GLASNIK MATEMATIČKI
Vol. 37(57)(2002), 259 – 268

STEINER 2-DESIGNS $S(2, 4, 28)$ WITH NONTRIVIAL AUTOMORPHISMS

Vedran Krčadinac
Department of Mathematics, University of Zagreb, Croatia

Abstract. In this article designs with parameters $S(2, 4, 28)$ and nontrivial automorphism groups are classified. A total of 4466 designs were found. Together with some $S(2, 4, 28)$'s with trivial automorphism groups found by A.Betten, D.Betten and V.D.Tonchev this sums up to 4653 nonisomorphic $S(2, 4, 28)$ designs.

1. Introduction

A Steiner 2-design is a finite incidence structure with v points and b lines, with k points on every line and a unique line through any pair of points. Lines can be identified with k-element point sets and incidence with inclusion. Line-degrees in a Steiner 2-design are constant by definition. Point-degrees also turn out to be constant, i.e. there are r lines through every point. The integers v, b, k and r are not independent; they satisfy $v - 1 = r(k - 1)$ and $bk = vr$. Usually v and k are taken as independent parameters and written in the form $S(2, k, v)$ or $2 - (v, k, 1)$.

Thus, a $S(2, 4, 28)$ design is a finite incidence structure with $v = 28$ points and $b = 63$ lines, with $k = 4$ points on any line, $r = 9$ lines through any point and a unique line through any pair of points. These designs belong to an important family, the *unitals*. Unitals are simply Steiner 2-designs with parameters $S(2, q+1, q^3+1)$. The classical (or Hermitian) unitals are obtained as absolute points and nonabsolute lines of a unitary polarity in the projective plane $PG(2, q^2)$. Another family of unitals is associated with the simple Ree groups (see [9]). Both constructions give rise to $S(2, 4, 28)$ designs. We will

2000 *Mathematics Subject Classification.* 05B05, 51E10.
Key words and phrases. Block design, Steiner system, automorphism group.

refer to them as the *classical* and the *Ree* $S(2,4,28)$. A detailed analysis of the two designs appears in [2].

An *automorphism* of a Steiner 2-design D is a permutation of points mapping lines on lines (here lines are taken to be k-element sets of points). The set of all automorphisms forms a group under composition, the *full automorphism group* $\operatorname{Aut} D$. Automorphisms are used for the construction of designs. By assuming a certain automorphism group $G \leq \operatorname{Aut} D$ the number of objects to look for is reduced and a computer search may become feasible.

The purpose of this article is to find all $S(2,4,28)$ designs with nontrivial automorphisms. Clearly it suffices to look for designs with automorphisms of prime order. In section 2 actions of such automorphisms on the points and lines of a $S(2,4,28)$ are studied. In section 3 the classification algorithm is described and the results are presented. Finally, in section 4 the designs are analysed. We look at resolvability, subdesigns, associated codes and other properties.

2. On Prime Order Automorphisms

Let α be an automorphism of prime order p of a $S(2,4,28)$ design. Two questions are of interest:

1. What are the possible values of p?
2. How many points and lines does α keep fixed and what configuration do they form?

The next two lemmas about Steiner 2-designs will be helpful.

Lemma 2.1. *If a* $S(2,k,v)$ *design possesses a* $S(2,k,v')$ *subdesign with* $v' < v$*, then* $v \geq k(k^2-2k+2)$.

Proof. Consider a point P not belonging to the subdesign. Any line through P contains at most one point of the subdesign, hence $r \geq v'$. Fisher's inequality applied to the subdesign yields $v' \geq k^2-k+1$ and consequently $r \geq k^2-k+1$. The inequality now follows from $r = \frac{v-1}{k-1}$. □

Lemma 2.2. *Let* α *be an automorphism of prime order* p *of a* $S(2,k,v)$ *design. If* $v < k(k^2-2k+2)$ *and* $p \geq k-1$*, then* α *either has no fixed points, or has a single fixed point, or keeps fixed* k *points on a line.*

Proof. A line joining two fixed points is necessarily fixed pointwise, since non-fixed points lie in orbits of size $p > k-2$. If the set of points kept fixed by α were to contain a triangle, it would form a $S(2,k,v')$ subdesign, contradicting lemma 2.1. Hence, all fixed points are collinear and their number is easily seen to be 0, 1 or k. □

The following known results easily follow from lemma 2.2.

