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PEDAL SETS OF UNITALS IN PROJECTIVE PLANES OF ORDER 9 AND 16

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Dedicated to the memory of Professor Svetozar Kurepa

Abstract. Let $\mathcal{U}$ be a unital in a projective plane $\mathcal{P}$. Given a point $P \in \mathcal{P} \setminus \mathcal{U}$, the points $F \in \mathcal{U}$ such that FP is tangent to $\mathcal{U}$ are the *feet* of P, and the set of all such points is the *pedal set* of P. In this paper we study pedal sets of unitals embedded in projective planes of order 9 and 16.

1. Introduction

A finite projective plane of order q can be defined as a block design with parameters $(q^2 + q + 1, q + 1, 1)$; that is, a finite incidence structure with $q^2 + q + 1$ points, $q + 1$ points on every line, and a unique line through every pair of points. Similarly, a unital of order q is a $(q^3 + 1, q + 1, 1)$ block design. For more on finite projective planes we refer to the book [10], and for unitals to [2].

The most interesting unitals are the ones embedded in a projective plane $\mathcal{P}$ of order q^2. In this setting a unital $\mathcal{U}$ is a set of $q^3 + 1$ points of $\mathcal{P}$ with the property that every line of $\mathcal{P}$ meets $\mathcal{U}$ either in one point, or in $q + 1$ points. In the first case the line is *tangent* to $\mathcal{U}$, and in the second case it is *secant*. The set of all tangents is a unital in the dual plane $\mathcal{P}^*$, called the *dual unital* and denoted by $\mathcal{U}^*$.

Through any point $P \in \mathcal{U}$ there is a unique tangent and q^2 secant lines. If P is a point of $\mathcal{P} \setminus \mathcal{U}$, there are $q + 1$ tangents and $q^2 - q$ secants through P [2, Theorem 2.3]. For every tangent t there is a single point $F \in t \cap \mathcal{U}$; such points F are the *feet* of P, and the set of all feet is called the *pedal set* of P. Thus, the pedal set of a point $P \in \mathcal{P} \setminus \mathcal{U}$ consists of $q + 1$ points of $\mathcal{U}$.

If $\mathcal{P}$ is the desarguesian projective plane $PG(2, q^2)$, a unital $\mathcal{U}$ can be obtained from a unitary polarity of $\mathcal{P}$ as the set of absolute points, i.e.

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points that are incident with their polars. This is the *hermitian* unital of $PG(2,q^2)$. A point $P \in \mathcal{P} \setminus \mathcal{U}$ is not incident with its polar $\ell = P^*$. By the definition of a polarity, the polar of a point on ℓ is a line through P. Therefore, the polar of an absolute point $F \in \ell$ is the line FP, and the pedal set of P is $\ell \cap \mathcal{U}$. In fact, the property that pedal sets are collinear holds for any unital arising from a unitary polarity, not just for the hermitian unitals in $PG(2,q^2)$. We shall refer to unitals obtained from unitary polarities as *classical*.

There are many non-classical unitals, and in this case the pedal sets need not be collinear. The most general known constructions of unitals are due to Buekenhout [4]. They apply to translation planes and utilize the Bruck-Bose representation [3] (discovered earlier by André [1] in a group-theoretic setting). Briefly, a translation plane of order q^2 is represented in the projective space $PG(4,q)$ by a hyperplane $\Sigma_\infty \cong PG(3,q)$ and a spread $\mathcal{S}$ of Σ_∞. Unitals are obtained either from nonsingular quadrics of $PG(4,q)$ meeting Σ_∞ in a regulus contained in $\mathcal{S}$ (these are the *hyperbolic Buekenhout unitals*, or *nonsingular Buekenhout unitals* as they are called in [2]), or from ovoidal cones tangent to Σ_∞ in a line of $\mathcal{S}$ (called the *parabolic Buekenhout unitals*, or *ovoidal-Buekenhout-Metz unitals* in [2]). A detailed treatment of Buekenhout's constructions is given in [2].

