A class of quasigroups associated with a cubic Pisot number

Vedran Krčadinac and Vladimir Volenec

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Abstract

In this paper idempotent medial quasigroups satisfying the identity $(ab \cdot a)a = b$ are studied. An example are the complex numbers with multiplication defined by $a \cdot b = (1-q)a + qb$, where q is a solution of $q^3 - 2q^2 + q - 1 = 0$. The positive root of this cubic equation can be viewed as a generalization of the golden ratio. It turns out that the quasigroups under consideration have many similar properties to the so-called golden section quasigroups.

1 Introduction

Let $q \neq 0, 1$ be a complex number and define a binary operation on $\mathbb{C}$ by $a \cdot b = (1-q)a + qb$. It is known that $(\mathbb{C}, \cdot)$ is an IM-quasigroup, i.e. satisfies the laws of *idempotency* and *mediality*:

$$a \cdot a = a, \tag{1}$$

$$ab \cdot cd = ac \cdot bd. \tag{2}$$

Immediate consequences are the identities known as *elasticity*, *left* and *right distributivity*:

$$ab \cdot a = a \cdot ba, \tag{3}$$

$$a \cdot bc = ab \cdot ac, \tag{4}$$

$$ab \cdot c = ac \cdot bc. \tag{5}$$

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This quasigroup will be denoted by $C(q)$. For some special values of q, the quasigroup satisfies additional identities. If $q = \frac{1+\sqrt{5}}{2}$ is the golden ratio, $C(q)$ is a representative example of the *golden section* or *GS-quasigroups*. GS-quasigroups were defined in [8] as idempotent quasigroups satisfying the (equivalent) identities $a(ab \cdot c) \cdot c = b$, $a \cdot (a \cdot bc)c = b$; see also [2], [3], [4] and [10]. An alternative definition would be as IM-quasigroups with the simpler identity $a(ab \cdot b) = b$. In this paper we study IM-quasigroups satisfying a similar identity:

$$(ab \cdot a)a = b. \tag{6}$$

Representative examples are the quasigroups $C(q)$ with q a root of $q^3 - 2q^2 + q - 1 = 0$. Denote by $r_{1,2} = \sqrt[3]{\frac{25 \pm \sqrt{69}}{2}}$. The roots of this cubic equation are $q_1 = \frac{1}{3}\,(2 + r_1 + r_2) \approx 1.755$ and $q_{2,3} = \frac{1}{6}\left(4 - r_1 - r_2 \pm i\sqrt{3}\,(r_1 - r_2)\right) \approx 0.123 \pm 0.745\,i$. The number q_1 is a Pisot number, i.e. an algebraic integer greater than 1 whose algebraic conjugates $q_{2,3}$ have absolute values less than 1. This number was considered in [5] as a generalization of the golden ratio and was called the *second upper golden ratio*. Therefore, we will refer to IM-quasigroups satisfying the identity (6) as *G_2-quasigroups*.

In the context of [5], the *second lower golden ratio* was the positive root of $p^3 - p - 1 = 0$. This is the smallest Pisot number $p_1 \approx 1.325$; note that $q_1 = p_1^2$. For more details about Pisot numbers see [1].

In this paper it is shown that G_2-quasigroups have many properties similar to those of GS-quasigroups. For example, they allow a simple definition of parallelograms using an explicit formula for the fourth vertex. In the last section G_2-quasigroups are characterized in terms of Abelian groups with a certain type of automorphism.

2 Basic properties and further identities

The following lemma will be used quite often.

Lemma 2.1. *In an IM-quasigroup, identity (6) is equivalent with either of the identities*

$$(a \cdot ba)a = b, \tag{7}$$

$$a(ba \cdot a) = b. \tag{8}$$

Proof. By using elasticity we get $(ab \cdot a)a \overset{(3)}{=} (a \cdot ba)a \overset{(3)}{=} a(ba \cdot a)$. □

Note that the equivalence holds even in a groupoid satisfying (1) and (2). Elasticity follows directly from idempotency and mediality, without using solvability or cancellativity. Consequently, the definition of G_2-quasigroups can be relaxed to the identities alone.

