The CB-norm approximation of generalized skew derivations by elementary operators

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Operator Days
Zagreb, February 1, 2019

This work has been fully supported by the Croatian Science Foundation under the project IP-2016-06-1046.

Introduction

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Examples of semiprime rings:

- Any prime ring is obviously a semiprime ring.
- Any reduced ring is a semiprime ring.
- Any J-semisimple ring is semiprime. In particular, semisimple rings, von Neumann regular rings and C^*-algebras are all semiprime.
- Any direct product of semiprime rings is semiprime.
- Any matrix ring over a semiprime ring is a semiprime ring.

An interesting class of additive maps $d : R \to R$ including both ring epimorphisms and generalized derivations is the class of **generalized skew derivations**, that is, those satisfying

$$d(xy) = \delta(x)y + \sigma(x)d(y) \qquad \forall x, y \in R,$$

for some map $\delta : R \to R$ and a ring epimorphism $\sigma : R \to R$. Since R is semiprime, it is easy to see that the map δ is automatically additive and it is uniquely determined by d. Moreover, δ is a σ-**derivation** (skew-derivation), i.e. δ satisfies

$$\delta(xy) = \delta(x)y + \sigma(x)\delta(y) \qquad \forall x \in R.$$

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$$\delta(xy) = \delta(x)y + \sigma(x)\delta(y) \qquad \forall x \in R.$$

We decompose d as

$$d = \delta + \rho,$$

where $\rho := d - \delta$. Note that ρ is a **left** R**-module** σ**-homomorphism**, that is

$$\rho(xy) = \sigma(x)\rho(y) \qquad \forall x \in R.$$

On the other hand, an attractive and fairly large class of additive maps $\phi : R \to R$ is the class of **elementary operators**, that is, those that can be expressed as a finite sum

$$\phi = \sum_i M_{a_i,b_i}$$

of two-sided multiplications $M_{a_i,b_i} : x \mapsto a_i x b_i$ such that a_i, b_i are elements of the multiplier ring $M(R)$.

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Motivated by the fact that elementary operators comprise both inner automorphisms $x \mapsto pxp^{-1}$ and inner generalized derivations $x \mapsto qx + xr$, we consider the following question:

Problem 1.

Describe the form of generalized skew derivations that are simultaneously elementary operators.

Recall that an **essentially defined double centralizer** on R is a triple $(\mathcal{L}, \mathcal{R}, I)$, where I is an essential ideal of R, $\mathcal{L} : I \to R$ is a left R-module homomorphism, $\mathcal{R} : I \to R$ is a right R-module homomorphism such that

$$\mathcal{L}(x)y = x\mathcal{R}(y) \qquad \forall x, y \in I.$$

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One can form the **symmetric ring of quotients** $Q_s = Q_s(R)$ of R which is characterized (up to isomorphism) by the following properties:

(i) R is a subring of Q_s;

(ii) for any $q \in Q_s$ there is an essential ideal I of R such that $qI + Iq \subseteq R$;

(iii) if $0 \neq q \in Q_s$ and I is an essential ideal of R, then $qI \neq 0$ and $Iq \neq 0$;

(iv) for any essentially defined double centralizer $(\mathcal{L}, \mathcal{R}, I)$ on R there exists $q \in Q_s$ such that $\mathcal{L}(x) = qx$ and $\mathcal{R}(x) = xq$ for all $x \in I$.

The center of Q_s is called the **extended centroid** of R and it is denoted by C. It is well known that C is a von Neumann regular self-injective ring. Moreover, C is a field if and only if R is a prime ring.

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For any subset $S \subseteq Q_s$ there exists a unique idempotent $\varepsilon(S)$ in C such that

$$\mathrm{ann}_{Q_s}(Q_s S Q_s) = (1 - \varepsilon(S))Q_s \quad \text{and} \quad \varepsilon(S)x = x \quad \forall x \in S,$$

where for $X \subset Q_S$, $\mathrm{ann}_{Q_s}(X)$ denotes the (two-sided) annihilator of X in Q_s. The idempotent $\varepsilon(S)$ is called the **central support** of S. Whenever $S = \{x\}$ for some $x \in Q_s$ we write $\varepsilon(x)$ for $\varepsilon(S)$.