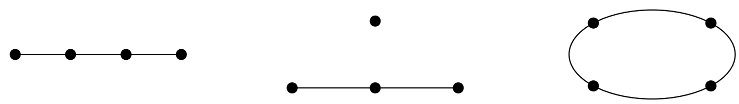
Dover [6] showed that for parabolic Buekenhout unitals the points on the tangent ℓ_∞, corresponding to Σ_∞ in the Bruck-Bose representation, have collinear pedal sets. Not many other general results about pedal sets of non-classical unitals are known. The goal of this paper is to study pedal sets of unitals embedded in projective planes of order 9 and 16.

We shall call points of $\mathcal{P} \setminus \mathcal{U}$ with collinear pedal sets *special points*. A tangent with q^2 special points, such as the tangent ℓ_∞ of a parabolic Buekenhout unital, will be called a *special tangent*. Dover [6] also showed that special points give rise to parallel classes (spreads) of the unital, and special tangents to resolutions (packings). Therefore, we shall pay attention to the distribution of special points in the unitals under consideration.

2. Unitals in the planes of order 9

By [12], there are exactly four projective planes of order 9: the desarguesian plane $PG(2,9)$, the Hall plane and its dual, and the self-dual Hughes plane. We will denote them by PG(2,9), HALL, HALL* and HUGH. Penttila and Royle [14] classified all unitals in these planes. Up to equivalence, there are two unitals in PG(2,9), four unitals in HALL and HALL*, and eight unitals in HUGH. Unitals are equivalent if they can be mapped onto each other by a collineation of the ambient plane. The unitals will be denoted by PG(2,9).1, PG(2,9).2, HALL.1,..., HALL.4, etc. Since they are not reproduced in [14], we repeated the classification and got the same numbers of unitals. The graph isomorphism program *nauty* [13] was used to check equivalence of (partial) unitals. The unitals can be downloaded from `http://web.math.hr/~krcko/results/steiner.html`.

In a plane of order 9, pedal sets consist of 4 points of $\mathcal{U}$. There are three possible configurations of 4 points: either they are collinear, or 3 points are on a line and the fourth point is not on this line, or no 3 of the 4 points are collinear and the configuration is an *arc*. We shall denote these configurations by giving sizes of their intersections with lines, i.e. by (4), $(3, 2^3)$ and (2^6), respectively. They are depicted in Figure 1.



(4) $(3, 2^3)$ (2^6)

FIGURE 1. Configurations of 4 points in a unital of order 3.

Interestingly, all three configurations occur as pedal sets. In Table 1, we report numbers of pedal sets of each type for unitals in the planes of order 9. The dual unitals have identical pedal set counts, therefore the plane HALL* is omitted.

Some of unitals can be identified from the numbers in Table 1: PG(2,9).1 is clearly the hermitian unital, and PG(2,9).2 is the (non-classical) orthogonal Buekenhout-Metz unital [2, Theorem 4.18]. The computer algebra system GAP [8] was used to identify some other unitals: HALL.1 is a hyperbolic Buekenhout unital, HALL.2 is a parabolic Buekenhout unital, and HALL.3 and HALL.4 are not Buekenhout unitals. HUGH.1 is the classical-Rosati unital [16]. Wantz [19] found a more general class of unitals in the Hughes plane of order q^2, but for $q = 3$ it comprises only HUGH.1.

As already mentioned, all tangents of the hermitian unital PG(2,9).1 are special. Among the remaining unitals, only the parabolic Buekenhout unitals PG(2,9).2 and HALL.2 posses a special tangent ℓ_∞. For PG(2,9).2 there are no special points outside ℓ_∞, and for HALL.2 the six special points outside ℓ_∞ lie on a secant through $P_\infty = \ell_\infty \cap \mathcal{U}$.

The 31 special points of the hyperbolic Buekenhout unital HALL.1 form an interesting configuration with the tangents. Through every special point there are exactly four tangents. Four of the 28 tangents contain one special point, while each of the remaining 24 tangents contains 5 special points.