Proposition 2.3. *If a* $S(2,4,28)$ *design admits an automorphism of prime order* p*, then* $p = 2$*,* 3 *or* 7*.*

Proposition 2.4. *Automorphisms of order* 7 *act fixed point- and line-free on* $S(2,4,28)$ *designs.*

It is also not difficult to answer question 2 for $p = 3$.

Proposition 2.5. *Let* α *be an automorphism of order* 3 *of a* $S(2,4,28)$ *design. The set of points and lines kept fixed by* α *is one of the following:*

(a) *a single point and* g *lines through it, where* $g = 0$, 3, 6 *or* 9,

(b) *four points on a line and two more lines through each of the points.*

Proof. The number of fixed lines through a fixed point is clearly 0, 3, 6 or 9. Dually, on a fixed line there is a single fixed point or 4 fixed points. The automorphism keeps at least one point fixed, because $v = 28 \equiv 1 (\text{mod } 3)$. According to lemma 2.2 this point is either unique or there are four fixed points on a line ℓ. The first case amounts to the configurations described in (a). In the second case through any of the fixed points there is a fixed line (ℓ) and consequently at least two more fixed lines. Since 8 fixed lines and ℓ already cover 28 points, there can be no more than three fixed lines through a fixed point. Thus, in the second case the fixed points and lines form configuration (b). ◻

Automorphisms of order $p = 2$ need to be considered separately. Lemma 2.2 does not apply here; non-collinear sets of fixed points are possible.

Theorem 2.6. *Let* α *be an automorphism of order* 2 *of a* $S(2,4,28)$ *design. The set of points and lines kept fixed by* α *is one of the following:*

(a) *seven lines and* f *points on one of the lines, where* $f = 0$, 2, *or* 4,

(b) *four points in general position (no three collinear), six lines joining pairs of points and three more lines not incident with any of the points,*

(c) *six points, four of them on a line; nine lines joining pairs of points and one more line not incident with any of the points.*

Configurations (b) and (c) are shown in figure 1.

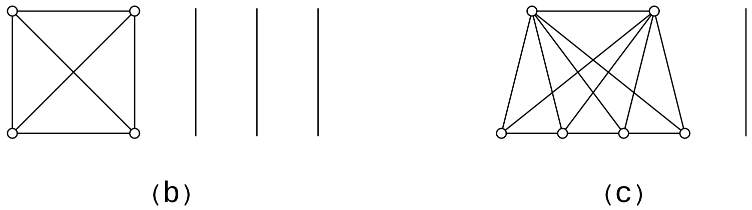

Figure 1. Configurations (b) and (c) of theorem 2.6.

Proof. Denote by m and n the number of point and line orbits of size two, respectively. Then the number of fixed points is $f = 28 - 2m$ and the number of fixed lines $g = 63 - 2n$. Furthermore, let $\mathcal{B}_i$ be the set of all lines incident with i fixed points and $b_i = |\mathcal{B}_i|$, for $i = 0, \ldots, 4$. The set $\mathcal{B}_3$ is obviously empty and hence $b_3 = 0$. Lines in $\mathcal{B}_2$ and $\mathcal{B}_4$ are fixed; lines in $\mathcal{B}_1$ are not fixed and lines in $\mathcal{B}_0$ can be either. Denote by b_0' the number of fixed lines in $\mathcal{B}_0$ and by b_0'' the number of non-fixed lines in $\mathcal{B}_0$. The total number of lines and the number of fixed lines can now be expressed as:

$$b_0' + b_0'' + b_1 + b_2 + b_4 \;=\; b \;=\; 63 \tag{2.1}$$

$$b_0' + b_2 + b_4 \;=\; g \;=\; 63 - 2n \tag{2.2}$$

Every line in $\mathcal{B}_2$ contains a point orbit of size two, and fixed lines in $\mathcal{B}_0$ contain two point orbits of size two each:

$$2b_0' + b_2 \;=\; m \tag{2.3}$$

By counting pairs of fixed points and pairs of non-fixed points we get the next two equations:

$$b_2 + 6b_4 = \frac{f(f-1)}{2} = (14-m)(27-2m) \tag{2.4}$$

$$6(b_0' + b_0'') + 3b_1 + b_2 = m\,(2m-1) \tag{2.5}$$

The system of equations (1)–(5) has a unique solution in terms of m and n.

