NAPOLEON'S QUASIGROUPS

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Abstract. Napoleon's quasigroups are idempotent medial quasigroups satisfying the identity $(ab \cdot b)(b \cdot ba) = b$. In works by V.Volenec geometric terminology has been introduced in medial quasigroups, enabling proofs of many theorems of plane geometry to be carried out by formal calculations in a quasigroup. This class of quasigroups is particularly suited for proving Napoleon's theorem and other similar theorems about equilateral triangles and centroids.

1. Introduction

Consider the Euclidean plane E^2 and define multiplication of its points by $a \cdot b = c$, where c is the centroid of the positively oriented equilateral triangle over $\overline{ab}$. The groupoid $(E^2, \cdot)$ is an example of the so-called *Napoleon's quasigroups.*

Definition 1.1. *An idempotent medial quasigroup* $(Q, \cdot)$ *is called a* Napoleon's quasigroup *if the following identity holds:*

$$(ab \cdot b)(b \cdot ba) = b. \tag{1}$$

This means that $(Q, \cdot)$ is a uniquely left and right solvable groupoid, i.e. for every $a, b \in Q$ there are unique $x, y \in Q$ such that $ax = b$ and $ya = b$ hold (denoted by $x = a \setminus b$ and $y = b \,/\, a$). Furthermore, $(Q, \cdot)$ satisfies the identities of *idempotency* and *mediality*:

$$a \cdot a = a, \tag{2}$$

$$ab \cdot cd = ac \cdot bd. \tag{3}$$

Immediate consequences are the identities known as *elasticity*, *left* and *right distributivity*:

$$ab \cdot a = a \cdot ba, \tag{4}$$

$$a \cdot bc = ab \cdot ac, \tag{5}$$

$$ab \cdot c = ac \cdot bc. \tag{6}$$

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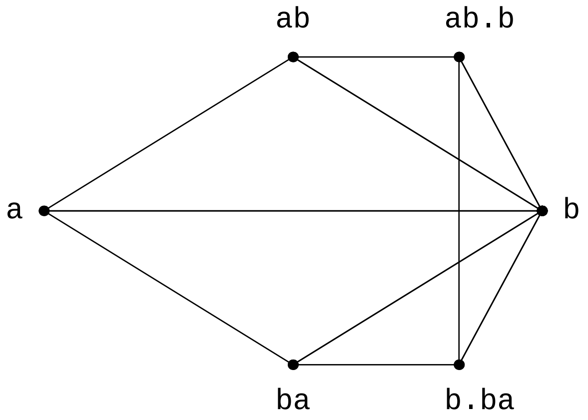

FIGURE 1. Geometric interpretation of identity (1).

The operation $\cdot$ is also left and right distributive over $\backslash$ and $/$, e.g.

$$(a \backslash b)c = ac \backslash bc. \tag{7}$$

Regarding our introductory example, the axioms are most easily checked by using complex coordinates in the plane. By identifying $E^2 \equiv \mathbb{C}$, the binary operation can be written as $a \cdot b = (1-q)a + qb$, for $q = \frac{1}{2} + \frac{i\sqrt{3}}{6}$. This is evidently an idempotent medial quasigroup, and identity (1) follows from $3q^2 - 3q + 1 = 0$. We could also take $a \cdot b$ to be the centroid of the negatively oriented equilateral triangle over $\overline{ab}$, which would correspond to choosing the other root $q = \frac{1}{2} - \frac{i\sqrt{3}}{6}$.

A more general example is obtained by taking an Abelian group $(Q, +)$ with an automorphism φ such that $3\varphi^2(x) - 3\varphi(x) + x = 0$, for all $x \in Q$, and defining a new binary operation by $a \cdot b = a + \varphi(b - a)$. This operation is obviously idempotent, medial and uniquely right solvable. The equation for φ can be written as $\mathbf{1}_Q = 3\varphi \circ (\mathbf{1}_Q - \varphi)$. Hence, $\mathbf{1}_Q - \varphi$ is a bijection and the operation is uniquely left solvable. Identity (1) also follows directly from the equation.