Proposition 2.2. *Any groupoid satisfying (1), (2) and (6) is necessarily a quasigroup.*

Proof. Given a and b define $x = ab \cdot a$ and $y = ba \cdot a$. From (6) and (8) we see that $xa = b$ and $ay = b$, i.e. the groupoid is left and right solvable. Now assume $ax_1 = ax_2$ and $y_1 a = y_2 a$. Then, $x_1 \stackrel{(6)}{=} (ax_1 \cdot a)a = (ax_2 \cdot a)a \stackrel{(6)}{=} x_2$ and $y_1 \stackrel{(8)}{=} a(y_1 a \cdot a) = a(y_2 a \cdot a) \stackrel{(8)}{=} y_2$, so the groupoid is left and right cancellative. □

The next proposition is similar to [8, Theorem 5].

Proposition 2.3. *In a G_2-quasigroup, any two of the equalities $ab = c$, $ca = d$ and $da = b$ imply the third.*

Proof. Denote the equalities by (i), (ii) and (iii), respectively. Then we have:

$$\begin{aligned} (i), (ii) \Rightarrow (iii): &\quad da \stackrel{(ii)}{=} ca \cdot a \stackrel{(i)}{=} (ab \cdot a)a \stackrel{(6)}{=} b, \\ (i), (iii) \Rightarrow (ii): &\quad ca \stackrel{(i)}{=} ab \cdot a \stackrel{(iii)}{=} (a \cdot da)a \stackrel{(7)}{=} d, \\ (ii), (iii) \Rightarrow (i): &\quad ab \stackrel{(iii)}{=} a \cdot da \stackrel{(ii)}{=} a(ca \cdot a) \stackrel{(8)}{=} c. \end{aligned}$$

□

We list some more identities valid in G_2-quasigroups. They are accompanied by pictures illustrating the example of the complex plane with multiplication defined by $a \cdot b = (1 - q_1)a + q_1 b$.

Proposition 2.4. *The following identity holds in any G_2-quasigroup:*

$$(a \cdot ab)c \cdot a = ac \cdot b. \tag{9}$$

Proof. $(a \cdot ab)c \cdot a \stackrel{(5)}{=} (a \cdot ab)a \cdot ca \stackrel{(5)}{=} (a \cdot ab)(ca) \cdot (a \cdot ca) \stackrel{(3)}{=} (a \cdot ab)(ca) \cdot (ac \cdot a) \stackrel{(2)}{=} (ac)(ab \cdot a) \cdot (ac \cdot a) \stackrel{(4)}{=} ac \cdot (ab \cdot a)a \stackrel{(6)}{=} ac \cdot b.$ □

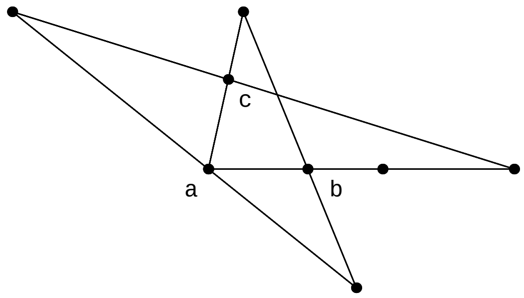

Figure 1: Identity (9) in the complex plane.