Theorem 1.1 (Eremita, G., Ilišević; 2014).

If a generalized σ-derivation d of R is also a generalized elementary operator, then there are elements $q, r \in Q_s$ such that $d(x) = qx + \sigma(x)r$ for all $x \in R$, and for every such pair of elements (q, r) there is an invertible element $p \in Q_s^\times$ such that

$$\varepsilon(r)\sigma(x) = \varepsilon(r)pxp^{-1} \qquad \forall x \in R.$$

In particular,

$$d(x) = qx + pxp^{-1}r \qquad \forall x \in R.$$

Results

In a case when $R = A$ is a C^*-algebra we now that all elementary operators $\phi : A \to A$ are completely bounded, i.e.

$$\|\phi\|_{cb} := \sup_{n\in\mathbb{N}} \|\phi_n\| < \infty,$$

where for each n, ϕ_n is an induced map on $M_n(A)$; $\phi_n([a_{ij}]) = [\phi(a_{ij})]$.

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where for each n, ϕ_n is an induced map on $M_n(A)$; $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. Moreover, we have the the following estimate for their cb-norm:

$$\left\|\sum_i M_{a_i,b_i}\right\|_{cb} \leq \left\|\sum_i a_i \otimes b_i\right\|_h, \tag{1}$$

where $\|\cdot\|_h$ is the Haagerup tensor norm on the algebraic tensor product $M(A) \otimes M(A)$, i.e.

$$\|u\|_h = \inf\left\{\left\|\sum_i a_i a_i^*\right\|^{\frac{1}{2}} \left\|\sum_i b_i^* b_i\right\|^{\frac{1}{2}} \ : \ u = \sum_i a_i \otimes b_i\right\}.$$

Hence, if by $\mathcal{E}\ell(A)$ we denote the set of all elementary operators on A and by $\mathrm{CB}(A)$ the set of all completely bounded operators on A, inequality (1) implies that the mapping

$$(M(A) \otimes M(A), \|\cdot\|_h) \to (\mathcal{E}\ell(A), \|\cdot\|_{cb})$$

given by

$$\sum_i a_i \otimes b_i \mapsto \sum_i M_{a_i,b_i}.$$

is a well-defined contraction. Its continuous extension to the completed Haagerup tensor product $M(A) \otimes_h M(A)$ is known as a **canonical contraction** from $M(A) \otimes_h M(A)$ to $\mathrm{CB}(A)$ and is denoted by Θ_A.

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Motivated by the observation that elementary operators on C^*-algebras are completely bounded we can consider the following problem:

Problem 2.

If A is a C^*-algebra, describe all generalized skew derivations $d : A \to A$ that can be approximated in the cb-norm by elementary operators of A, that is, that lie in the cb-norm closure $\overline{\mathcal{E}\ell(A)}_{cb}$.

In the case when A is a prime C^*-algebra we have the following result:

Theorem 2.3 (G.; preprint).

Let A be a prime C^*-algebra and let $d : A \to A$ be a generalized σ-derivation. The following conditions are equivalent:

(i) $d \in \overline{\overline{\mathcal{E}\ell(A)}}_{cb}$.

(ii) *Either d is a left multiplication implemented by some element of $M(A)$ or there is an invertible element $a \in M(A)$ such that $\sigma = \mathrm{Ad}(a)$ (so that σ is an inner automorphism of A). In the latter case, if $d = \delta + \rho$ is a usual decomposition, then there are elements $b, c \in M(A)$ such that*

$$\delta(x) = cx - \sigma(x)c = cx - axa^{-1}c \quad \text{and} \quad \rho(x) = axa^{-1}b \qquad \forall x \in A, \tag{2}$$

so that

$$d(x) = cx + axa^{-1}(b - c) \qquad \forall x \in A.$$

Remark 2.4.