As a consequence of Toyoda's representation theorem [7], this is in fact the most general example of Napoleon's quasigroups.

Theorem 1.2. *For every Napoleon's quasigroup $(Q, \cdot)$ there is an Abelian group $(Q, +)$ with an automorphism φ such that $3\varphi^2 - 3\varphi + \mathbf{1}_Q = 0$ and $a \cdot b = a + \varphi(b - a)$, for all $a, b \in Q$.*

Proof. According to a special version of Toyoda's theorem for idempotent medial quasigroups, there is a commutative group $(Q, +)$ with automorphism φ such that $a \cdot b = a + \varphi(b - a)$. Identity (1) is equivalent with the property $3\varphi^2 - 3\varphi + \mathbf{1}_Q = 0$, which is easily verified by direct computation. $\square$

This theorem completely describes the structure of Napoleon's quasigroups and reduces them to the study of Abelian groups with a special type of automorphism. Our main motivation is to use these quasigroups as a language for proving Napoleon's theorem and some related theorems of plane geometry. It turns out that all necessary ingredients are encoded in the properties of a single binary operation, most importantly identity (1).

In the next section, geometric concepts such as equilateral triangles and midpoints are defined in Napoleon's quasigroups. We fall back to Toyoda's representation theorem to prove some of the more technical results in this section. These results could also be proved directly, by somewhat tedious calculations in the quasigroup.

A surprisingly large number of results related to Napoleon's theorem have been published over the years. A survey up to 1996 can be found in [5], and Napoleon-like theorems have kept appearing since. In the third section we prove Napoleons's theorem and a well-known fact about centroids in the general context of Napoleon's quasigroups. The solution to an old problem by E. Lemoine [4] is provided in this context. Finally, two more recent theorems by B. Grünbaum [1] and F. van Lamoen [3] are stated and proved in Napoleon's quasigroups.

The challenge here lies not so much in the proofs, but in how these results should be formulated in the more general context. As illustrated by Theorem 3.6, a literal translation is not always correct. Once the result is formulated correctly, the proof in the quasigroup context is usually fairly straightforward.

This approach could make geometric theorems accessible to automated theorem provers. We do not pursue it in this work, but the use of automated theorem provers has become quite widespread in quasigroup and loop theory (see section 3 of [6] for a catalogue of results obtained in this way). The method of translating problems in plane geometry to the language of medial quasigroups is due to V. Volenec [8, 9]. Another class of idempotent medial quasigroup related to Napoleon's are the hexagonal quasigroups [10]. As we shall see in the third section, a special hexagonal quasigroup can be obtained from an arbitrary Napoleon's quasigroup by the formula (11).

2. Equilateral triangles and midpoints

We first state an auxiliary lemma.

Lemma 2.1. *In an idempotent medial quasigroup* $(Q,\cdot)$*, identity* (1) *is equivalent with either of the identities*

$$ab\cdot ba = ba\cdot b, \tag{8}$$

$$ab\cdot ca = ba\cdot cb. \tag{9}$$

Proof. Using Toyoda's theorem, the quasigroup can be represented as $a\cdot b = a+\varphi(b-a)$ in an Abelian group $(Q,+)$ with automorphism φ. The identities (1), (8) and (9) are seen to be equivalent with $3\varphi^2 - 3\varphi + \mathbf{1}_Q = 0$. □

The following observation will turn out as an algebraic statement of Napoleon's theorem.