Proposition 2.5. *The following identity holds in any G_2-quasigroup:*

$$(ab \cdot a)c \cdot b = (ab \cdot c)a. \tag{10}$$

Proof. $(ab{\cdot}a)c{\cdot}b \overset{(5)}{=} (ab{\cdot}b)(ab){\cdot}cb \overset{(3)}{=} (ab)(b{\cdot}ab){\cdot}cb \overset{(2)}{=} (ab{\cdot}c){\cdot}(b{\cdot}ab)b \overset{(7)}{=} (ab{\cdot}c)a.$ □

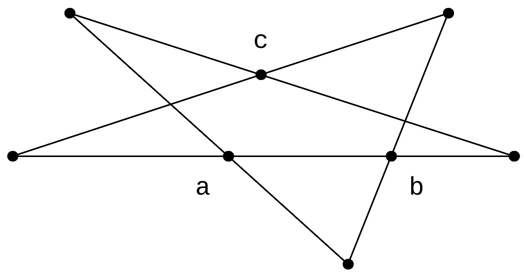

Figure 2: Identity (10) in the complex plane.

Proposition 2.6. *The following identity holds in any G_2-quasigroup:*

$$a \cdot (ba \cdot c)d = b(ac \cdot d). \tag{11}$$

Proof. $a \cdot (ba \cdot c)d \overset{(5)}{=} a \cdot (ba \cdot d)(cd) \overset{(4)}{=} (a \cdot ba)(ad) \cdot (a \cdot cd) \overset{(2)}{=} (a \cdot ba)a \cdot (ad \cdot cd) \overset{(7)}{=} b(ad \cdot cd) \overset{(5)}{=} b(ac \cdot d).$ □

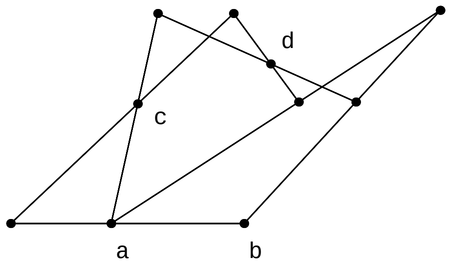

Figure 3: Identity (11) in the complex plane.

3 Parallelograms and other geometric concepts

The points a, b, c, d of a medial quasigroup are said to form a *parallelogram*, denoted by $\mathrm{Par}(a,b,c,d)$, if there are points p, q such that $pa = qb$ and $pd = qc$. In [7] it was proved that this relation satisfies the axioms of *parallelogram space*:

1. For any three points a, b, c there is a unique point d such that $\mathrm{Par}(a,b,c,d)$.
2. $\mathrm{Par}(a,b,c,d)$ implies $\mathrm{Par}(e,f,g,h)$, where (e,f,g,h) is any cyclic permutation of (a,b,c,d) or (d,c,b,a).
3. $\mathrm{Par}(a,b,c,d)$ and $\mathrm{Par}(c,d,e,f)$ imply $\mathrm{Par}(a,b,f,e)$.

In an IM-quasigroup, the unique point d of axiom 1 satisfies the following equation [9, Theorem 12]:

$$ab \cdot dc = ac. \tag{12}$$

This equation can be explicitly solved for d in GS-quasigroups: $d = a{\cdot}b(ca{\cdot}a)$ [8, Theorem 6]. Here we prove a similar result for G_2-quasigroups.

Proposition 3.1. *In a G_2-quasigroup, for any a, b, c we have*

$$\mathrm{Par}(a,b,c,(ba \cdot cb)b).$$

Proof. By substituting $d = (ba \cdot cb)b$ into the equation (12) we get

$$ab \cdot [(ba \cdot cb)b \cdot c] = ac.$$

It suffices to show that this is a valid identity in any G_2-quasigroup:

$$ab \cdot [(ba \cdot cb)b \cdot c] \overset{(5)}{=} ab \cdot [(ba \cdot c)(cb \cdot c) \cdot bc] \overset{(2)}{=} ab \cdot [(ba \cdot c)b \cdot (cb \cdot c)c] =$$
$$\overset{(6)}{=} ab \cdot [(ba \cdot c)b \cdot b] \overset{(5)}{=} ab \cdot [(ba \cdot b) \cdot cb]b \overset{(5)}{=} ab \cdot [(ba \cdot b)b \cdot (cb \cdot b)] =$$
$$\overset{(6)}{=} ab \cdot a(cb \cdot b) \overset{(4)}{=} a \cdot b(cb \cdot b) \overset{(8)}{=} ac.$$