$$\begin{aligned}
b_0' &= \frac{m(m-25)}{2} + 3n\\
b_0'' &= 16m - 6n\\
b_1 &= -16m + 8n\\
b_2 &= m(26-m) - 6n\\
b_4 &= 63 + \frac{m(m-27)}{2} + n
\end{aligned}$$

Only nine pairs $(m,n) \in \{1,\ldots,14\} \times \{1,\ldots,31\}$ yield non-negative values for the b_i's, as reported in table 1. The top three rows obviously correspond to configurations (a). Row 4 corresponds to configuration (b) and row 5 to configuration (c). It remains to be shown that rows 6 – 9 of table 1 describe impossible configurations.

In row 6 we have $b_4 = 2$, i.e. there are two lines containing four fixed points each. These two lines meet in at most one point, so there should be at least 7 fixed points. However, we have $f = 6$ in row 6. Similarly, three or more $\mathcal{B}_4$-lines require at least 9 fixed points, while we have $f = 8$ in rows 7 and 8. To prove row 9 impossible consider the incidence structure formed by the 10 fixed points and six $\mathcal{B}_4$-lines. Denote the point degrees in this configuration $r_1, \ldots, r_{10}$. The total number of incidences is 24, hence $r_1 + \ldots + r_{10} = 24$.

No.	m	n	b_0'	b_0''	b_1	b_2	b_4	f	g
1	14	28	7	56	0	0	0	0	7
2	13	28	6	40	16	1	0	2	7
3	12	28	6	24	32	0	1	4	7
4	12	27	3	30	24	6	0	4	9
5	11	26	1	20	32	9	1	6	11
6	11	27	4	14	40	3	2	6	9
7	10	26	3	4	48	4	4	8	11
8	10	25	0	10	40	10	3	8	13
9	9	24	0	0	48	9	6	10	15

TABLE 1. Pairs (m,n) yielding non-negative values for the b_i's.

The number of triples (P,ℓ_1,ℓ_2), where ℓ_1 and ℓ_2 are two lines through a point P is clearly

$$\sum_{i=1}^{10}\binom{r_i}{2}=\sum_{i=1}^{10}\frac{r_i(r_i-1)}{2}$$

Under the constraint $r_1+\ldots+r_{10}=24$ this quadratic function reaches minimal value 16.8 when $r_1=\ldots=r_{10}=2.4$; hence there are at least 17 triples. However, because any two lines meet in at most one point there can be no more than $\binom{6}{2}=15$ triples. Thus configurations corresponding to row 9 of table 1 are not possible, too. $\square$

3. The Classification

A construction method for block designs with automorphism groups was developed in the 1980s and used by many authors throughout the 1980s and 1990s (see, for example, [3], [6], [7], [13] and [17]). It is based on the notion of an *orbit matrix*. Let D be a Steiner 2-design with parameters v, b, k, r and automorphism group $G\le \operatorname{Aut} D$. Denote the point and line orbits $\mathcal{P}_1,\ldots,\mathcal{P}_m$ and $\mathcal{L}_1,\ldots,\mathcal{L}_n$, respectively. The orbits form a tactical decomposition, i.e. the number $a_{ij}=\left|\left\{\ell\in\mathcal{L}_j \;\middle|\; P\in\ell\right\}\right|$ of lines from $\mathcal{L}_j$ incident with a point $P\in\mathcal{P}_i$ does not depend on the choice of P. If $\nu_i=|\mathcal{P}_i|$ and $\beta_j=|\mathcal{L}_j|$ are the orbit sizes, the matrix $A=[a_{ij}]$ has following properties:

(1) $\displaystyle\sum_{j=1}^{n}a_{ij}=r$, for $1\le i\le m$,

(2) $\displaystyle\sum_{i=1}^{m}\frac{\nu_i}{\beta_j}a_{ij}=k$, for $1\le j\le n$,

$$\textbf{(3)}\ \sum_{j=1}^{n} \frac{\nu_i}{\beta_j} a_{ij} a_{i'j} = \begin{cases} \nu_i, & \text{if } i \neq i' \\ \nu_i + r - 1, & \text{if } i = i' \end{cases}, \text{ for } 1 \leq i, i' \leq m.$$

Any such matrix is called an *orbit matrix.* The first step of the classification is to find all orbit matrices.