Note that any left multiplier $d = M_{c,1}$, where $c \in M(A)$, is a generalized σ-derivation with respect to any ring epimorphism $\sigma : A \to A$, since for any such σ we have $d(x) = cx - \sigma(x)c + \sigma(x)c$. Hence, if $\delta(x) = cx - \sigma(x)c$ and $\rho(x) = \sigma(x)c$, then clearly δ is a σ-derivation of A and ρ is a left σ-module homomorphism of A such that $d = \delta + \rho$. Therefore in this case we cannot say anything about the epimorphism σ and corresponding maps δ and ρ.

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For the proof of Theorem 2.3 we will need some auxiliary results.

Lemma 2.5.

Let A be a prime C^*-algebra. Suppose that $\sigma : A \to A$ is a ring epimorphism for which there exists a non-zero element $a \in M(A)$ such that

$$ax = \sigma(x)a \qquad \forall x \in A. \tag{3}$$

Then $a \in M(A)^\times$, so that $\sigma = \mathrm{Ad}(a)$ is an inner automorphism of A.

Before proving Lemma 2.5, first recall that in a case when $R = A$ is a C^*-algebra, $Q_s(A)$ has a natural structure as a unital complex $*$-algebra, whose involution is positive definite. An element $q \in Q_s(A)$ is called **bounded** if there is $\lambda \in \mathbb{R}_+$ such that $q^*q \leq \lambda 1$ in a sense that there is a finite number of elements $q_1, \ldots, q_n \in Q_s(A)$ such that

$$q^*q + \sum_{i=1}^{n} q_i^* q_i = \lambda 1.$$

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The set $Q_b(A)$ of all bounded elements of $Q_s(A)$ has a pre-C^*-algebra structure with respect to the norm

$$\|q\|^2 = \inf\{\lambda \in \mathbb{R} : \ q^*q \leq \lambda 1\},$$

which clearly extends the norm of A. One can easily check that an element $q \in Q_s(A)$ is bounded if and only if it can be represented by a bounded (continuous) essentially defined double centralizer. We call $Q_b(A)$ the **bounded symmetric algebra of quotients** of A and its completion $M_{\mathrm{loc}}(A)$ the **local multiplier algebra** of A. Note that $M_{\mathrm{loc}}(A)$ has a structure of a C^*-algebra as a completion of a pre-C^*-algebra.

Proof of Lemma 2.5. First note that such an element a cannot be a zero-divisor. Indeed, if there exists $x \in M(A)$ such that $ax = 0$ then for each $y \in A$ we have $ayx = \sigma(y)ax = 0$, so that $aAx = 0$. Since A is prime, A is an essential ideal of $M(A)$ and $a \neq 0$, we conclude that $x = 0$. Similarly, if $xa = 0$ then for all $y \in A$ we have $xya = xa\sigma(y) = 0$. Since σ is surjective, this is equivalent to $xAa = 0$, which again implies $x = 0$.

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$$AaA = \{\sigma(x)ay : \ x, y \in A\} = \{axy : \ x, y \in A\} \subseteq aA.$$

We define maps $\mathcal{L}, \mathcal{R} : aA \to A$ by

$$\mathcal{L}(ax) = \sigma(x) \quad \text{and} \quad \mathcal{R}(ax) = x.$$

That the maps $\mathcal{L}$ and $\mathcal{R}$ are well-defined follows from the fact that a is not a (left) zero-divisor. It is easy to check that $(\mathcal{L}, \mathcal{R}, aI)$ is an essentially defined double centralizer on A.