□

Now we have a direct definition of parallelograms in G_2-quasigroups, without using auxiliary points:

$$\mathrm{Par}(a, b, c, d) \iff d = (ba \cdot cb)b. \tag{13}$$

Using the parallelogram relation geometric concepts such as midpoints, vectors and translations can be introduced. Of course, in the special case of the quasigroups $C(q)$ the concepts agree with the usual definitions of plane geometry. Thus, geometric theorems can be proved by formal calculations in a quasigroup. We give an example particular to G_2-quasigroups (Theorem 3.4).

In any medial quasigroup, b is said to be the *midpoint* of the pair of points a, c if $\mathrm{Par}(a, b, c, b)$ holds. This is denoted by $M(a, b, c)$. The following proposition provides a characterization in G_2-quasigroups.

Proposition 3.2. *In a G_2-quasigroup, $M(a, b, c)$ is equivalent with*

$$c = (ab \cdot ba)a. \tag{14}$$

Proof. By axiom 2 of parallelogram spaces, $M(a, b, c)$ is equivalent with $\mathrm{Par}(b, a, b, c)$, and the claim follows from (13). □

To facilitate notation, we introduce a new binary operation:

$$a * b = (ba \cdot a)b. \tag{15}$$

Starting from the quasigroup $C(q_1)$, this defines the binary operation in the quasigroup $C(p_1)$, i.e. $a * b = (1 - p_1)a + p_1 b$. If $ab = c$ (resp. $a * b = c$), we say that b *divides the pair of points a, c in the second upper (resp. lower) golden ratio.* Here are some properties of the new binary operation. It is assumed that the original binary operation has higher priority than '$*$', e.g. $a * bc$ means $a * (bc)$.

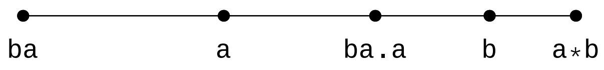

Figure 4: A new binary operation defined by (15).

Lemma 3.3. *The operation defined by (15) in a G_2-quasigroup satisfies the following identities:*

$$a * a = a, \tag{16}$$

$$ab * cd = (a * c)(b * d), \tag{17}$$

$$(a * (a * b)c)c = b. \tag{18}$$

Proof. Idempotency of the new operation (16) follows directly from (1). Identity (17) follows by repeated application of mediality:

$$\begin{aligned} ab * cd &\overset{(15)}{=} (cd \cdot ab)(ab) \cdot cd \overset{(2)}{=} (ca \cdot db)(ab) \cdot cd \overset{(2)}{=} (ca \cdot a)(db \cdot b) \cdot cd = \\ &\overset{(2)}{=} (ca \cdot a)c \cdot (db \cdot b)d \overset{(15)}{=} (a * c)(b * d). \end{aligned}$$

Here is the proof of identity (18):

$$\begin{aligned} (a * (a * b)c)c &\overset{(15)}{=} \{[(ba \cdot a)b \cdot c]a \cdot a\}[(ba \cdot a)b \cdot c] \cdot c = \\ &\overset{(2)}{=} \{[(ba \cdot a)b \cdot c]a \cdot (ba \cdot a)b\}(ac) \cdot c = \\ &\overset{(2)}{=} \{[(ba \cdot a)b \cdot c](ba \cdot a) \cdot ab\}(ac) \cdot c = \\ &\overset{(2)}{=} \{[(ba \cdot a)b \cdot ba](ca) \cdot ab\}(ac) \cdot c = \\ &\overset{(5)}{=} \{[(ba \cdot b)(ab) \cdot ba](ca) \cdot ab\}(ac) \cdot c = \\ &\overset{(2)}{=} \{[(ba \cdot b)b \cdot (ab \cdot a)](ca) \cdot ab\}(ac) \cdot c = \\ &\overset{(6)}{=} \{[a(ab \cdot a) \cdot ca](ab) \cdot ac\}c \overset{(2)}{=} \{[ac \cdot (ab \cdot a)a](ab) \cdot ac\}c = \\ &\overset{(6)}{=} [(ac \cdot b)(ab) \cdot ac]c \overset{(5)}{=} [(ac \cdot a)b \cdot ac]c \overset{(2)}{=} [(ac \cdot a)a \cdot bc]c = \\ &\overset{(6)}{=} (c \cdot bc)c \overset{(7)}{=} b. \end{aligned}$$