The 15 special points of the classical-Rosati unital HUGH.1 form a triangle. The sides of the triangle are three secant lines of HUGH.1, containing

Unital	Pedal set		
	(4)	$(3,2^3)$	(2^6)
PG(2,9).1	63	0	0
PG(2,9).2	9	0	54
HALL.1	31	32	0
HALL.2	15	0	48
HALL.3	1	32	30
HALL.4	1	8	54
HUGH.1	15	24	24
HUGH.2	7	8	48
HUGH.3	4	17	42
HUGH.4	4	17	42
HUGH.5	3	18	42
HUGH.6	1	28	34
HUGH.7	1	0	62
HUGH.8	1	0	62

Table 1. Pedal sets of unitals in the planes of order 9.

six special points each. For HUGH.2, six of the seven special points lie on a secant line. Similarly, three of the four special points of HUGH.3 and HUGH.4 are on a secant line, while the three special points of HUGH.5 are not collinear.

We noticed that all parallel classes of unitals embedded in planes of order 9 arise from special points, and all resolutions arise from special tangents. This is not the case with unitals in planes of order 16, as we shall see in the next section. Thus, PG(2,9).1, PG(2,9).2, HALL.2, and HALL.2* are resolvable, and the remaining unitals in the planes of order 9 are not resolvable. There are two more resolvable 2-$(28,4,1)$ designs, one of which is the (non-embedable) Ree unital of order 3; see [11].

3. Unitals in the planes of order 16

There are 22 known projective planes of order 16. They are described in [15] and are available for download from Gordon Royle's web page [17]. We will use the same names for the planes as in [15]: PG(2,16), SEMI2, SEMI4, HALL, LMRH, JOWK, DSFP, DEMP, BBH1, BBH2, JOHN, BBS4, and MATH. The first eight are translation planes, and this list is known to be complete [5]. The planes PG(2,16), SEMI2, SEMI4, and BBH1 are self-dual, while the remaining planes have non-isomorphic duals HALL*, LMRH*, JOWK*, DSFP*, DEMP*, BBH2*, JOHN*, BBS4*, and MATH*.

Stoichev and Tonchev [18] performed a non-exhaustive search for unitals in the known planes of order 16. They found 38 examples, which are explicitly listed in [18] and are denoted by PG(2,16).1, PG(2,16).2, SEMI2.1, etc. The same notation is used in this paper. We noticed that the set of points HALL.4 represents a unital not only in the Hall plane, but also in the plane SEMI4. This unital is not equivalent to SEMI4.1 and will be denoted by SEMI4.2. Similarly, SEMI4.1 represents a unital in the Hall plane, but it is equivalent to HALL.4. Moreover, the dual set SEMI2.2* represents a unital in two dual planes, SEMI2* and LMRH*. By dualizing SEMI2.2* in LMRH*, we get another unital in LMRH. It will be denoted by LMRH.2 and is reproduced in Table 2 according to [18, Table 2]. Two more unitals in HALL, not listed in [18], are given in Table 2. They will be described shortly.

Thus, Stoichev's and Tonchev's data together with Table 2 represents 42 unitals in the projective planes of order 16 (not counting the dual unitals). These unitals are also available from our web page. Notice that each of the

Solution	Unital
LMRH.2	2 5 12 13 19 20 25 31 34 37 44 45 48
	49 55 62 67 68 73 79 80 81 87 94 99 100
	105 111 114 117 124 125 134 136 138 139 144 145 151
	158 162 165 172 173 176 177 183 190 192 193 199 206
	210 213 220 221 227 228 233 239 243 244 249 255 272
HALL.5	0 1 8 13 15 18 21 24 30 31 37 49 57
	59 60 61 65 71 72 76 78 81 110 112 114 117
	121 125 130 138 139 140 143 149 151 153 155 158 160
	164 167 168 169 189 194 198 200 201 204 208 211 215
	219 223 224 225 226 235 238 245 247 252 253 255 258
HALL.6	0 3 8 10 11 18 20 22 25 31 37 38 39
	42 44 51 57 58 60 63 72 81 83 84 87 92
	96 98 99 100 110 121 130 135 136 138 143 148 150
	151 152 155 160 164 168 172 175 176 178 182 186 189
	192 198 201 203 204 210 211 215 217 219 235 255 258

TABLE 2. Some new unitals in the Hall plane and Lorimer-Rahilly plane of order 16.

planes contains at least one unital. To date, no projective plane of square order without unitals is known.