It is well known that all essentially defined double centralizers on prime C^*-algebras are automatically continuous, so in particular there exists an element $b \in Q_b(A) \subseteq M_{\mathrm{loc}}(A)$ such that

$$\sigma(x) = \mathcal{L}(ax) = axb = \sigma(x)ab \quad \text{and} \quad x = \mathcal{R}(ax) = bax$$

for all $x \in A$. Since $\sigma(A) = A$, this is equivalent to $A(1 - ab) = 0$ and $(1 - ba)A = 0$. Hence, a is invertible in $M_{\mathrm{loc}}(A)$ and $a^{-1} = b$. $\square$

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Recall that a sequence (a_n) in a C^*-algebra A such that the series $\sum_{n=1}^{\infty} a_n^* a_n$ is norm convergent is said to be **strongly independent** if for every sequence $(\alpha_n) \in \ell^2$, equality $\sum_{n=1}^{\infty} \alpha_n a_n = 0$ implies $\alpha_n = 0$ for all n.

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Remark 2.6.

(i) Every tensor $t \in A \otimes_h A$ can be represented as convergent series $t = \sum_{n=1}^{\infty} a_n \otimes b_n$, where (a_n) and (b_n) are sequences in A such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent. Moreover, the sequence (b_n) can be chosen to be strongly independent.

(ii) If $t = \sum_{n=1}^{\infty} a_n \otimes b_n$ is a representation of t as above, with (b_n) strongly independent, then $t = 0$ if and only if $a_n = 0$, for all $n \in \mathbb{N}$.

Corollary 2.7.

Let A be a prime C^*-algebra. If (a_n), (b_n) are sequences in $M(A)$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with (b_n) strongly independent. If

$$\sum_{n=1}^{\infty} a_n x b_n = 0, \tag{4}$$

for all $x \in A$, then $a_n = 0$ for all $n \in \mathbb{N}$.

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for all $x \in A$, then $a_n = 0$ for all $n \in \mathbb{N}$.

Proof. If $t := \sum_{n=1}^{\infty} a_n \otimes b_n \in M(A) \otimes_h M(A)$, then (4) is equivalent to $\Theta_A(t) = 0$. Since A is prime, by Mathieu's theorem Θ_A is isometric (hence injective), so $t = 0$. The claim now follows form Remark 2.6. □

Prposition 2.8.

Let A be a prime C^*-algebra and let $\sigma : A \to A$ be a ring epimorphism. If $\rho : A \to A$ is a non-zero left A-module σ-homomorphism, then the following conditions are equivalent:

(i) $\rho \in \overline{\overline{\mathcal{E}\ell(A)}}_{cb}$.

(ii) *There are elements $a, p \in M(A)$, with a invertible and $p \neq 0$, such that $\sigma = \mathrm{Ad}(a) \in \mathrm{InnAut}(A)$ and*

$$\rho(x) = \sigma(x)p = axa^{-1}p$$

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Proof. Since A is prime, by Mathieu's theorem the canonical contraction $\Theta_A : M(A) \otimes_h M(A)$ is isometric. In particular, the image of Θ_A is closed in the cb-norm so $\overline{\overline{\mathcal{E}\ell(A)}}_{cb}$ coincides with the image of Θ_A. Hence, there is a tensor $t \in M(A) \otimes_h M(A)$ such that $\rho = \Theta_A(t)$. We can write $t = \sum_{n=1}^{\infty} a_n \otimes b_n$, where (a_n) and (b_n) are sequences in $M(A)$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with (b_n)

strongly independent. Since $\rho(xy) = \sigma(x)\rho(y)$, we have

$$\sum_{n=1}^{\infty}(a_n x - \sigma(x)a_n)yb_n = 0$$

for all $x, y \in A$. Corollary 2.7 implies

$$a_n x = \sigma(x)a_n \tag{5}$$

for all $n \in \mathbb{N}$. Since ρ is non-zero, there is $n_0 \in \mathbb{N}$ such that $a_{n_0} \neq 0$. By the previous Lemma $a := a_{n_0}$ is invertible in $M(A)$. Hence $\sigma = \mathrm{Ad}(a)$ is an inner automorphism of A. Finally, if $p := \sum_{n=1}^{\infty} a_n b_n \in M(A)$, using (5) we get

$$\rho(x) = \sum_{n=1}^{\infty} a_n x b_n = \sigma(x)\left(\sum_{n=1}^{\infty} a_n b_n\right) = \sigma(x)p = axa^{-1}p.$$

$\square$

As a direct consequence of Proposition 2.8 we get:

Corollary 2.9.

