6.3 OPTIMALNO UPRAVLJANJE

Promotrimo PR. 1.3 s početka kolegija:

$$\begin{cases} \begin{pmatrix} x_1'(t)\\ x_2'(t)\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}u(t), \quad t\in\langle 0,1\rangle & (1.6)\\[2mm] \begin{pmatrix} x_1(0)\\ x_2(0)\end{pmatrix} = \begin{pmatrix}-2\sqrt{2}\\ 5\sqrt{2}\end{pmatrix} & (1.7)\\[2mm] x_1(1)^2 + x_2(1)^2 - 1 = 0 & (1.8)\\[2mm] f(u) := \int_0^1 (u(t))^2\,dt & (1.9)\end{cases}$$

$\min\limits_u f(u)$ t.d. (x,u) zadovoljavaju gornje uvjete.

Gornji problem se također može zapisati kao min. problem (6.1) uz pretpostavke (6.P2).

~~Najprije uvedimo~~

Dajmo općenitiju situaciju:

(6.P3)

- $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$
  Tada za fiksnu $u\in L^\infty(t_0,t_1;\mathbb{R}^m)$ i $x_0\in\mathbb{R}^n$ postoji jedinstveno rj. $x\in W^{1,\infty}(t_0,t_1;\mathbb{R}^n)$ zadaće

$$(6.9)\ \boxed{\begin{cases} x'(t) = Ax(t) + Bu(t), & t\in\langle t_0,t_1\rangle\\ x(t_0) = x_0 \end{cases}}$$

  dano s $x(t) = x_0 + \int_{t_0}^{t} (Ax(s) + Bu(s))\,ds, \quad t\in[t_0,t_1]$

  kako su A i B konstantne matrice, možemo napisati eksplicitnu formulu

  $\left(\text{odnosno}\quad x(t) = x_0 + \int_{t_0}^{t} e^{A(t-s)} Bu(s)\,ds, \quad t\in[t_0,t_1]\right)$

- Dodatno tražimo $\boxed{g(x(t_1)) = 0_{\mathbb{R}^r}}$ za $g\in C^1(\mathbb{R}^n;\mathbb{R}^r)$, $\underline{r\le n}$.

- Dakle, naši prostori (u terminima (6P.2)) su

$$X = W^{1,\infty}(\langle t_0,t_1\rangle;\mathbb{R}^n)\times L^\infty(\langle t_0,t_1\rangle;\mathbb{R}^m) = \hat{S}$$

$$Z = W^{1,\infty}(\langle t_0,t_1\rangle;\mathbb{R}^n)\times\mathbb{R}^r$$

$$h(x,u) := \begin{pmatrix} x(\cdot) - x_0 - \int_{t_0}^{\cdot} (Ax(s) - Bu(s))\,ds \\ g(x(t_1)) \end{pmatrix}, \quad \forall x,u\in X$$

Funkcija g (pa onda ni prostor Y) nam se ne javljaju.

Funkcija koju minimiziramo na $\hat{S} = X$ je ~~dana~~ oblika

(6.P3)
$$\boxed{f(x,u) = f_1(x(t_1)) + \int_{t_0}^{t_1} f_2(x(s), u(s))\, ds},$$

pri čemu su $f_1 \in C^1(\mathbb{R}^n)$, $f_2 \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$.

Iz Teorema 6.6 slijedi nužan uvjet optimalnosti za ovaj konkretan problem, a ti uvjeti se često zovu PONTRJAGINOV PRINCIP MAKSIMUMA. Prije njihovog iskazivanja uvedimo pojam upravljivosti ~~sustava~~ zadaće (6.9).

DEF. 6.15 Kažemo da je zadaća (6.9) upravljiva ako za svaki $x_1 \in \mathbb{R}^n$ postoji $u \in L^\infty(\langle t_0, t_1\rangle; \mathbb{R}^m)$ t.d. rj. (6.9) x zadovoljava $x(t_1) = x_1$.

LEMA 6.16 (Kalmanov uvjet) Ako je $\dim[\{B, AB, A^2B, \dots, A^{n-1}B\}] = n$, tada je zadaća (6.9) upravljiva.