□

Identity (17) could be called *mutual mediality* of the two binary operations. By identifying two factors various kinds of distributivities follow:

$a * bc = (a * b)(a * c)$, $a(b * c) = ab * ac$ and their right counterparts. Identity (18) is an analogue of the defining identity for GS-quasigroups [8]. It is used in the proof of the following theorem.

Theorem 3.4. *In a G_2-quasigroup, suppose that $a * e = c$, $a * f = b$ and $cg = f$. Then, $bg = e$. Furthermore, suppose $M(a, h, g)$ and $h * g = d$. Then, $dh = a$ and $M(b, d, c)$.*

Proof. The first claim follows by substitution:

$$bg = (a * f)g = (a * cg)g = (a * (a * e)g)g \overset{(18)}{=} e.$$

If, in addition, $M(a, h, g)$ and $h * g = d$ hold, we get $g = (ah \cdot ha)a$ by (14), and the remaining claims follow by tedious, but straightforward computations:

$$\begin{aligned}
dh &= (h * g)h = [h * (ah \cdot ha)a]h \overset{(15)}{=} \{[(ah \cdot ha)a \cdot h]h \cdot (ah \cdot ha)a\}h = \\
&\overset{(2)}{=} \{[(ah \cdot ha)a \cdot h](ah \cdot ha) \cdot ha\}h \overset{(2)}{=} \{[(ah \cdot ha)a \cdot ah](h \cdot ha) \cdot ha\}h = \\
&\overset{(5)}{=} \{[(ah \cdot a)(ha \cdot a) \cdot ah](h \cdot ha) \cdot ha\}h = \\
&\overset{(2)}{=} \{[(ah \cdot a)a \cdot (ha \cdot a)h](h \cdot ha) \cdot ha\}h = \\
&\overset{(6)}{=} \{[h \cdot (ha \cdot a)h](h \cdot ha) \cdot ha\}h \overset{(4)}{=} \{h[(ha \cdot a)h \cdot ha] \cdot ha\}h = \\
&\overset{(5)}{=} \{h[(ha \cdot h)(ah) \cdot ha] \cdot ha\}h \overset{(2)}{=} \{h[(ha \cdot h)h \cdot (ah \cdot a)] \cdot ha\}h = \\
&\overset{(6)}{=} [h \cdot a(ah \cdot a)](ha) \cdot h \overset{(4)}{=} h[a(ah \cdot a) \cdot a] \cdot h \overset{(3)}{=} h[a \cdot (ah \cdot a)a] \cdot h = \\
&\overset{(6)}{=} (h \cdot ah)h \overset{(7)}{=} a.
\end{aligned}$$

To prove $M(b, d, c)$, we utilize (14) once more:

$$\begin{aligned}
(bd \cdot db)b &\overset{(4)}{=} (bd \cdot d)(bd \cdot b) \cdot b \overset{(5)}{=} (bd \cdot d)b \cdot (bd \cdot b)b \overset{(6)}{=} (bd \cdot d)b \cdot d = \\
&= (bd \cdot d)b \cdot (h * g) \overset{(15)}{=} (bd \cdot d)b \cdot (gh \cdot h)g = \\
&\overset{(2)}{=} (bd \cdot d)(gh \cdot h) \cdot bg \overset{(2)}{=} (bd \cdot gh)(dh) \cdot bg = \\
&\overset{(2)}{=} (bg \cdot dh)(dh) \cdot bg = (ea \cdot a)e \overset{(15)}{=} a * e = c.
\end{aligned}$$

