$$\left(W_0^{m,p}(\Omega)\right)' =: W^{-m,p}(\Omega), \quad \text{uz normu}$$

$$\|T\|_{-m,p} = \min\{\|(f_\alpha)\|_{p'} : (f_\alpha) \text{ zadovoljava } (**)\}$$

Aproksimacija glatkim funkcijama na Ω

- particija jedinice (glatka)

$A \subseteq \mathbb{R}^d$, $A \subseteq \bigcup_{U \in \mathcal{O}} U$ ($\mathcal{O}$ kolekcija otvorenih skupova)

tada postoji kolekcija Ψ funkcija $\psi \in C_c^\infty(\mathbb{R}^d)$ t.d.

(i) $0 \leq \psi(x) \leq 1$

(ii) $K \Subset A \Rightarrow$ samo konačno mnogo članova kolekcije Ψ ne iščezava na K

(iii) $(\forall \psi \in \Psi)(\exists U \in \mathcal{O}) \ \operatorname{supp} \psi \subset U$

(iv) $\sum_{\psi \in \Psi} \psi(x) = 1, \quad x \in A$

- izglađivači

$J \in C_c^\infty(\mathbb{R}^d)$, $\operatorname{supp} J \subseteq K[0,1]$, $\int_{\mathbb{R}^d} J(x)\,dx = 1$

$$J_\varepsilon(x) = \varepsilon^{-d} J\left(\frac{x}{\varepsilon}\right)$$

$$J_\varepsilon * u(x) = \int_{\mathbb{R}^d} J_\varepsilon(x-y)\,u(y)\,dy$$

a) $u \in L^1_{loc} \Rightarrow J_\varepsilon * u \in C^\infty$

b) $u \in L^p$, $p \in [1,\infty\rangle \Rightarrow \|J_\varepsilon * u - u\|_p \to 0$

- izglađivanje na $W^{m,p}(\Omega)$

$u \in W^{m,p}(\Omega)$, $p \in [1, \infty\rangle$

$\Omega' \Subset \Omega \Rightarrow J_\varepsilon * u \to u$ u $W^{m,p}(\Omega')$

<u>Dokaz.</u>

$\varepsilon < d(\Omega', \partial\Omega)$, $\tilde{u}$ proširenje nulom izvan Ω

Tada je $\partial^\alpha (J_\varepsilon * u) = J_\varepsilon * \partial^\alpha u$ na Ω'.

$$\int_{\Omega'} (J_\varepsilon * u)\, \partial^\alpha \varphi = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{u}(x-y) J_\varepsilon(y)\, \partial^\alpha \varphi(x)\, dx\, dy$$

$$= (-1)^{|\alpha|} \int_{\mathbb{R}^d}\int_{\Omega'} \partial_x^\alpha u(x-y) J_\varepsilon(y) \varphi(x)\, dx\, dy$$

$$= (-1)^{|\alpha|} \int_{\Omega'} (J_\varepsilon * \partial^\alpha u)\varphi$$

$$\Rightarrow \|\partial^\alpha (J_\varepsilon * u) - \partial^\alpha u\|_{p,\Omega'} = \|J_\varepsilon * \partial^\alpha u - \partial^\alpha u\|_{p,\Omega'} \xrightarrow{\varepsilon \searrow 0} 0$$

$$\Rightarrow \|J_\varepsilon * u - u\|_{m,p,\Omega'} \xrightarrow{\varepsilon \searrow 0} 0.$$

- Sada dokazujemo da je $H^{m,p}(\Omega) = W^{m,p}(\Omega)$, $p \in [1,\infty\rangle$, a kao posljedicu dobivamo i da je $C^\infty(\Omega)$ gust u $W^{m,p}(\Omega)$.

1) $H^{m,p} \subseteq W^{m,p}$ ← Banachov

$\Rightarrow H^{m,p}$ je zatvarač (definiran kao upotpunjenje) skupa $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\} \subseteq W^{m,p}(\Omega)$ u odnosu na $\|\cdot\|_{m,p}$ normu

2) $W^{m,p} \subseteq H^{m,p}$

Dovoljno je pokazati da je $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}$ gust u $W^{m,p}(\Omega)$, a pokazat ćemo više:

$\{u \in C^\infty(\Omega) : \|u\|_{m,p} < \infty\}$ je gust u $W^{m,p}(\Omega)$

Definiramo $\Omega_0 = \Omega_{-1} = \emptyset$, a za $k \geq 1$:

$$\Omega_k = \{x \in \Omega : |x| < k \text{ i } d(x, \partial\Omega) > \tfrac{1}{k}\}$$

$$\mathcal{O} = \{U_k : U_k = \Omega_{k+1} \cap (\overline{\Omega_{k-1}})^c\}$$

i neka je Ψ C^∞-particija jedinice podređena $\mathcal{O}$.