(glavnina dokaza je vezana uz karakterizaciju funkcionala ℓ_2)

Sada donosimo nužne uvjete optimalnosti bez dokaza (za dokaz v. [1, T-5.1

TEOREM 6.17 Uz (6.P3) neka ~~...~~ je $(\bar{x}, \bar{u}) \in X$ rj. min. problema ($\bar{u}$ optimalna upravljačka f-ja, $\bar{x}$ dobiveno stanje), te neka je $\frac{\partial g}{\partial x}(\bar{x}(t_1))$ ~~...~~ punog ranga r i neka je Kalmanov uvjet zadovoljen.

Tada postoji $p \in W^{1,\infty}([t_0,t_1]; \mathbb{R}^n)$ i $\alpha \in \mathbb{R}^r$ t.d.

ADJUNGIRANA JED. a) $-\dot{p}(t)^T = p(t)^T A - \frac{\partial f_2}{\partial x}(\bar{x}(t), \bar{u}(t))$, s.s. $t \in [t_0, t_1]$

UVJET TRANSVERZALNOSTI b) $-p(t_1)^T = \alpha^T \frac{\partial g}{\partial x}(\bar{x}(t_1)) + \frac{\partial f_1}{\partial x}(\bar{x}(t_1))$

PONTRJAGINOV PRINCIP MAKSIMUMA c) $(\forall u \in L^\infty([t_0,t_1]; \mathbb{R}^m))$ $p(t)^T B - \frac{\partial f_2}{\partial u}(\bar{x}(t), \bar{u}(t)) = 0$, s.s. $t \in [t_0, t_1]$.

NAP. 6.18. Ako npr. tražimo $u_i(t) \geq \alpha_i$, $i \in \{1, 2, \dots, m\}$, tada je u c) i-ta jednadžba zadovoljena za s.s. $t \in \{s \in [t_0, t_1] : u_i(t) > \alpha_i\}$.

Vratimo se sada na početni primjer. Imamo:

- $n=2$, $m=1$, $r=1$
- $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $B=\begin{pmatrix}0\\1\end{pmatrix}$, $x_0=\begin{pmatrix}-2\sqrt{2}\\5\sqrt{2}\end{pmatrix}$, $t_0=0$, $t_1=1$
- $j(x_1,x_2)=x_1^2+x_2^2-1$ (glatka f-ja)
- $f_1\equiv 0$, $f_2(x,u)=u^2$ (glatke f-je)

$\{B,AB\}=\left\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\} \Rightarrow$ ispunjen je Kalmanov uvjet

$\frac{\partial j}{\partial x}(x_1,x_2)=\begin{pmatrix}2x_1 & 2x_2\end{pmatrix}$ ... ranga 1 van ishodišta, a kako $\bar{x}(t_1)$ zadovoljava $j(\bar{x}(t_1))=0$ (leži na kružnici radijusa 1) slučaj $x_1=x_2=0$ se ne može dogoditi pa je gornja matrica sigurno punog ranga.

Dakle, ispunjene su pretpostavke prethodnog teorema.

$$(a)\Rightarrow \quad \begin{aligned} -p_1'(t) &= 0-0 \Rightarrow p_1(t)=C_1\\ -p_2'(t) &= p_1(t)-0 \Rightarrow p_2(t)=-C_1t+C_2\end{aligned}$$

$$(b)\Rightarrow \quad \begin{aligned} -p_1(1) &= 2a\bar{x}_1(1)\\ -p_2(1) &= 2a\bar{x}_2(1)\end{aligned} \Rightarrow \begin{aligned} &p_1(1)=-2a\bar{x}_1(1)\Rightarrow p_1\equiv -2a\bar{x}_1(1)\\ &p_2(1)=-2a\bar{x}_2(1)\Rightarrow p_2(t)=+2a\bar{x}_1(1)t\\ &\qquad -2a(\bar{x}_1(1)+\bar{x}_2(1))\end{aligned}$$

$$(c)\Rightarrow \quad p_2(t)-2\bar{u}(t)=0 \Rightarrow \bar{u}(t)=\frac{1}{2}p_2(t) = a\overbrace{\bar{x}_1(1)}^{\alpha}t-a(\overbrace{\bar{x}_1(1)}^{\alpha}+\overbrace{\bar{x}_2(1)}^{\beta}) = a\alpha t-a(\alpha+\beta)$$