Corollary 2.2. *If* $(Q,\cdot)$ *is a Napoleon's quasigroup and* $a,b,c\in Q$*, then*

$$ab\cdot ca = ac\cdot ba = ba\cdot cb = bc\cdot ab = ca\cdot bc = cb\cdot ac. \tag{10}$$

Proof. Follows from (9) by using mediality (3). □

Let $(Q,\cdot)$ be a Napoleon's quasigroup. By a *triangle* we mean an ordered triple of points $(a,b,c)\in Q^3$. Using the binary operation we can define *equilateral triangles.*

Definition 2.3. *The triangle* (a,b,c) *is called* left equilateral *if* $ab = bc = ca$ *holds. This is denoted by* $\Delta(a,b,c)$ *or* $\Delta_o(a,b,c)$*, where* $o = ab = bc = ca$ *is the* centroid. *Similarly,* (a,b,c) *is called* right equilateral *if* $ba = cb = ac = o$ *holds. This is denoted by* $\nabla(a,b,c)$ *or* $\nabla_o(a,b,c)$*.*

Positive and negative orientation cannot be distinguished in this abstract setting. In the quasigroup $(\mathbb{C},\cdot)$ defined by $a\cdot b = (1-q)a+qb$ for $q=\frac{1}{2}+\frac{i\sqrt{3}}{6}$, left equilateral triangles are positively oriented and right equilateral triangles are negatively oriented, and vice versa for $q=\frac{1}{2}-\frac{i\sqrt{3}}{6}$. Here are some properties of the ternary relations Δ and ∇.

Proposition 2.4. *The statements* $\Delta_o(a,b,c)$*,* $\Delta_o(b,c,a)$*,* $\Delta_o(c,a,b)$*,* $\nabla_o(a,c,b)$*,* $\nabla_o(c,b,a)$ *and* $\nabla_o(b,a,c)$ *are equivalent.*

Proof. Obvious from the definition. □

Because of this equivalence, the next two propositions and some other results in the sequel are stated only for left equilateral triangles. Analogous results hold for right equilateral triangles.

Proposition 2.5. *If* $ab = bc = o$*, then* $ca = o$ *and* $\Delta_o(a,b,c)$ *holds.*

Proof. If $bc = o$, then $c = b \setminus o$. We have $ca \cdot o = (b \setminus o)a \cdot o \overset{(7)}{=} (ba \cdot o) \setminus (oa \cdot o) = (ba \cdot ab) \setminus (oa \cdot o) \overset{(8)}{=} (ab \cdot a) \setminus (oa \cdot o) = oa \setminus (oa \cdot o) = o \overset{(2)}{=} oo$. By canceling o from the right we get $ca = o$. □

Proposition 2.6. *For all $a, b \in Q$ there is a unique $c \in Q$ such that $\Delta(a, b, c)$ holds.*

Proof. Denote $o = ab$ and $c = b \setminus o$. Then, $ab = bc = o$ and, according to Proposition 2.5, $\Delta_o(a, b, c)$ holds. From $\Delta(a, b, c)$ we see that $c = b \setminus ab$, so c is uniquely determined by a and b. □

Equilateral triangles can also be defined in hexagonal quasigroups [11]. However, in that context centroids of equilateral triangles cannot be expressed explicitly, making them less suitable for proving Napoleon-like theorems. In [8], midpoints were defined in arbitrary medial quasigroups by using parallelograms. Because of [9, Theorem 12], this is equivalent with the following more direct definition in idempotent medial quasigroups.

Definition 2.7. *Let $(Q, \cdot)$ be an idempotent medial quasigroup. The point $m \in Q$ is the* midpoint *of the pair of points $(a, b) \in Q^2$, denoted by $M(a, m, b)$, if $am \cdot mb = ab$ holds.*

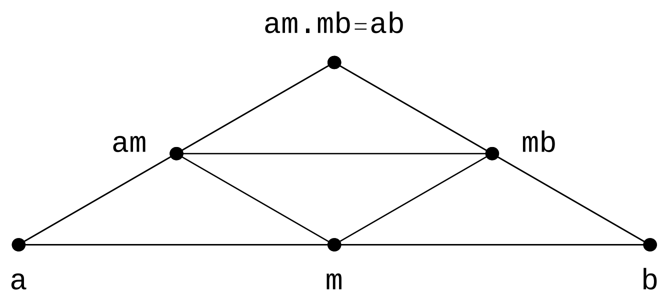

FIGURE 2. The midpoint relation $M(a, m, b)$.