ψ_k ... suma funkcija iz Ψ čiji je nosač sadržan u U_k

$$\psi_k \in C_c^\infty(U_k) \quad \text{i} \quad \sum \psi_k(x) = 1, \quad x \in \Omega$$

Za dovoljno mali ε_k:

$$\operatorname{supp}(J_{\varepsilon_k} * (\psi_k u)) \subseteq V_k := \Omega_{k+2} \cap (\overline{\Omega_{k-2}})^c \Subset \Omega,$$

te

$$\| J_{\varepsilon_k} * (\psi_k u) - \psi_k u \|_{m,p,\Omega} = \| J_{\varepsilon_k} * (\psi_k u) - \psi_k u \|_{m,p,V_k} < \frac{\varepsilon}{2^k}.$$

$$\varphi = \sum_{k=1}^{\infty} J_{\varepsilon_k} * (\psi_k u)$$

konačna suma (u okolini svake točke) $\Rightarrow \varphi \in C^\infty(\Omega)$

$$x \in \Omega_k, \quad u(x) = \sum_{j=1}^{k+2} \psi_j(x) u(x)$$

$$\varphi(x) = \sum_{j=1}^{k+2} J_{\varepsilon_j} * (\psi_j u)(x)$$

$$\Rightarrow \|u - \varphi\|_{m,p,\Omega_k} \leq \sum_{j=1}^{k+2} \underbrace{\| J_{\varepsilon_j} * (\psi_j u) - \psi_j u \|_{m,p,\Omega}}_{< \frac{\varepsilon}{2^j}} < \varepsilon$$

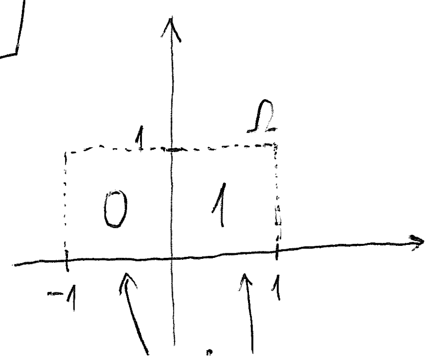
$$\Rightarrow \|u - \varphi\|_{m,p,\Omega} \leq \varepsilon.$$

Aproksimacija glatkim funkcijama na $\mathbb{R}^d$

Da li je za neki k $C^k(\overline{\Omega})$ gusto u $W^{m,p}(\Omega)$?

Općenito NE!

Pr.



$\Omega = L \cup D$

$$u(x,y) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

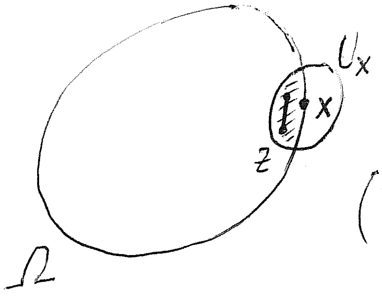
$u \in W^{1,p}(\Omega)$

ne postoji $\varphi \in C^1(\overline{\Omega})$ t.d. $\|u - \varphi\|_{1,p} < \varepsilon$

$\|\varphi\|_{L^1(L)} < \varepsilon$, $\|\varphi\|_{L^1(D)} > 1 - \varepsilon$

$\phi(x) = \int_0^1 \varphi(x,y)\,dy$, kontradikcija s $\phi(b) - \phi(a) = \int_a^b \phi'(x)\,dx$

UVJET SEGMENTA



$$(\forall x \in \partial\Omega)(\exists \text{ okolina } U_x)(\exists y_x \neq 0) \text{ t.d.}$$

$$(\forall z \in \overline{\Omega} \cap U_x)(\forall t \in \langle 0,1\rangle) \quad z + t y_x \in \Omega$$

TEOREM.

Ako Ω zadovoljava uvjet segmenta, onda je skup restrikcija na Ω funkcija iz $C_c^\infty(\mathbb{R}^d)$ gust u $W^{m,p}(\Omega)$, $p \in [1, \infty\rangle$.

Dz.

