A NEW GENERALIZATION OF THE GOLDEN RATIO

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ABSTRACT

We propose a generalization of the golden section based on division in mean and extreme ratio. The associated integer sequences have many interesting properties.

1 GENERALIZED GOLDEN RATIOS

There have been many generalizations of the number known as *golden ratio* or *golden section*, $\phi = \frac{1+\sqrt{5}}{2}$. Examples are G.A. Moore's golden numbers [10] and S. Bradley's nearly golden sections [5] (see also [7] and [9]). A generalization that has been considered by several authors are the positive roots of $x^{k+1} - x^k - 1 = 0$; see [12] and [14]. In this paper, a similar generalization is proposed. It is based on the original definition of ϕ, division of a line segment in mean and extreme ratio.

Let G be a point dividing the segment $\overline{AB}$ in parts of length $a = |AG|$ and $b = |GB|$; suppose $a > b$. The division is *mean and extreme* if the ratio of the larger to the smaller part equals the ratio of the whole segment to the larger part:

$$\frac{a}{b} = \frac{a+b}{a}.$$

Given a positive integer k, we consider divisions satisfying

$$\left(\frac{a}{b}\right)^k = \frac{a+b}{a}.$$

For $k > 1$, we have not one but two ratios: $\varphi_k = \frac{a}{b}$ and $\phi_k = \frac{a+b}{a} = 1 + \frac{1}{\varphi_k}$. These numbers will be called the k-th *lower* and *upper golden ratio*,

k	φ_k	ϕ_k
1	1.6180339887	1.6180339887
2	1.3247179572	1.7548776662
3	1.2207440846	1.8191725134
4	1.1673039783	1.8566748839
5	1.1347241384	1.8812714616

Table 1: Lower and upper golden ratios.

respectively. Obviously, $(\varphi_k)^k = \phi_k$. It is also evident that φ_k is a root of the polynomial $p_k(x) = x^{k+1} - x - 1$ and ϕ_k is a root of the polynomial $P_k(x) = x(x-1)^k - 1$.

Proposition 1.1 *For every positive integer k, the polynomials $p_k(x)$ and $P_k(x)$ have a unique positive root. If k is even this is the only real root, and if k is odd the polynomials have another negative root.*

Proof. The equation $p_k(x) = 0$ can be rewritten as $x^k - 1 = \frac{1}{x}$. Thus, real roots correspond to intersections of the hyperbola $y = \frac{1}{x}$ and the graph of the power function translated one unit downwards, $y = x^k - 1$. Similarly, real roots of P_k correspond to intersections of the hyperbola and the graph of the power function translated one unit to the right, $y = (x-1)^k$. The claims follow from elementary properties of the functions involved. $\square$

Therefore, φ_k is the unique positive root of p_k and ϕ_k is the unique positive root of P_k. The only instance when φ_k and ϕ_k coincide is $k = 1$, when both are equal to the ordinary golden ratio ϕ. The second lower golden ratio φ_2 has been called *plastic number* by the Benedictine monk and architect Dom Hans van der Laan [1]. This is the smallest Pisot-Vijayaraghavan number (see [4]). Its square, ϕ_2, is also a cubic Pisot-Vijayaraghavan number. In Table 1, we list decimal approximations to the first five lower and upper golden ratios. As k grows, the lower golden ratios tend to 1 and the upper golden ratios tend to 2.

Proposition 1.2 $\lim_{k\to\infty} \varphi_k = 1, \ \lim_{k\to\infty} \phi_k = 2.$

Proof. By direct computation, p_k is strictly increasing on $\left[1, \sqrt[k+1]{3}\,\right]$, attains a negative value at $x = 1$ and a positive value at $x = \sqrt[k+1]{3}$. Hence, p_k has a unique zero in this interval, i.e. $\varphi_k \in \left(1, \sqrt[k+1]{3}\right)$. The proposition follows from $\lim\limits_{k\to\infty} \sqrt[k+1]{3} = 1$ and $\phi_k = 1 + \frac{1}{\varphi_k}$. $\square$

2 ASSOCIATED INTEGER SEQUENCES

The connection between the golden ratio and Fibonacci numbers is well known. We can define integer sequences associated with the generalized golden ratios in a similar manner. The k-th *lower Fibonacci sequence* $f_n^{(k)}$ is defined by $f_1^{(k)} = f_2^{(k)} = \ldots = f_{k+1}^{(k)} = 1$ and the linear recurrence with characteristic polynomial p_k:

$$f_n^{(k)} = f_{n-k}^{(k)} + f_{n-k-1}^{(k)}.$$

The k-th *upper Fibonacci sequence* $F_n^{(k)}$ satisfies the same initial conditions and the linear recurrence with characteristic polynomial P_k. By the binomial theorem, we get

$$F_n^{(k)} = \sum_{i=1}^{k} \binom{k}{i} (-1)^{i+1} F_{n-i}^{(k)} + F_{n-k-1}^{(k)}.$$