Here is a characterization, making symmetry of the midpoint relation in a and b apparent.

Proposition 2.8. *In an idempotent medial quasigroup $(Q, \cdot)$, $M(a, m, b)$ is equivalent with $ma \cdot bm = m$.*

Proof. Using Toyoda's representation theorem, both of the equations $am \cdot mb = ab$ and $ma \cdot bm = m$ are easily seen to be equivalent with $a + b = 2m$ in the underlying Abelian group $(Q, +)$. □

Corollary 2.9. *In an idempotent medial quasigroup, $M(a,m,b)$ holds if and only if $M(b,m,a)$ holds.*

Proof. Follows from Proposition 2.8 and mediality (3). □

Generally, it is not true that each pair of points has a unique midpoint. To prove this, we need division by 2 in the underlying Abelian group. Our first example of Napoleon's quasigroup constructed from the complex numbers has this property. However, if an example is defined in the same way from a field of characteristic 2, then every $m \in Q$ is the midpoint of the pairs (a,a), while pairs (a,b) with $a \neq b$ do not possess midpoints. However, if two sides (a,b) and (a,c) of a triangle (a,b,c) in a medial quasigroup possess midpoints, then the third side (b,c) also has a midpoint [8, Theorem 40].

3. Napoleon's theorem and its relatives

The last few claims of the previous section hold in general idempotent medial quasigroups. Now we turn back to Napoleon's quasigroups.

Theorem 3.1 (Napoleon's theorem)**.** *Let (a,b,c) be an arbitrary triangle in a Napoleon's quasigroup $(Q,\cdot)$. Then, $\nabla_o(ab,bc,ca)$ and $\Delta_o(ba,cb,ac)$ hold for some $o \in Q$.*

Proof. This is a direct consequence of Definition 2.3 and Corollary 2.2. □

It is known that the centroid of a triangle in the Euclidean plane coincides with the centroid of its Napoleon triangles. This motivates the following definition.

Definition 3.2. *The* centroid *of an arbitrary triangle $(a,b,c) \in Q^3$ in a Napoleon's quasigroup $(Q,\cdot)$ is the point $C(a,b,c) = ab \cdot ca$.*

The point o in Napoleons's theorem is precisely $C(a,b,c)$. Corollary 2.2 implies that $C(a,b,c) = C(d,e,f)$ for any permutation (d,e,f) of (a,b,c). Of course, if $\Delta_o(a,b,c)$ or $\nabla_o(a,b,c)$, then $C(a,b,c) = o$. Furthermore, Proposition 2.8 can now be reinterpreted as

$$M(a,m,b) \iff C(m,a,b) = m.$$

Proposition 3.3. *Let $(a,b,c) \in Q^3$ be a triangle in a Napoleon's quasigroup and suppose $m \in Q$ is the midpoint of (a,b), i.e. $M(a,m,b)$ holds. Then, $C(a,b,c) = C(m,m,c)$.*

Proof. According to Proposition 2.8 and Definition 3.2, $ma \cdot bm = m$, $C(a,b,c) = ab \cdot ca$ and $C(m,m,c) = mm \cdot cm \stackrel{(2)}{=} m \cdot cm$. Therefore,

$(m\cdot cm)(m\cdot cm) \overset{(2)}{=} m\cdot cm \overset{(4)}{=} mc\cdot m = mc\cdot(ma\cdot bm) \overset{(5)}{=} (mc\cdot ma)(mc\cdot bm) \overset{(5),(10)}{=} (m\cdot ca)(cm\cdot bc) \overset{(3)}{=} (m\cdot cm)(ca\cdot bc) \overset{(10)}{=} (m\cdot cm)(ab\cdot ca)$. By cancelling $m\cdot cm$ from the left we get $m\cdot cm = ab\cdot ca$, i.e. $C(m,m,c) = C(a,b,c)$. □

In the Napoleon's quasigroup defined from the Euclidean plane, $C(m,m,c)$ is the point dividing the segment $\overline{mc}$ in the ratio $1:2$. Therefore, the previous proposition can be interpreted as the well-known fact that the centroid trisects each median.