- dokazujemo prvo da možemo pretpostaviti da $u \in W^{m,p}(\Omega)$ ima kompaktan nosač

$$f \in C_c^\infty(\mathbb{R}^d), \quad f(x) = 1 \quad \text{za } |x| \leq 1$$
$$f(x) = 0 \quad \text{za } |x| \geq 2$$
$$|\partial^\alpha f(x)| \leq M, \quad 0 \leq |\alpha| \leq m$$

$$\varepsilon > 0, \quad f_\varepsilon(x) = f(\varepsilon x) \longrightarrow f_\varepsilon(x) = 1 \quad \text{za } |x| \leq \frac{1}{\varepsilon}$$
$$\longrightarrow |\partial^\alpha f_\varepsilon(x)| \leq M\varepsilon^{|\alpha|} \leq M \quad \text{za } \varepsilon \leq 1$$

$$u_\varepsilon = f_\varepsilon u$$

$$|\partial^\alpha u_\varepsilon(x)| = \left|\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta u(x) \partial^{\alpha-\beta} f_\varepsilon(x)\right| \leq M \cdot \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^\beta u(x)|$$

$$\Rightarrow u_\varepsilon \in W^{m,p}(\Omega)$$

$\Omega_\varepsilon = \{x \in \Omega : |x| > \frac{1}{\varepsilon}\}$,

$$\|u - u_\varepsilon\|_{m,p,\Omega} = \|u - u_\varepsilon\|_{m,p,\Omega_\varepsilon}$$

$$\leqslant \|u\|_{m,p,\Omega_\varepsilon} + \|u_\varepsilon\|_{m,p,\Omega_\varepsilon}$$

$$\leqslant C \cdot \|u\|_{m,p,\Omega_\varepsilon} \xrightarrow{\varepsilon \searrow 0} 0$$

BSO $K = \{x \in \Omega : u(x) \neq 0\}$ je ograničen

$F = \overline{K} \setminus \bigcup_{x \in \partial\Omega} U_x$, U_x - otvoreni skupovi iz definicije uvjeta segmenta

$F \subseteq \Omega$ kompaktan

$\Rightarrow \exists U_0$ otvoren t.d. $F \Subset U_0 \Subset \Omega$

$U_0, \{U_x\}$ - otvoren pokrivač kompaktnog skupa $\overline{K}$,

i neka je $U_0, U_1, U_2, \dots, U_k$ konačan potpokrivač,

i neka je $V_0, V_1, V_2, \dots, V_k$ profinjenje tog pokrivača

$V_j \Subset U_j$ otvoreni

$\Psi = C^\infty$-particija jedinice podređena pokrivaču $\{V_j : 0 \leq j \leq k\}$

ψ_j ... suma funkcija iz Ψ čiji je nosač sadržan u V_j

$\psi_j \in C_c^\infty(V_j)$ i $\sum \psi_j(x) = 1$, $x \in \overline{K}$

$u_j = \psi_j u \in W^{m,p}(\Omega)$, $u = \sum u_j$

Cilj nam je za svaki $j \in 0..k$ pronaći funkciju $\varphi_j \in C_c^\infty(\mathbb{R}^d)$ takvu da je

$$\|u_j - \varphi_j\|_{m,p,\Omega} < \frac{\varepsilon}{j+2},$$

jer je tada

$$\|u - \sum_{j=1}^{k} \varphi_j\|_{m,p,\Omega} < \varepsilon.$$

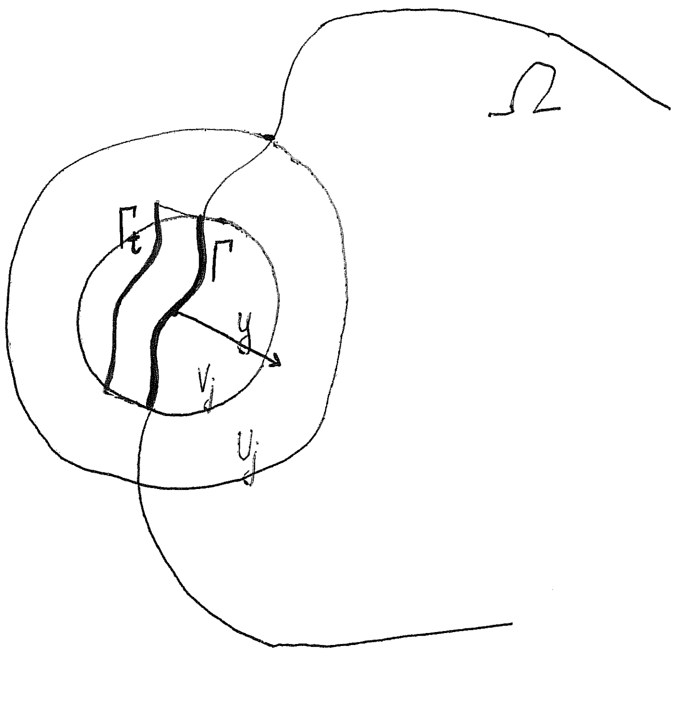
$\underline{j=0:}$ $\varphi_0 = J_\varepsilon * u_0$ za dovoljno mali ε