Of course, both $f_n^{(1)}$ and $F_n^{(1)}$ are just the Fibonacci numbers. The second lower Fibonacci sequence has been called the *Padovan sequence* in [13]:

$$(f_n^{(2)}) = (1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, \ldots).$$

This is sequence number A000931 in N. Sloane's *Encyclopedia of Integer Sequences* [11]. Another interesting sequence satisfying the same recurrence with different initial conditions is the Perrin sequence (Sloane's A001608), giving a necessary condition for primality [2]. The second upper Fibonacci sequence is Sloane's A005251:

$$(F_n^{(2)}) = (1, 1, 1, 2, 4, 7, 12, 21, 37, 65, 114, 200, 351, 616, 1081, \ldots).$$

Among other combinatorial interpretations, $F_n^{(2)}$ is the number of compositions of n without 2's [6] and the number of binary strings of length $n-3$ without isolated ones [3]. Notice that $F_{n+1}^{(2)} = f_{2n-1}^{(2)}$.

The third lower Fibonacci sequence is listed in [11] as A079398:

$$(f_n^{(3)}) = (1, 1, 1, 1, 2, 2, 2, 3, 4, 4, 5, 7, 8, 9, 12, 15, 17, 21, 27, 32, \ldots).$$

Upper Fibonacci sequences are currently listed up to $k = 5$. Here are the first few values of $F_n^{(3)}$, Sloane's A003522:

$$(F_n^{(3)}) = (1, 1, 1, 1, 2, 5, 11, 21, 37, 64, 113, 205, 377, 693, 1266, \ldots).$$

De Villiers [14] considered sequences defined by the recurrence $L_n^{(k)} = L_{n-1}^{(k)} + L_{n-k-1}^{(k)}$. When equipped with Fibonacci-like initial conditions, $L_1^{(k)} = \ldots = L_{k+1}^{(k)} = 1$, these are the *Lamé sequences of higher order* (according to [11]). De Villiers gave a partial proof that ratios of consecutive members tend to the positive root of $x^{k+1} - x^k - 1 = 0$, generalizing a famous property of the Fibonacci numbers. The proof was later completed by S. Falcon [8]. Not surprisingly, ratios of consecutive members of the lower and upper Fibonacci sequences tend to the corresponding golden ratios.

Theorem 2.1 $\lim_{n\to\infty} \dfrac{f_{n+1}^{(k)}}{f_n^{(k)}} = \varphi_k, \quad \lim_{n\to\infty} \dfrac{F_{n+1}^{(k)}}{F_n^{(k)}} = \phi_k.$

Proof. The polynomials p_k, P_k and their derivatives are relatively prime. Therefore, p_k and P_k have $k+1$ distinct complex roots each and formulae for the corresponding integer sequences are of the form $a_n = C_0 z_0^n + \ldots + C_k z_k^n$. Here, $z_0, \ldots, z_k$ are the roots of p_k or P_k and $C_0, \ldots, C_k$ are constants. The quotient of two consecutive sequence members can be expressed as

$$\frac{a_{n+1}}{a_n} = \frac{C_0 z_0^{n+1} + \ldots + C_k z_k^{n+1}}{C_0 z_0^n + \ldots + C_k z_k^n} = \frac{C_0 z_0 + C_1 z_1 \left(\dfrac{z_1}{z_0}\right)^n + \ldots + C_k z_k \left(\dfrac{z_k}{z_0}\right)^n}{C_0 + C_1 \left(\dfrac{z_1}{z_0}\right)^n + \ldots + C_k \left(\dfrac{z_k}{z_0}\right)^n}.$$

Suppose $|z_0| > |z_i|$ for $i = 1, \ldots, k$. Then, $\left(\frac{z_i}{z_0}\right)^n \to 0$ as $n \to \infty$ and $\frac{a_{n+1}}{a_n} \to z_0$, provided $C_0 \neq 0$. Thus, it remains to be shown that the coefficients $C_0, \ldots, C_k$ are not zero and φ_k, ϕ_k are greater than the absolute values of the remaining roots of p_k and P_k.