Pedal sets of unitals in planes of order 16 consist of 5 points. Possible configurations are (5), $(4, 2^4)$, $(3^2, 2^4)$, $(3, 2^7)$, and (2^{10}); see Figure 2. The number of points $P \in \mathcal{P} \setminus \mathcal{U}$ for which the pedal set forms each of these configurations is given in Table 3. For 30 of the 42 unitals, the duals have identical pedal set counts. The remaining 12 dual unitals are reported in Table 4.

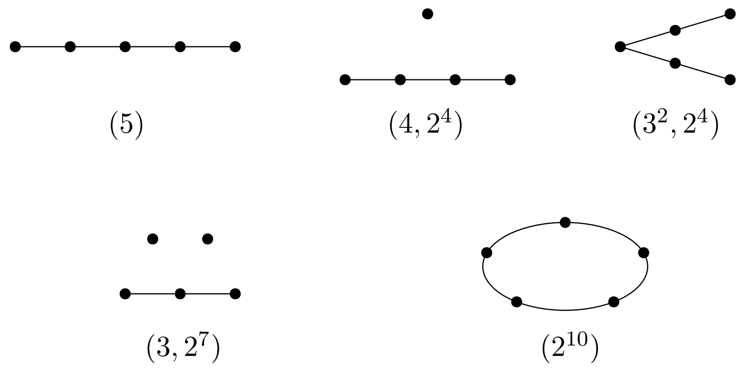


FIGURE 2. Configurations of 5 points in a unital of order 4.

The unital PG(2,16).1 is hermitian, and PG(2,16).2 is the non-classical Buekenhout-Metz unital. According to [18], HALL.1 is the Grüning unital [9], HALL.2 is another hyperbolic Buekenhout unital, HALL.3 is the (non-Buekenhout) Dover unital [7], and HALL.4 is a parabolic Buekenhout unital. Using GAP [8], we investigated Buekenhout unitals in HALL and found two more parabolic unitals HALL.5 and HALL.6.

In the Bruck-Bose representation, the desarguesian plane arises from a regular spread of Σ_∞. The Hall plane arises from a regular spread with one regulus replaced by the opposite regulus. We shall call this transformation "switching of a regulus". The unital HALL.1 corresponds to a nonsingular quadric of $PG(4,4)$ intersecting Σ_∞ in the switched regulus; see [2, Theorem 5.1]. If the intersection is a regulus disjoint from the switched regulus, a unital equivalent to HALL.2 is obtained [2, Theorem 5.2]. The unital HALL.4 corresponds to an ovoidal cone of $PG(4,4)$ tangent to Σ_∞ in a line belonging to the switched regulus. The unitals HALL.5 and HALL.6 are associated with ovoidal cones tangent to Σ_∞ in a spread line not belonging to the switched regulus [2, Theorem 5.4]. HALL.5 comes from a cone that would yield PG(2,16).1 if a regular spread was used, and HALL.6 from a cone that would yield PG(2,16).2. In [2, Chapter 5.1] it is remarked that there are exactly five inequivalent Buekenhout unitals in the Hall plane of order 16; thus, we have found all such unitals (however, there could be more non-Buekenhout unitals in HALL).

Regarding special points and special tangents, the most interesting unital is HALL.1. All of its pedal sets are collinear, despite the fact that it does not arise from a unitary polarity (recall that the Hall plane is not self-dual). Except this unital and the two unitals in $PG(2,16)$, other unitals with a special tangent are SEMI2.1, SEMI2.2, SEMI4.1, SEMI4.2, HALL.4, HALL.5, HALL.6, LMRH.1, LMRH.2, BBH1.1, BBH1.2, BBH2.2, JOHN.2, MATH.1, MATH.2, and MATH.3. The following unitals have exactly 16 special points, but they do not lie on a special tangent: HALL.3, JOWK.1, JOWK.2, JOWK.3, DSFP.1, DSFP.2, BBH1.3, BBH2.4, JOHN.1, JOHN.3, and JOHN.5. Instead, for these unitals the 16 special points constitute an affine plane of order 4, i.e. a 2-$(16,4,1)$ subdesign of the ambient plane. Such subdesigns also occur for some of the unitals with more than 16 special points, e.g. half of the 32 special points of HALL.4 form a special tangent, and the other half an affine plane of order 4.