$(\operatorname{supp} u_0 \subseteq V_0 \Subset \Omega \Rightarrow \operatorname{supp} \varphi_0 \subseteq \Omega)$

$\underline{j \in 1..k:}$

$\tilde{u}_j$ = proširenje nulom izvan Ω

$\tilde{u}_j \in W^{m,p}(\mathbb{R}^d \setminus \Gamma)$, $\Gamma = \overline{V_j} \cap \partial\Omega$



$$\Gamma_t = \{x - t\cdot y : x \in \Gamma\},$$

$$0 < t < \min\left\{1, \frac{d(V_j, U_j^c)}{|y|}\right\}$$

$$\Gamma_t \cap \overline{\Omega} = \emptyset \qquad \Gamma_t \subseteq U_j$$

Naime, $\tilde{x} \in \overline{\Omega} \cap \Gamma_t \subseteq \overline{\Omega} \cap U_j \Rightarrow \tilde{x} + ty \in \Omega$

$\Rightarrow$ kontradikcija s $\tilde{x} + ty \in \Gamma$.

$$u_{j,t}(x) = u_j(x + ty) \in W^{m,p}(\mathbb{R}^d \setminus \Gamma_t)$$

$$\|\partial^\alpha u_{j,t} - \partial^\alpha u_j\|_p \xrightarrow{t \searrow 0} 0, \quad 0 \le |\alpha| \le m$$

$$\Rightarrow \|u_{j,t} - u_j\|_{m,p} \xrightarrow{t \searrow 0} 0,$$

pa je dovoljno aproksimirati $u_{j,t}$.

$\varphi_{j,t} := J_\delta * u_{j,t}$ za dovoljno mali δ

Q.E.D.

Korolar.

$$W_0^{m,p}(\mathbb{R}^d) = W^{m,p}(\mathbb{R}^d)$$

Pitanje: Za koje domene Ω je $W_0^{m,p}(\Omega) = W^{m,p}(\Omega)$?

TEOREM.

$C_c^\infty(\Omega)$ je gust u $W^{m,p}(\mathbb{R}^d)$ ako i samo ako je Ω^c (m,p')-polaran.

Zašto $W^{m,p}(\underline{\underline{\mathbb{R}^d}})$?

1) pojam (m,p')-polarnosti definiramo za
$F \subseteq \mathbb{R}^d$ zatvoren

2) $u \in W_0^{m,p}(\Omega) \Rightarrow \tilde{u} \in W^{m,p}(\mathbb{R}^d)$ $(\partial^\alpha \tilde{u} = \widetilde{\partial^\alpha u})$

$\uparrow$
proširenje nulom

$\varphi_j \in C_c^\infty(\Omega)$, $\|u - \varphi_j\|_{m,p} \to 0$

$\psi \in C_c^\infty(\mathbb{R}^d)$,

$$(-1)^{|\alpha|} \int_{\mathbb{R}^d} \tilde{u}(x)\, \partial^\alpha \psi(x)\, dx = (-1)^{|\alpha|} \int_\Omega u(x)\, \partial^\alpha \psi(x)\, dx$$

$$= \lim_{j\to\infty} (-1)^{|\alpha|} \int_\Omega \varphi_j(x)\, \partial^\alpha \psi(x)\, dx$$

$$= \lim_{j\to\infty} \cancel{(-1)^{|\alpha|}} \int_\Omega \partial^\alpha \varphi_j(x)\, \psi(x)\, dx$$

$$= \int_\Omega (\partial^\alpha u\, \psi)(x)\, dx = \int_{\mathbb{R}^d} \widetilde{\partial^\alpha u}(x)\, \psi(x)\, dx$$

<u>Definicija (m,p')-polarnosti</u>

$F \subseteq \mathbb{R}^d$ zatvoren

$T \in \mathcal{D}'(\mathbb{R}^d)$ kažemo da ima nosač u F ako je $T(\varphi) = 0$ za svaki $\varphi \in C_c^\infty(F^c)$

F je (m,p')-polaran ako:
$T \in W^{-m,p'}(\mathbb{R}^d)$, $\operatorname{supp} T \subseteq F \Rightarrow T = 0$

F (m,p')-polaran $\Rightarrow \lambda(F) = 0$ $(K \subseteq F$ kompaktan, $\chi_F \in L^{p'}(\mathbb{R}^d))$