□

In the special case of the quasigroup $C(q_1)$, Theorem 3.4 proves:

Corollary 3.5. *Let ABC be a triangle and suppose the points E and F divide $\overline{AC}$ and $\overline{AB}$ in the second lower golden ratio, respectively. Then the cevians $\overline{BE}$ and $\overline{CF}$ intersect in a point G that divides them in the second upper golden ratio. Furthermore, the midpoint H of $\overline{AG}$ divides the third cevian $\overline{AD}$ in the second upper golden ratio.*

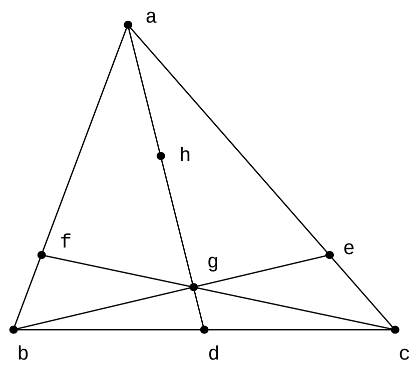

Figure 5: Geometric interpretation of Theorem 3.4.

The statement of Corollary 3.5 remains true if every instance of the second lower/upper golden ratio is replaced by the corresponding n-th golden ratio (for a definition see [5]). For $n = 1$, both the lower and the upper golden ratio are equal to $\frac{1+\sqrt{5}}{2}$ and we get the geometric interpretation of [8, Theorem 15].

4 Representation theorems

Let $(G, +)$ be an Abelian group with an automorphism φ such that the following equality holds for every $x \in G$:

$$\varphi^3(x) - 2\varphi^2(x) + \varphi(x) - x = 0. \tag{19}$$

Define another binary operation on G by the formula

$$a \cdot b = a + \varphi(b - a). \tag{20}$$

It is easy to verify that G is an IM-quasigroup with this new operation. Furthermore, the identity (6) follows from (19):

$$\begin{aligned}
(ab\cdot a)a &= ab\cdot a+\varphi(a)-\varphi(ab\cdot a)\\
&= ab+\varphi(a)-\varphi(ab)+\varphi(a)-\varphi(ab)-\varphi^2(a)+\varphi^2(ab)\\
&= 2\varphi(a)-\varphi^2(a)+ab-2\varphi(ab)+\varphi^2(ab)\\
&= 2\varphi(a)-\varphi^2(a)+(id-2\varphi+\varphi^2)(a+\varphi(b)-\varphi(a))\\
&= \big[a-\varphi(a)+2\varphi^2(a)-\varphi^3(a)\big]+\big[\varphi^3(b)-2\varphi^2(b)+\varphi(b)-b\big]+b\\
&\overset{(19)}{=} b.
\end{aligned}$$

Therefore, $(G,\cdot)$ is a G_2-quasigroup. The purpose of this section is to show that any G_2-quasigroup can be obtained in this way.

Theorem 4.1. *Let $(G,\cdot)$ be a G_2-quasigroup. Choose an arbitrary $o\in G$ and define a new binary operation on G by the formula*

$$a+b=(oa\cdot bo)o. \tag{21}$$

Then, $(G,+)$ is an Abelian group with neutral element o.

