UNIFORM CLOSURE OF TWO-SIDED MULTIPLICATIONS AND PHANTOM LINE BUNDLES

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Two-sided multiplications (TMs) $M_{a,b} : x \mapsto axb$ on C^*-algebras A, where a and b are elements of the multiplier algebra $M(A)$, are usually considered as basic building blocks for more general types of operators on A, as their finite sums (elementary operators) comprise both inner derivations and inner automorphisms. It is therefore natural to ask which operators $\phi : A \to A$ can be obtained as uniform (operator-norm) limits of TMs.

Although this question might seem simple at first, the answer is non-trivial even for n-homogeneous C^*-algebras A (i.e all irreducible representations of A have the same finite dimension n), where $n \geq 2$. A well-known theorem of Fell and Tomiyama-Takesaki asserts that for any n-homogeneous C^*-algebra A with spectrum $\widehat{A}$ there is a canonical (up to isomorphism) locally trivial bundle $\mathcal{E}$ over $\widehat{A}$ with fibre $\mathbb{M}_n$ and structure group $PU(n) = \mathrm{Aut}(\mathbb{M}_n)$ such that A is isomorphic to the algebra $\Gamma_0(\mathcal{E})$ of sections of $\mathcal{E}$ which vanish at infinity.

Let us denote by $\mathrm{TM}(A)$ the set of all TMs on a C^*-algebra A. The main result of this talk is the following:

Theorem. *Let A be an n-homogeneous C^*-algebra with the associated bundle $\mathcal{E}$. If $\widehat{A}$ is second-countable, then* $\mathrm{TM}(A)$ *is not uniformly closed if and only if there exists an open subset U of $\widehat{A}$ and a phantom complex line subbundle of $\mathcal{E}|_U$.*

Definition. A locally trivial fibre bundle $\mathcal{F}$ over a locally compact Hausdorff space X is said to be a *phantom bundle* if $\mathcal{F}$ is not globally trivial, but is trivial on each compact subset of X.

A prominent example of a C^*-algebra for which $\mathrm{TM}(A)$ is not uniformly closed is $A = C_0(X, \mathbb{M}_2)$, where X is the standard model of the Eilenberg-MacLane space $K(\mathbb{Q}, 2)$ (i.e. a mapping telescope of the sequence $\mathbb{S}^1 \xrightarrow{z} \mathbb{S}^1 \xrightarrow{z^2} \mathbb{S}^1 \xrightarrow{z^3} \cdots$). Furthermore, using standard arguments from algebraic topology, we show that $d = 3$ is the smallest possibble dimension such that there exists an open subset X of $\mathbb{R}^d$ with the property that $\mathrm{TM}(C_0(X, \mathbb{M}_n))$ is not uniformly closed for some n.

This is a joint work with Richard Timoney (Trinity College Dublin).