In 1868, E. Lemoine [4] posed the problem to construct vertices a, b, c of a triangle, given the vertices a_1, b_1, c_1 of the equilateral triangles erected over its sides. A solution was published by L. Kiepert [2] in the following year. Kiepert's solution is to erect equilateral triangles with vertices a_2, b_2, c_2 over the sides of the given triangle and to construct midpoints of (a_1,a_2), (b_1,b_2) and (c_1,c_2). Here is a more precise statement in the setting of Napoleon's quasigroups.

Theorem 3.4. *Let $(a,b,c)\in Q^3$ be a triangle in a Napoleon's quasigroup and denote by $a_1,b_1,c_1\in Q$ the unique points such that $\Delta(a_1,b,c)$, $\Delta(a,b_1,c)$ and $\Delta(a,b,c_1)$ hold. Furthermore, let $a_2,b_2,c_2\in Q$ be the unique points such that $\Delta(a_2,b_1,c_1)$, $\Delta(a_1,b_2,c_1)$ and $\Delta(a_1,b_1,c_2)$ hold. Then, $M(a_1,a,a_2)$, $M(b_1,b,b_2)$ and $M(c_1,c,c_2)$.*

To make the proof of this and the next theorem shorter, we introduce a new binary operation $*$ in a Napoleon's quasigroup $(Q,\cdot)$. Given $a,b\in Q$, denote by $a*b$ the unique point c such that $\Delta(a,b,c)$ holds (see Proposition 2.6). Thus,

$$a*b = b\setminus ab = (b\setminus a)b. \tag{11}$$

This new operation is obviously idempotent. Because it is defined by a formula involving multiplication and left division, it is mutually medial with the old operation [12]:

$$(a*b)\cdot(c*d) = ac*bd. \tag{12}$$

As a consequence, the new operation is medial itself and is left and right distributive over the old operation:

$$a*bc = (a*b)(a*c), \tag{13}$$

$$ab*c = (a*c)(b*c), \tag{14}$$

$$a(b*c) = ab*ac, \tag{15}$$

$$(a*b)c = ac*bc. \tag{16}$$

The following properties of the new operation will also be useful.

Lemma 3.5. *Let $(Q,\cdot)$ be a Napoleon's quasigroup and define $a*b = b\setminus ab$. Then, for any $a,b,c\in Q$,*

$$(a*b)a = b(a*b) = ab \tag{17}$$

and

$$ab*ca = bc. \tag{18}$$

Proof. From the definition of $*$ we have $\Delta(a,b,a*b)$. By Definition 2.3, this means $ab = b(a*b) = (a*b)a$. By Napoleon's theorem and Proposition 2.4, we have $\Delta(ab,ca,bc)$ and hence $bc = ab*ca$. $\square$

As a consequence of identity (17), $(Q,*)$ is in fact a hexagonal quasigroup. Dividing $ab = (a*b)a$ by a from the left we get $b = a\setminus(a*b)a \overset{(11)}{=} (a*b)*a$. This is the defining identity for hexagonal quasigroups (see [10]). Now we can also prove Theorem 3.4.