Matrices equivalent under rearrangements of rows and columns can be identified. More precisely, orbit matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be *isomorphic* provided there is a pair of permutations $(\pi, \sigma) \in S_m \times S_n$ such that $\nu_i = \nu_{\pi(i)}$, $\beta_j = \beta_{\sigma(j)}$ and $b_{ij} = a_{\pi(i)\,\sigma(j)}$, for $i = 1, \ldots, m$, $j = 1, \ldots, n$. The task is to find all orbit matrices up to isomorphism.

In the next step, usually called *indexing*, orbit matrices are transformed to incidence matrices of the designs. The (i, j)-entry of an orbit matrix is replaced by a $\nu_i \times \beta_j$ zero-one matrix having a_{ij} ones in each row and being invariant under the action of G.

Not every orbit matrix gives rise to designs. On the other hand, a single orbit matrix may produce several nonisomorphic designs. Thus, as the final step incidence matrices of the designs need to be checked for isomorphism.

Turning now to $S(2, 4, 28)$ designs, the first step of the classification proved to be computationally most difficult. This is due to a large number of nonisomorphic orbit matrices for $p = 2$ and $p = 3$. A classification algorithm used by E.Spence in [14], [15] and [16] was adapted to find orbit matrices.

The set of all orbit matrices needs to be totally ordered; we use lexicographical ordering on vectors obtained by concatenating the rows. A matrix is said to be *canonical* if it is the greatest among all isomorphic matrices. The algorithm produces canonical orbit matrices by adjoining a row and eliminating non-canonical matrices in each step. To see that this will indeed produce a canonical representative of each orbit matrix note that a canonical matrix with last row deleted remains canonical.

The second step of the classification, indexing, proceeded at a much faster pace; a straightforward backtracking algorithm was used. To eliminate isomorphic designs B.D.McKay's `nauty` [11] was used, and proved to be very efficient.

Seven nonisomorphic orbit matrices corresponding to automorphisms of order 7 were found. Five of the matrices could be indexed and they gave rise to 11 nonisomorphic $S(2, 4, 28)$ designs. Thus we can conclude:

THEOREM 3.1. *There are* 11 *designs* $S(2, 4, 28)$ *with automorphisms of order* 7*.*

Brouwer [2] gives 8 as the number of $S(2, 4, 28)$'s with automorphisms of order 7. By comparing the table in section C of [2] with table 4 we conclude that he missed two designs with full automorphism group of order 21 and one with full automorphism group of order 7.

Type of aut.	# of orbit mat.	# of designs
(a), $g=0$	13083	1635
(a), $g=3$	9017	297
(a), $g=6$	267	0
(a), $g=9$	11	8
(b)	29	43

TABLE 2. Designs with automorphisms of order 3.

Classification of $S(2,4,28)$ designs with automorphisms of order 3 involved much more computation. An automorphism of order 3 keeps fixed one of five different configurations (proposition 2.5), i.e. there are five types of orbit matrices. Matrices of each type were classified separately; the results are summarised in table 2. Since a single design may well have several automorphisms fixing different configurations, the total number of $S(2,4,28)$'s with automorphisms of order 3 cannot be determined simply by summing up numbers in the last column of table 2. Lists of representatives were concatenated and `nauty` was used to eliminate isomorphic copies; 1978 designs remained.

THEOREM 3.2. *There are* 1978 *designs* $S(2,4,28)$ *with automorphisms of order* 3.

The search for $S(2,4,28)$ designs with automorphisms of order 2 was even more involved. Five cases corresponding to fixed structures of theorem 2.6 had to be examined. Results are presented in table 3. The total number of designs with automorphisms of order 2 was determined with `nauty`.

THEOREM 3.3. *There are* 2590 *designs* $S(2,4,28)$ *with automorphisms of order* 2.

Our main result was obtained by concatenating lists of representatives for $p=2$, 3 and 7 and applying `nauty` once more; 4466 nonisomorphic designs remained.

