(E\phi)(x',x_d) = (-1)^{\alpha_d}(\partial^\alpha\phi)(x',0+)\underbrace{\sum_{k=0}^{\infty}2^{k\alpha_d}a_k}_{\overset{(**)}{=}(-1)^{\alpha_d}}$$

$$= (\partial^\alpha\phi)(x',0+)$$

$$= \lim_{x_d\to 0+}\partial^\alpha E\phi(x',x_d) .$$

Dakle, $E\phi\in C_c^\infty(\mathbb{R}^d)$.

Još je preostalo pokazati neprekinutost.
Za $x\in\mathbb{R}^d_-$, slično kao i ranije, imamo

$$|\psi_k(x)| \le \|f\|_{C^{\alpha_d}([0,\infty\rangle)}\, 2^{k\alpha_d}|a_k|\, 2^{\alpha_d}\sum_{|\beta|\le|\alpha|}|\partial^\beta\phi(x',-2^kx_d)| .$$

Neka je $\alpha\in\mathbb{N}_0^d$ t.d. $|\alpha|\le m$ ($\Rightarrow \alpha_d\le m$), pa slijedi za $p\in[1,\infty\rangle$

$$|\psi_k(x)|^p \le 2^{mp}\|f\|^p_{C^m([0,\infty\rangle)}\, 2^{kmp}|a_k|^p\Big(\sum_{|\beta|\le m}|\partial^\beta\phi(x',-2^kx_d)|\Big)^p$$

$$\le \underbrace{\Big(2^{mp}\, 2^{1-\frac{1}{p}}\|f\|^p_{C^m([0,\infty\rangle)}\Big)}_{=:K_1^p} 2^{kmp}|a_k|^p\sum_{|\beta|\le m}|\partial^\beta\phi(x',-2^kx_d)|^p$$

$$\boxed{\sum|b_k|\le 2^{1-\frac{1}{p}}\Big(\sum|b_k|^p\Big)^{1/p}}$$

$$\Rightarrow \|\psi_k\|_{0,p,\mathbb{R}^d_-}\le K_1\, 2^{km}|a_k|\Big(\sum_{|\beta|\le m}\|(\partial^\beta\phi)(\cdot,-2^k\cdot)\|^p_{0,p,\mathbb{R}^d_-}\Big)^{1/p}$$

$$\left\{\begin{array}{l} y=-2^kx_d \\ dy=-2^k dx_d \\ x_d\to-\infty \Rightarrow y\to+\infty \\ x_d=0 \Rightarrow y=0 \end{array}\right\}$$

$$= K_1\, 2^{km}|a_k|\underbrace{2^{-k/p}}_{\le 1}\|\phi\|_{m,p,\mathbb{R}^d_+}$$

$$\le K_1\, 2^{km}|a_k|\,\|\phi\|_{m,p,\mathbb{R}^d_+}$$

$$\Rightarrow \quad \| \partial^\alpha E\phi \|_{0,p,\mathbb{R}^d_-} \leq K_1 \|\phi\|_{m,p,\mathbb{R}^d_+} \sum_{k=0}^{\infty} 2^{km} |a_k|$$

$$\Rightarrow \|E\phi\|_{m,p,\mathbb{R}^d} \leq 2^{\frac{1}{p}} \left(\|E\phi\|_{m,p,\mathbb{R}^d_+} + \|E\phi\|_{m,p,\mathbb{R}^d_-} \right)$$

$$= 2^{1/p} \left(\|\phi\|_{m,p,\mathbb{R}^d_+} + \left(\sum_{|\beta| \leq m} \|\partial^\beta E\phi\|^p_{0,p,\mathbb{R}^d_-} \right)^{1/p} \right)$$

$$\leq 2^{1/p} \left(\|\phi\|_{m,p,\mathbb{R}^d_+} + K_1 \|\phi\|_{m,p,\mathbb{R}^d_+} \left(\sum_{k=0}^{\infty} 2^{km} |a_k| \right) \left(\sum_{|\beta| \leq m} 1 \right)^{1/p} \right)$$

$$= \underbrace{2^{1/p} \left(1 + K_1 \left(\sum_{k=0}^{\infty} 2^{km} |a_k| \right) \binom{m+d}{m}^{1/p} \right)}_{\text{konstanta ovisi o } m, p, d \text{ i } f} \|\phi\|_{m,p,\mathbb{R}^d_+}$$

$\blacksquare$