Proof. We first prove associativity, commutativity and that o is the neutral element:

$$\begin{aligned}
(a+b)+c &\overset{(21)}{=} [o\cdot(oa\cdot bo)o](co)\cdot o \overset{(5)}{=} [o\cdot(oa\cdot bo)o]o\cdot(co\cdot o)=\\
&\overset{(7)}{=} (oa\cdot bo)(co\cdot o) \overset{(2)}{=} (ob\cdot ao)(co\cdot o) \overset{(2)}{=} (ob\cdot co)(ao\cdot o)=\\
&\overset{(7)}{=} [o\cdot(ob\cdot co)o]o\cdot(ao\cdot o) \overset{(5)}{=} [o\cdot(ob\cdot co)o](ao)\cdot o=\\
&\overset{(2)}{=} (oa)[(ob\cdot co)o\cdot o]\cdot o \overset{(21)}{=} a+(b+c),
\end{aligned}$$

$$a+b \overset{(21)}{=} (oa\cdot bo)o \overset{(2)}{=} (ob\cdot ao)o \overset{(21)}{=} b+a,$$

$$a+o \overset{(21)}{=} (oa\cdot oo)o \overset{(1)}{=} (oa\cdot o)o \overset{(6)}{=} a.$$

For any $a\in G$ define $-a=o\cdot(o\cdot oa)a$. This is the inverse of a:

$$\begin{aligned}
a+(-a) &\overset{(21)}{=} \{oa\cdot[o\cdot(o\cdot oa)a]o\}o \overset{(5)}{=} (oa\cdot o)\{[o\cdot(o\cdot oa)a]o\cdot o\}=\\
&\overset{(6)}{=} (oa\cdot o)\cdot(o\cdot oa)a \overset{(2)}{=} (oa)(o\cdot oa)\cdot oa \overset{(7)}{=} o.
\end{aligned}$$

□

Theorem 4.2. *The mappings $\varphi : x \mapsto ox$ and $\psi : x \mapsto xo$ are automorphisms of the group $(G,+)$ of Theorem 4.1 and satisfy the identity*

$$\psi(a) + \varphi(b) = ab. \tag{22}$$

Proof. The following shows that φ is an automorphism:

$$\begin{aligned} \varphi(a)+\varphi(b) &= oa+ob \overset{(21)}{=} (o\cdot oa)(ob\cdot o)\cdot o \overset{(3)}{=} (o\cdot oa)(o\cdot bo)\cdot o = \\ &\overset{(4)}{=} o(oa\cdot bo)\cdot o \overset{(3)}{=} o\cdot(oa\cdot bo)o \overset{(21)}{=} o(a+b) = \varphi(a+b). \end{aligned}$$

The proof that ψ is an automorphism is similar. Finally,

$$\begin{aligned} \psi(a)+\varphi(b) &= ao+ob \overset{(21)}{=} (o\cdot ao)(ob\cdot o)\cdot o \overset{(3)}{=} (o\cdot ao)(o\cdot bo)\cdot o = \\ &\overset{(4)}{=} o(ao\cdot bo)\cdot o \overset{(5)}{=} o(ab\cdot o)\cdot o \overset{(7)}{=} ab. \end{aligned}$$

□

Theorem 4.3. *Equations (19) and (20) are satisfied in the setting of the previous two theorems.*

Proof. As a special case of (22), we see that $\psi(x)+\varphi(x) = xx \overset{(1)}{=} x$, i.e. $\psi(x) = x - \varphi(x)$. Now equation (20) follows directly from (22):

$$ab = \psi(a)+\varphi(b) = a - \varphi(a) + \varphi(b) = a + \varphi(b-a).$$

To prove equation (19), note that

$$\psi^2(x) = \psi(x-\varphi(x)) = x - \varphi(x) - \varphi(x-\varphi(x)) = \varphi^2(x) - 2\varphi(x) + x.$$

Therefore, $\varphi^3(x) - 2\varphi^2(x) + \varphi(x) = \varphi(\psi^2(x)) = o(xo\cdot o) \overset{(8)}{=} x$. □

This is a direct proof of a G_2-version of Toyoda's representation theorem for medial quasigroups [6].

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Department of Mathematics
University of Zagreb
Bijenička 30
HR-10000 Zagreb
Croatia
e-mail: krcko@math.hr, volenec@math.hr