If A is a prime C^*-algebra then every ring epimorphism $\sigma : A \to A$ that lies in $\overline{\mathcal{E}\ell(A)}_{cb}$ must be an inner automorphism of A.

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In the proof of Theorem 2.3 we will also use the following technical result:

Lemma 2.10.

Let B be a unital C^*-algebra and let $f, g, h : B \to B$ be any functions with $f \neq 0$. Suppose that for all $x \in B$ we have the following equality

$$f(x) \otimes 1 = \sum_{n=1}^{\infty} (a_n g(x) + h(x) a_n) \otimes b_n \tag{6}$$

of tensors in $B \otimes_h B$, where (a_n) and (b_n) are sequences in B such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with (b_n) strongly independent. Then there is a non-zero element $c \in B$ such that

$$f(x) = cg(x) + h(x)c \qquad \forall x \in B. \tag{7}$$

Proof. Choose $x_0 \in B$ such that $f(x_0) \neq 0$ and let $\varphi \in B^*$ be an arbitrary bounded linear functional such that $\varphi(f(x_0)) \neq 0$. If for $x = x_0$ we act on the equality (6) with the right slice map $R_\varphi : B \otimes_h B \to B$, $R_\varphi : a \otimes b \mapsto \varphi(a)b$, we obtain

$$\varphi(f(x_0)) \cdot 1 = \sum_{n=1}^{\infty} \varphi(a_n g(x_0) + h(x_0)a_n)b_n. \tag{8}$$

For $n \in \mathbb{N}$ let

$$\alpha_n := \frac{\varphi(a_n g(x_0) + h(x_0)a_n)}{\varphi(\delta(x_0))}.$$

Note that $(\alpha_n) \in \ell^2$, since bounded linear functionals on C^*-algebras are completely bounded and the series $\sum_{n=1}^{\infty}(a_n g(x_0) + h(x_0)a_n)(a_n g(x_0) + h(x_0)a_n)^*$ is norm convergent. Then (8) can be rewritten as $\sum_{n=1}^{\infty} \alpha_n b_n = 1$, so by (6) we have

$$\sum_{n=1}^{\infty}(\alpha_n f(x) - a_n g(x) - h(x)a_n) \otimes b_n = 0 \qquad \forall x \in B.$$

Consequently, since (b_n) is strongly independent, we conclude that

$$\alpha_n f(x) = a_n g(x) + h(x) a_n$$

for all $n \in \mathbb{N}$ and $x \in B$. Since $\sum_{n=1}^{\infty} \alpha_n b_n = 1$, there is some $n_0 \in \mathbb{N}$ such that $\alpha_{n_0} \neq 0$. If $c := (1/\alpha_{n_0}) a_{n_0}$, then the above equation is obviously equivalent to (7). Also, $c \neq 0$ since $f \neq 0$. □

Proof of Theorem 2.3. (ii) $\Longrightarrow$ (i). This is trivial (see also Remark 2.4).

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Proof of Theorem 2.3. (ii) $\Longrightarrow$ (i). This is trivial (see also Remark 2.4).
(ii) $\Longrightarrow$ (i). Assume that $d \in \overline{\overline{\mathcal{E}\ell(A)}}_{cb}$ and that d is not a left multiplier implemented by some element of $M(A)$. In particular $d \neq 0$. Using the same arguments as before, we see that there is a tensor $t \in M(A) \otimes_h M(A)$ such that $d = \Theta_A(t)$. We can write $t = \sum_{n=1}^{\infty} a_n \otimes b_n$, where (a_n) and (b_n) are sequences in $M(A)$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with (b_n) strongly independent. Using the functional identity $d(xy) = \delta(x)y + \sigma(x)d(y)$ we get

$$\delta(x)y = \sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) y b_n,$$

or equivalently

$$\Theta_A(\delta(x) \otimes 1)) = \Theta_A \left(\sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) \otimes b_n \right). \qquad (9)$$