Type of aut.	# of orbit mat.	# of designs
(a), $f=0$	52281	121
(a), $f=2$	58538	226
(a), $f=4$	3513	788
(b)	28835	473
(c)	16259	1333

TABLE 3. Designs with automorphisms of order 2.

THEOREM 3.4. *There are* 4466 *designs* $S(2,4,28)$ *with nontrivial automorphisms.*

The complete list of representatives is far too big to be reproduced here, but it can be downloaded from the author's Web page

`http://www.math.hr/~krcko`

The *CRC Handbook of Combinatorial designs* [4] gives 145 as a lower bound for the number of nonisomorphic $S(2,4,28)$ designs. Recently A.Betten, D.Betten and V.Tonchev [1] constructed 909 nonisomorphic $S(2,4,28)$'s by considering tactical decompositions defined by vectors from the dual code associated with the designs. Among them are 187 designs with trivial automorphism groups, hence there are at least 4653 unitals $S(2,4,28)$. Brouwer [2] also found 26 unitals with trivial automorphism groups but it is difficult to assess if they are different from those constructed in [1].

There are probably many more $S(2,4,28)$'s with trivial automorphism groups. Considering recent advances in computer hardware a complete classification may soon be coming within reach.

4. PROPERTIES OF THE DESIGNS

In this section properties of the 4466 constructed designs are briefly examined. First we give the distribution by order of full automorphism group (table 4). The classical $S(2,4,28)$ has $P\Gamma U(3,9)$ (of order 12096) as its full automorphism group, while the Ree $S(2,4,28)$ has $P\Gamma L(2,8)$ (of order 1512). Both groups act doubly transitively on the points. These are the only transitive $S(2,4,28)$ designs.

Six of the designs are resolvable, among them both the classical and the Ree $S(2,4,28)$. The Ree unital admits two nonisomorphic resolutions, hence there are 7 different resolutions of $S(2,4,28)$'s with nontrivial automorphisms.

It can be shown that only designs with parameters $S(2,3,7)$ (Fano planes) and $S(2,3,9)$ (affine planes of order 3) can appear as subdesigns in

|Aut|	#	|Aut|	#	|Aut|	#	|Aut|	#
12096	1	48	12	18	1	6	60
1512	1	42	1	16	10	4	374
216	1	32	2	12	12	3	1849
192	2	27	1	9	18	2	2028
72	1	24	12	8	71		
64	1	21	6	7	2		

TABLE 4. Distribution by order of full automorphism group.

$S(2,4,28)$'s. Of our 4466 designs 39 (all with full automorphism group of order 3) possess both kinds of subdesigns; 1511 possess only Fano subplanes, and 89 only $S(2,3,9)$ subdesigns. Neither the classical nor the Ree $S(2,4,28)$ have any subdesigns.

Finally, we look at codes spanned by incidence vectors of the lines. Only binary codes are of interest. Brouwer [2] notes that only vectors of weights 0, 10, 12, 14, 16, 18 and 28 can occur in the dual code. Since it contains the all-one vector j, the weight enumerator $W(x) = \sum_{i=0}^{28} a_i x^i$ of the dual code is completely determined by a_{10}, a_{12} and a_{14}. The weight enumerator of the code itself can then be computed from MacWilliams' relation; 45 different weight enumerators occurred. Distribution of the designs by 2-rank and (a_{10}, a_{12}, a_{14}) is given in table 5.

The design of 2-rank 19 is the Ree unital. It was proved in [10] that 19 is the lowest possible 2-rank of a $S(2,4,28)$ and that it is attained only for the Ree unital. Furthermore, in [5] it was proved that there are no $S(2,4,28)$'s of 2-rank 20 and precisely four of 2-rank 21 (the classical unital and three more unitals with automorphism groups of size 192, 24 and 6).