Imamo trenutno 3 nepoznanice. Jedna jednadžba je dana s $j(\alpha,\beta)=0$, dok dvije dobivamo iz ~~xxxxx~~ $x(t_1)=\begin{pmatrix}\alpha\\\beta\end{pmatrix}$, ali moramo najprije odrediti x da iskoristimo taj uvjet.

$$x_2'(t)=u(t) \Rightarrow \bar{x}_2(t)=\frac{a\alpha t^2}{2}-a(\alpha+\beta)t+c_1$$
$$x_1'(t)=x_2(t) \Rightarrow \bar{x}_1(t)=\frac{a\alpha t^3}{6}-\frac{a(\alpha+\beta)t^2}{2}+c_1t+c_2$$
$$\bar{x}_1(0)=x_0 \Rightarrow \begin{aligned}\bar{x}_2(t)&=\frac{a\alpha t^2}{2}-a(\alpha+\beta)t+5\sqrt{2}\\ \bar{x}_1(t)&=\frac{a\alpha t^3}{6}-\frac{a(\alpha+\beta)t^2}{2}+5\sqrt{2}t-2\sqrt{2}\end{aligned}$$

$$\alpha=\bar{x}_1(1)=\frac{a\alpha}{6}-\frac{a(\alpha+\beta)}{2}+3\sqrt{2}$$
$$\beta=\bar{x}_2(1)=\frac{a\alpha}{2}-a(\alpha+\beta)+5\sqrt{2}$$
$$\alpha^2+\beta^2=1$$

$$\alpha=a\left(\frac{\alpha}{6}-\frac{\alpha+\beta}{2}\right)+3\sqrt{2} \iff \boxed{\begin{aligned}\alpha&=-\frac{a}{6}(2\alpha+3\beta)+3\sqrt{2}\\ \beta&=-\frac{a}{2}(\alpha+2\beta)+5\sqrt{2}\end{aligned}}$$
$$\alpha^2+\beta^2=1$$

$$\boxed{\begin{aligned}(6+2a)\alpha+3a\beta&=18\sqrt{2}\\ a\alpha+(2+2a)\beta&=10\sqrt{2}\end{aligned}}$$

$\hookrightarrow$ sustav ima jedinstveno rj. dano s

$$\alpha = \frac{6\sqrt{2}(a+6)}{a^2+16a+12}, \quad \beta = \frac{2\sqrt{2}(30+a)}{a^2+16a+12}$$

$$1 = \alpha^2+\beta^2 = \frac{72(a+6)^2+8(30+a)^2}{(a^2+16a+12)^2}$$

$$(a^2+16a+12)^2 = 72(a+6)^2+8(30+a)^2$$

$$a^4+256a^2+144+\cancel{16}\,32a^3+24a^2+16\cdot 24a = 72(a^2+12a+36)+8(900+60a+a^2)$$

$$\boxed{a^4+32a^3+200a^2-960a-9648=0}$$

Ova jednadžba ima 2 realna i 2 kompleksna rješenja.

Realna su: $a_1=6$

$a_2 \approx -21{,}679$

Za $a_1=6$ dobivamo $\alpha=\beta=\frac{\sqrt{2}}{2}$. $\Rightarrow \bar{u}(t) = 6\cdot\frac{\sqrt{2}}{2}t - 6\cdot\sqrt{2}$

$= 3\sqrt{2}t - 6\sqrt{2}$

$\int_0^1 (\bar{u}(t))^2 dt = 42$

Za a_2 dobivamo približno: $\left.\begin{array}{l}\alpha \approx 0{,}174\\ \beta \approx 0{,}969\end{array}\right\} \Rightarrow \bar{u}(t) = -3{,}772t + 24{,}779$

$\int_0^1 (\bar{u}(t))^2 dt \approx 525{,}275$

očito ovo drugo rj. ne daje optimalnu upravljačku f-ju!

U ~~teor~~ [1, Teorem 5.22] nije zadovoljena pretpostavka da je $a g(\cdot)$ kvazikonveksna jer je a negativan, a g je kvadratna f-ja.