Since Θ_A is isometric (hence injective), (9) is equivalent to the equality

$$\delta(x) \otimes 1 = \sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) \otimes b_n$$

of tensors in $M(A) \otimes_h M(A)$ for all $x \in A$. If $\delta = 0$, then d must be a non-zero left A-module σ-homomorphism of A so the claim follows from Proposition 2.8. If $\delta \neq 0$, Lemma 2.10 implies that there is a non-zero element $c \in M(A)$ such that

$$\delta(x) = cx - \sigma(x)c$$

for all $x \in A$.

If we make a usual decomposition $d = \delta + \rho$, the map $\rho' : A \to A$ defined by

$$\rho'(x) := \rho(x) - \sigma(x)c = d(x) - cx$$

is obviously a left A-module σ-homomorphism of A that lies in $\overline{\overline{\mathcal{E}\ell(A)}}_{cb}$ (since d does). Since, by assumption, d is not a left multiplier, ρ' is non-zero. Hence, by Proposition 2.8 there are elements $a, p \in M(A)$ with a invertible and $p \neq 0$ such that $\sigma = \mathrm{Ad}(a)$ and

$$\rho'(x) = \sigma(x)p = axa^{-1}p$$

for all $x \in A$. In particular, if we put $b := p + c$ we get (2). $\square$

Final remarks

In my previous papers, I showed that <u>any</u> derivation of a C^*-algebra A with $\mathrm{Prim}(A)$ Hausdorff that lies in $\overline{\overline{\mathcal{E}\ell(A)}}_{cb}$ must be an inner derivation. This result was further extended for unital C^*-algebras whose every Glimm ideal is prime.

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In light of this, it is natural to ask if one can extend Corollary 2.9 in it's original form (and consequently Theorem 2.3) for a similar class of C^*-algebras. However, this will no longer be possible. In fact, there are unital C^*-algebras A of the form $C(X, M_n(\mathbb{C}))$, where X is a suitable compact Hausdorff space, that admit outer automorphisms which are simultaneously elementary operators on A.

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First recall that if A is a unital homogeneous C^*-algebra with $X = \mathrm{Prim}(A)$, there is an exact sequence

$$0 \longrightarrow \mathrm{InnAut}^*(A) \longrightarrow \mathrm{Aut}^*_{C(X)}(A) \xrightarrow{\eta} \check{H}^2(X;\mathbb{Z})$$

of abelian groups and that the image of η is contained in the torsion subgroup of $\check{H}^2(X;\mathbb{Z})$.

Example 3.1.

For $n \geq 2$ let X_n be the projective unitary group $PU(n) = U(n)/\mathbb{S}^1$. It is well known that $\check{H}^2(X_n;\mathbb{Z}) \cong \mathbb{Z}_n$, so that $\check{H}^2(X_n;\mathbb{Z})$ is a torsion group. Using this fact and som additional calculations, Kadison and Ringrose showed that for $A_n = C(X_n, M_n(\mathbb{C}))$ we have $\mathrm{InnAut}^*(A) \subsetneq \mathrm{Aut}^*_{C(X)}(A)$.

On the other hand, $\mathrm{Aut}_{C(X_n)}(A_n) \subseteq \mathcal{E}\ell(A_n)$. This follows from the fact that for any homogeneous C^*-algebra A with $X = \mathrm{Prim}(A)$, any bounded $C_0(X)$-linear map $\phi : A \to A$ preserves all two-sided ideals of A (i.e. $\phi(I) \subseteq I$ for any such ideal I). But, by Magajna's theorem, any such map on A is an elementary operator on A.

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Problem 3.

Is (some variant of) Theorem 2.3 true for all boundedly centrally closed C^*-algebras (in particular for AW^*-algebras)?