The coefficients $C_0, \ldots, C_k$ satisfy the system of linear equations

$$\begin{bmatrix} z_0 & z_1 & \cdots & z_k \\ z_0^2 & z_1^2 & \cdots & z_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_0^{k+1} & z_1^{k+1} & \cdots & z_k^{k+1} \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Let A be the square matrix on the left. By Cramer's rule we have

$$C_i = \frac{1}{\det A} \begin{vmatrix} z_0 & \cdots & 1 & \cdots & z_k \\ z_0^2 & \cdots & 1 & \cdots & z_k^2 \\ \vdots & & \vdots & & \vdots \\ z_0^{k+1} & \cdots & 1 & \cdots & z_k^{k+1} \end{vmatrix}.$$

The Vandermonde determinant in the numerator is not zero because the roots are all distinct and 1 is neither a root of p_k nor of P_k.

Finally, let $z = x + i\,y \neq \varphi_k$ be a root of p_k and denote its absolute value by $r = |z| = \sqrt{x^2 + y^2}$. By Proposition 1.1, z is either the unique negative root (for odd k) or else $y \neq 0$; in both cases $x < r$. Taking the absolute value of $p_k(z) = 0$ we have:

$$|z|^{k+1} = |z+1| = \sqrt{(x+1)^2 + y^2} < \sqrt{x^2 + y^2 + 2r + 1}.$$

Equivalently, $r^{k+1} < \sqrt{r^2 + 2r + 1} = r + 1$, i.e. $p_k(r) < 0$. The polynomial p_k is strictly increasing on $[1, +\infty)$ and $p_k(\varphi_k) = 0$. Therefore, $p_k(x) > 0$ for all $x > \varphi_k$ and we conclude $r < \varphi_k$. Similarly, if $z = x + i\,y \neq \phi_k$ is a root of P_k, we get

$$1 = |z| \cdot |z-1|^k = r\sqrt{r^2 - 2x + 1}^{\,k} > r(r-1)^k \implies P_k(r) < 0.$$

Again, $P_k(x) > 0$ for all $x > \phi_k$ and $r < \phi_k$ follows. This completes the proof. $\square$

Corollary 2.2 $\lim\limits_{n\to\infty} \dfrac{f_{n+k}^{(k)}}{f_n^{(k)}} = \phi_k.$

Proof. By the preceding theorem, consecutive ratios of the k-th lower Fibonacci sequences tend to φ_k so we have

$$\frac{f_{n+k}^{(k)}}{f_n^{(k)}} = \frac{f_{n+k}^{(k)}}{f_{n+k-1}^{(k)}} \cdot \frac{f_{n+k-1}^{(k)}}{f_{n+k-2}^{(k)}} \cdots \frac{f_{n+1}^{(k)}}{f_n^{(k)}} \to \varphi_k \cdot \varphi_k \cdots \varphi_k = (\varphi_k)^k = \phi_k.$$

$\square$

Just like ordinary Fibonacci numbers, their upper "cousins" can be expressed as sums of binomial coefficients. We will need the following lemma.

Lemma 2.3 *For any $k \le l \le m$,* $\displaystyle\sum_{j=0}^{k}(-1)^j \binom{k}{j}\binom{m-j}{l} = \binom{m-k}{l-k}.$

Proof. Let M be a set of m elements and suppose a subset of k elements is given. The right side enumerates all l-element subsets of M containing the given k elements. On the other hand, $\binom{k}{j}\binom{m-j}{l}$ is the number of l-subsets avoiding at least j of the k given elements. The sum on the left equals the binomial coefficient on the right by inclusion-exclusion. □

Proposition 2.4 $F^{(k)}_{n+1} = \displaystyle\sum_{i\ge 0}\binom{n-i}{k\,i}$

Proof. Obviously, $\sum_{i\ge 0}\binom{n-i}{k\,i} = 1$ for all $n \le k$. The recurrence for the upper Fibonacci numbers can be rewritten as

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}F^{(k)}_{n+k+1-j} = F^{(k)}_n.$$

By substituting appropriate sums of binomial coefficients we get

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{i\ge 0}\binom{n+k-j-i}{k\,i} = \sum_{i\ge 0}\binom{n-1-i}{k\,i} = \sum_{i\ge 1}\binom{n-i}{k(i-1)}.$$

Equivalently,

$$\sum_{i\ge 1}\left[\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{n+k-i-j}{k\,i} - \binom{n-i}{k(i-1)}\right] = 0.$$

The terms in the square brackets are all zero by Lemma 2.3 for $m = n+k-i$ and $l = k\,i$. Therefore, the considered sums satisfy the the initial conditions and the recurrence for the upper Fibonacci sequence. □

Members of the Lamé sequences can also be expressed as sums of binomial coefficients [11]:

$$L^{(k)}_{n+1} = \sum_{i=0}^{\lfloor n/k\rfloor}\binom{n-k\,i}{i}.$$

It would be of interest to find a similar formula for the lower Fibonacci sequences and to generalize other known properties of Fibonacci numbers.

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