Proof of Theorem 3.4. By the definition of a_1,b_1,c_1 and a_2,b_2,c_2, we have

$$a_1 = b*c,\quad b_1 = c*a,\quad c_1 = a*b \tag{19}$$

and

$$a_2 = b_1*c_1,\quad b_2 = c_1*a_1,\quad c_2 = a_1*b_1. \tag{20}$$

Multiplying equations (19) by b,c,a respectively and using (17), we get

$$a_1b = bc,\quad b_1c = ca,\quad c_1a = ab. \tag{21}$$

Now we prove $M(a_1,a,a_2)$ by using Proposition 2.8: $aa_1\cdot a_2a \overset{(20)}{=} aa_1\cdot(b_1*c_1)a \overset{(16)}{=} aa_1\cdot(b_1a*c_1a) \overset{(21)}{=} aa_1\cdot(b_1a*ab) \overset{(15)}{=} (aa_1\cdot b_1a)*(aa_1\cdot ab) \overset{(3),(5)}{=} (ab_1\cdot a_1a)*(a\cdot a_1b) \overset{(19),(21)}{=} (a(c*a)\cdot a_1a)*(a\cdot bc) \overset{(17)}{=} (ca\cdot a_1a)*(a\cdot bc) \overset{(6)}{=} (ca_1\cdot a)*(a\cdot bc) \overset{(19)}{=} (c(b*c)\cdot a)*(a\cdot bc) \overset{(17)}{=} (bc\cdot a)*(a\cdot bc) \overset{(18)}{=} aa \overset{(2)}{=} a$. The relations $M(b_1,b,b_2)$ and $M(c_1,c,c_2)$ are proved similarly. $\square$

Branko Grünbaum [1] discovered another Napoleon-like theorem involving midpoints. We state and prove it here in the context of Napoleon's quasigroups:

Theorem 3.6. *Let $(a,b,c)\in Q^3$ be a triangle in a Napoleon's quasigroup and denote by $a',b',c'\in Q$ the unique points such that $\nabla(a',b,c)$, $\nabla(a,b',c)$ and $\nabla(a,b,c')$ hold. Suppose there are points $a_1,b_1,c_1\in Q$ such that $\Delta(a,b_1,c_1)$, $\Delta(a_1,b,c_1)$ and $\Delta(a_1,b_1,c)$ hold; denote the centroids of these three left equilateral triangles by a_2, b_2 and c_2. Then, $M(b',a_1,c')$, $M(a',b_1,c')$, $M(a',c_1,b')$ and $\nabla(a_2,b_2,c_2)$ hold.*

Proof. By the assumptions of the theorem and Proposition 2.4, we have

$$a' = c * b, \quad b' = a * c, \quad c' = b * a \tag{22}$$

and

$$a = b_1 * c_1, \quad b = c_1 * a_1, \quad c = a_1 * b_1. \tag{23}$$

The relation $M(b', a_1, c')$ follows from Proposition 2.8: $a_1c'{\cdot}b'a_1 \overset{(22)}{=} a_1(b*a){\cdot}(a*c)a_1 \overset{(15),(16)}{=} (a_1b*a_1a){\cdot}(aa_1*ca_1) \overset{(12)}{=} (a_1b{\cdot}aa_1)*(a_1a{\cdot}ca_1) \overset{(23)}{=} (a_1(c_1*a_1) \cdot aa_1) * (a_1a \cdot (a_1 * b_1)a_1) \overset{(17)}{=} (c_1a_1 \cdot aa_1) * (a_1a \cdot a_1b_1) \overset{(5),(6)}{=} (c_1a \cdot a_1) * (a_1 \cdot ab_1) \overset{(23)}{=} (c_1(b_1 * c_1) \cdot a_1) * (a_1 \cdot (b_1 * c_1)b_1) \overset{(17)}{=} (b_1c_1 \cdot a_1) * (a_1 \cdot b_1c_1) \overset{(18)}{=} a_1a_1 \overset{(2)}{=} a_1$. Proofs of the relations $M(a', b_1, c')$ and $M(a', c_1, b')$ are similar.

By Definition 2.3, the centroids of $\Delta(a, b_1, c_1)$, $\Delta(a_1, b, c_1)$ and $\Delta(a_1, b_1, c)$ are $a_2 = b_1c_1$, $b_2 = c_1a_1$ and $c_2 = a_1b_1$. According to Propositions 2.4 and 2.5, to prove $\nabla(a_2, b_2, c_2)$ it suffices to show $b_2a_2 = c_2b_2$, i.e. $c_1a_1 \cdot b_1c_1 = a_1b_1 \cdot c_1a_1$. This follows directly from Corollary 2.2. □

The preceding theorem is actually a kind of converse of Grünbaum's original theorem [1]. Grünbaum assumed a_1, b_1, c_1 to be the midpoints of (b', c'), (a', c'), (a', b') and proved that (a, b_1, c_1), (a_1, b, c_1) and (a_1, b_1, c) are equilateral triangles. This is not true in general Napoleon's quasigroups. It may happen that a', b', c' coincide and then any $a_1, b_1, c_1 \in Q$ would be midpoints in a Napoleon's quasigroup constructed from a field of characteristic 2. However, given a and b_1, there is only one c_1 such that $\Delta(a, b_1, c_1)$ holds.