dim	(a_{10}, a_{12}, a_{14})	#	dim	(a_{10}, a_{12}, a_{14})	#	dim	(a_{10}, a_{12}, a_{14})	#
19	$(84,63,216)$	1	23	$(6,7,4)$	8	24	$(4,3,0)$	32
21	$(0,63,0)$	1	23	$(7,3,10)$	3	24	$(5,1,2)$	7
21	$(20,31,24)$	1	23	$(8,3,8)$	9	24	$(6,1,0)$	2
21	$(24,15,48)$	2	23	$(8,7,0)$	5	25	$(0,0,6)$	28
22	$(8,15,16)$	1	23	$(9,3,6)$	3	25	$(0,1,4)$	68
22	$(12,7,24)$	4	24	$(0,1,12)$	9	25	$(0,3,0)$	87
22	$(12,15,8)$	4	24	$(0,3,8)$	30	25	$(1,0,4)$	21
22	$(14,7,20)$	3	24	$(0,7,0)$	19	25	$(1,1,2)$	106
23	$(0,3,24)$	7	24	$(1,1,10)$	9	25	$(2,0,2)$	16
23	$(0,15,0)$	3	24	$(1,3,6)$	14	25	$(2,1,0)$	179
23	$(3,3,18)$	2	24	$(2,1,8)$	3	25	$(3,0,0)$	47
23	$(4,3,16)$	3	24	$(2,3,4)$	58	26	$(0,0,2)$	174
23	$(4,7,8)$	11	24	$(3,1,6)$	29	26	$(0,1,0)$	249
23	$(5,3,14)$	5	24	$(3,3,2)$	25	26	$(1,0,0)$	532
23	$(6,3,12)$	15	24	$(4,1,4)$	23	27	$(0,0,0)$	2608

TABLE 5. Distribution by 2-rank and weight enumerator.

ACKNOWLEDGEMENTS

The author wishes to thank the referees for valuable suggestions.

References

[1] A. Betten, D. Betten, V. D. Tonchev, Unitals and Codes, to appear in Discrete Math.

[2] A. E. Brouwer, Some unitals on 28 points and their embeddings in projective planes of order 9, Geometries and Groups, Proc. Colloq. Berlin 1981, Lecture Notes in Math. **893** (1981), 183–188.

[3] V. Ćepulić and M. Essert, Biplanes $(56, 11, 2)$ with automorphisms of order 4 fixing some points, Discrete Math. **71** (1988), 9–17.

[4] C. J. Colbourn and J. H. Dinitz (eds.), The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, 1996.

[5] D. B. Jaffe, V. D. Tonchev, Computing linear codes and unitals, Des. Codes Cryptogr. **14** (1998), 39–52.

[6] Z. Janko and T. van Trung, Construction of a new symmetric block design for $(78, 22, 6)$ with the help of tactical decompositions, J. Combin. Theory Ser. A **40** (1985), 451–455.

[7] E. S. Kramer, S. S. Magliveras and R. Mathon, The Steiner systems $S(2, 4, 25)$ with nontrivial automorphism group, Discrete Math. **77** (1989), 137–157.

[8] V. Krčadinac, Steiner 2-designs $S(2, 5, 41)$ with automorphisms of order 3, to appear in J. Combin. Math. Combin. Comput.

[9] H. Lüneburg, Some remarks concerning the Ree groups of type (G_2), J. Algebra **3** (1966), 256–259.

[10] G. McGuire, V. D. Tonchev, H. N. Ward, Characterizing the Hermitian and Ree unitals on 28 points, Des. Codes Cryptogr. **13** (1998), 57–61.

[11] B. D. McKay, `nauty` User's guide (version 1.5), Technical Report TR-CS-90-02, Department of Computer Science, Australian National University, 1990.

[12] M. Schönert et al., GAP – Groups, Algorithms and Programming, Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, 1995.

[13] J. Šiftar and B. Shita, On rigidity of $2-(46, 6, 1)$ designs, Glas. Mat. Ser. III **26(46)** (1991), 3–11.

[14] E. Spence, A complete classification of symmetric $(31, 10, 3)$ designs, Des. Codes Cryptogr. **2** (1992), 127–136.

[15] E. Spence, Classification of Hadamard matrices of orders 24 and 28, Discrete Math. **140** (1995), 185–243.

[16] E. Spence, The complete classification of Steiner Systems $S(2, 4, 25)$, J. Combin. Des. **4** (1996), 295–300.

[17] V. D. Tonchev, Steiner triple systems of order 21 with automorphisms of order 7, Ars Combin. **23** (1987), 93-96.

Department of Mathematics
University of Zagreb
p.p. 335
HR-10002 Zagreb, Croatia
E-mail address: `krcko@math.hr`

Received: 01.06.2001.