Many unitals in the planes of order 16 admit more parallel classes than there are special points. Resolutions not associated with special tangents are also possible. For example, the unital BBH1.2 has 68 special points and one special tangent. Regarded as a design, BBH1.2 admits 138 parallel classes and two resolutions.

4. Some questions

We end this paper with some questions about pedal sets of unitals. Some of them may be addressed in future works.

In our opinion, a question worth further exploration is the relationship between pedal sets of a unital $\mathcal{U}$ and the dual unital $\mathcal{U}^*$. For all unitals in the planes of order 9 and for many unitals in the planes of order 16, the numbers of pedal sets of each type in $\mathcal{U}$ and $\mathcal{U}^*$ agree. However, for the unitals in Table 4 the pedal set counts of $\mathcal{U}$ and $\mathcal{U}^*$ are different.

The number of collinear pedal sets, i.e. the number of special points, always seems to agree for $\mathcal{U}$ and $\mathcal{U}^*$. Presently we do not know whether this holds in general, or just for the unitals considered in this paper. All considered unitals have at least one special point. Are there unitals without special points?

For the classical unitals all points outside $\mathcal{U}$ are special, but the non-classical unital HALL.1 also has this property. Are there more examples, and what can be said about unitals with the property that all points outside $\mathcal{U}$ have collinear pedal sets?

Unital	Pedal set				
	(5)	$(4, 2^4)$	$(3^2, 2^4)$	$(3, 2^7)$	(2^{10})
PG(2,16).1	208	0	0	0	0
PG(2,16).2	16	0	0	0	192
SEMI2.1	16	0	0	192	0
SEMI2.2	16	0	32	32	128
SEMI2.3	4	48	12	96	48
SEMI4.1	20	16	0	96	76
SEMI4.2	16	0	0	192	0
HALL.1	208	0	0	0	0
HALL.2	48	0	50	100	10
HALL.3	16	0	12	0	180
HALL.4	32	32	0	96	48
HALL.5	68	0	0	120	20
HALL.6	16	24	0	112	56
LMRH.1	16	0	0	0	192
LMRH.2	16	0	40	16	136
JOWK.1	16	0	12	0	180
JOWK.2	16	0	12	72	108
JOWK.3	16	8	12	100	72
JOWK.4	13	7	0	76	112
DSFP.1	16	24	24	84	60
DSFP.2	16	12	0	84	96
DEMP.1	4	12	24	120	48
DEMP.2	8	24	28	88	60
BBH1.1	32	32	0	64	80
BBH1.2	68	0	0	104	36
BBH1.3	16	0	24	88	80
BBH2.1	68	0	60	80	0
BBH2.2	16	16	0	112	64
BBH2.3	48	0	50	80	30
BBH2.4	16	32	28	88	44
BBH2.5	52	16	8	120	12
BBH2.6	18	0	30	76	84
JOHN.1	16	0	12	48	132

TABLE 3. Pedal sets of unitals in planes of order 16.

Unital	Pedal set				
	(5)	$(4, 2^4)$	$(3^2, 2^4)$	$(3, 2^7)$	(2^{10})
JOHN.2	32	0	0	112	64
JOHN.3	16	0	24	24	144
JOHN.4	24	0	28	88	68
JOHN.5	16	24	16	76	76
BBS4.1	24	24	16	84	60
MATH.1	16	64	0	128	0
MATH.2	16	0	0	128	64
MATH.3	16	32	0	64	96
MATH.4	4	0	28	96	80

TABLE 3. Pedal sets of unitals in planes of order 16 (continued).

Unital	Pedal set				
	(5)	$(4, 2^4)$	$(3^2, 2^4)$	$(3, 2^7)$	(2^{10})
SEMI2.2*	16	0	0	96	96
LMRH.2*	16	0	8	80	104
JOWK.2*	16	0	36	24	132
JOWK.3*	16	8	20	84	80
JOWK.4*	13	7	8	60	120
DEMP.1*	4	12	48	72	72
BBH1.3*	16	0	16	104	72
BBH2.4*	16	32	12	120	28
BBH2.6*	18	0	32	72	86
JOHN.2*	32	0	16	80	80
JOHN.5*	16	24	28	52	88
MATH.4*	4	0	12	128	64

TABLE 4. Pedal sets of the dual unitals in planes of order 16.

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