Floor van Lamoen [3] proved a generalization of Napoleon's theorem. Here is a slightly modified version in our setting. Napoleon's theorem is the special case $(a_1, b_1, c_1) = (c_2, a_2, b_2)$.

Theorem 3.7. *Let $\Delta(a_1, a_2, a_3)$, $\Delta(b_1, b_2, b_3)$ and $\Delta(c_1, c_2, c_3)$ be equilateral triangles in a Napoleon's quasigroup $(Q, \cdot)$. Denote by $z_i = C(a_i, b_i, c_i)$, $i = 1, 2, 3$, and $d_1 = C(a_1, b_2, c_3)$, $e_1 = C(a_2, b_3, c_1)$, $f_1 = C(a_3, b_1, c_2)$, $d_2 = C(a_1, b_3, c_2)$, $e_2 = C(a_2, b_1, c_3)$, $f_2 = C(a_3, b_2, c_1)$. Then, $\Delta_o(z_1, z_2, z_3)$, $\Delta_o(d_1, e_1, f_1)$ and $\Delta_o(d_2, e_2, f_2)$ hold for some $o \in Q$.*

Proof. Since the three triangles are left equilateral, $a_1a_2 = a_2a_3 = a_3a_1$, $b_1b_2 = b_2b_3 = b_3b_1$ and $c_1c_2 = c_2c_3 = c_3c_1$. Denote $o = (a_1a_2 \cdot b_1b_2)(c_1c_2 \cdot a_1a_2)$. Because of Proposition 2.5, it suffices to show that $o = z_1z_2 = z_2z_3 = d_1e_1 = e_1f_1 = d_2e_2 = e_2f_2$. This follows by repeated application of mediality, e.g.

$$\begin{aligned} z_1z_2 &= (a_1b_1 \cdot c_1a_1)(a_2b_2 \cdot c_2a_2) \stackrel{(3)}{=} (a_1b_1 \cdot a_2b_2)(c_1a_1 \cdot c_2a_2) \stackrel{(3)}{=} \\ &= (a_1a_2 \cdot b_1b_2)(c_1c_2 \cdot a_1a_2) = o, \end{aligned}$$

$$\begin{aligned} d_1e_1 &= (a_1b_2 \cdot c_3a_1)(a_2b_3 \cdot c_1a_2) \stackrel{(3)}{=} (a_1b_2 \cdot a_2b_3)(c_3a_1 \cdot c_1a_2) \stackrel{(3)}{=} \\ &= (a_1a_2 \cdot b_2b_3)(c_3c_1 \cdot a_1a_2) = (a_1a_2 \cdot b_1b_2)(c_1c_2 \cdot a_1a_2) = o, \end{aligned}$$

$$\begin{aligned} d_2e_2 &= (a_1b_3 \cdot c_2a_1)(a_2b_1 \cdot c_3a_2) \stackrel{(3)}{=} (a_1b_3 \cdot a_2b_1)(c_2a_1 \cdot c_3a_2) \stackrel{(3)}{=} \\ &= (a_1a_2 \cdot b_3b_1)(c_2c_3 \cdot a_1a_2) = (a_1a_2 \cdot b_1b_2)(c_1c_2 \cdot a_1a_2) = o. \end{aligned}$$

The proofs of $z_2z_3 = e_1f_1 = e_2f_2 = o$ are analogous. □

The duals of Theorems 3.4, 3.6 and 3.7, obtained by exchanging Δ with ∇, are also true.

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