Elementary operators on Hilbert C^*-modules

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Introduction

Let A be a C^*-algebra. An attractive and fairly large class of bounded linear maps $\phi : A \to A$ that preserve all ideals of A is the class of **elementary operators**, that is, those that can be expressed as a finite sum

$$\phi = \sum_i M_{a_i,b_i}$$

of two-sided multiplications $M_{a_i,b_i} : x \mapsto a_i x b_i$, where $a_i, b_i \in M(A)$ (the multiplier algebra of A).

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In fact, elementary operators are completely bounded (cb), i.e.

$$\|\phi\|_{cb} := \sup_{n\in\mathbb{N}} \|\phi_n\| < \infty,$$

where for each n, ϕ_n is an induced map on $\mathrm{M}_n(A)$ (the C^*-algebra of $n \times n$ matrices over A), i.e.

$$\phi_n([a_{ij}]) = [\phi(a_{ij})].$$

Indeed, if $\phi = \sum_{i=1}^{k} M_{a_i,b_i}$ then, working inside $\mathrm{M}_k(M(A))$, for each $x \in A$ we have

$$\begin{aligned}
\|\phi(x)\| &= \left\| \begin{bmatrix} a_1 & a_2 & \dots & a_k \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x & 0 & \dots & 0 \\ 0 & x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x \end{bmatrix} \begin{bmatrix} b_1 & 0 & \dots & 0 \\ b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_k & 0 & \dots & 0 \end{bmatrix} \right\| \\
&\le \left\| \begin{bmatrix} a_1 & a_2 & \dots & a_k \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} x & 0 & \dots & 0 \\ 0 & x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x \end{bmatrix} \right\| \left\| \begin{bmatrix} b_1 & 0 & \dots & 0 \\ b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_k & 0 & \dots & 0 \end{bmatrix} \right\| \\
&= \left\| \sum_{i=1}^{k} a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^{k} b_i^* b_i \right\|^{\frac{1}{2}} \|x\|.
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&= \left\| \sum_{i=1}^k a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^k b_i^* b_i \right\|^{\frac{1}{2}} \|x\|.
\end{aligned}$$

This shows

$$\|\phi\| \leq \left\| \sum_{i=1}^k a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^k b_i^* b_i \right\|^{\frac{1}{2}}. \tag{1}$$

Similarly, for each $n \in \mathbb{N}$ and $[x_{ij}] \in \mathrm{M}_n(A)$ we have

$$\phi_n([x_{ij}]) = \sum_{i=1}^{k} a_i^{(n)} [x_{ij}] \, b_i^{(n)},$$

where for each $a \in M(A)$, $a^{(n)} = \mathrm{diag}(a, \ldots, a) \in \mathrm{M}_n(M(A))$. Hence, by (1)

$$\begin{aligned} \|\phi_n\| &\leq \left\| \sum_{i=1}^{k} a_i^{(n)} (a_i^*)^{(n)} \right\|_{\mathrm{M}_n(M(A))}^{\frac{1}{2}} \left\| \sum_{i=1}^{k} (b_i^*)^{(n)} b_i^{(n)} \right\|_{\mathrm{M}_n(M(A))}^{\frac{1}{2}} \\ &= \left\| \sum_{i=1}^{k} a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^{k} b_i^* b_i \right\|^{\frac{1}{2}}, \end{aligned}$$

Similarly, for each $n \in \mathbb{N}$ and $\left[x_{ij}\right] \in \mathrm{M}_n(A)$ we have

$$\phi_n\left(\left[x_{ij}\right]\right) = \sum_{i=1}^{k} a_i^{(n)} \left[x_{ij}\right] b_i^{(n)},$$

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which shows

$$\|\phi\|_{cb} \leq \left\| \sum_{i=1}^{k} a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^{k} b_i^* b_i \right\|^{\frac{1}{2}}. \tag{2}$$

In particular, if $\|\cdot\|_h$ is the Haagerup tensor norm on $M(A) \otimes M(A)$, i.e.

$$\|t\|_h = \inf\left\{ \left\|\sum_i u_i u_i^*\right\|^{\frac{1}{2}} \left\|\sum_i v_i^* v_i\right\|^{\frac{1}{2}} \ : \ t = \sum_i u_i \otimes v_i \right\},$$

(2) implies that the natural map

$$(M(A) \otimes M(A), \|\cdot\|_h) \to (\mathrm{CB}(A), \|\cdot\|_{cb})$$

given by

$$\sum_i a_i \otimes b_i \mapsto \sum_i M_{a_i,b_i}.$$

is a well-defined contraction. Its continuous extension to the completed Haagerup tensor product $M(A) \otimes_h M(A)$ is known as a **canonical contraction** from $M(A) \otimes_h M(A)$ to $\mathrm{CB}(A)$ and is denoted by Θ_A.

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Problem

When is Θ_A isometric or injective?

Remark

A necessary condition for the injectivity of Θ_A is that A is a prime C^*-algebra. Indeed, if A is not prime, then there are two non-zero ideals I and J of A such that $IJ = \{0\}$. Choose any non-zero elements $a \in I$ and $b \in J$. Then $a \otimes b \neq 0$ in $M(A) \otimes_h M(A)$, while $\Theta_A(a \otimes b) = 0$.

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Theorem (Mathieu 2003)

Let A be a C^*-algebra. TFAE:

(i) *Θ_A is isometric.*

(ii) *Θ_A is injective.*

(iii) *A is a prime C^*-algebra.*

Preliminaries

- Throughout, X will be a (right) Hilbert module over a C^*-algebra A.

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- if Y is another Hilbert A-module, by $\mathbb{B}(X, Y)$ we denote the set of all **adjointable operators** from X to Y, that is those $u : X \to Y$ for which there is $u^* : Y \to X$ with the property

$$\langle ux, y \rangle = \langle x, u^* y \rangle \qquad \forall x \in X,\ y \in Y.$$

  It is well-known that all adjointable operators are bounded and A-linear (i.e. $u(xa) = (ux)a$ for all $x \in X$ and $a \in A$).

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- By $\mathbb{K}(X, Y)$ we denote the closed linear subspace of $\mathbb{B}(X, Y)$ generated by the maps $z \mapsto y\langle x, z\rangle$ ($x \in X$, $y \in Y$).
- If $X = Y$ we write $\mathbb{B}(X)$ (or $\mathbb{B}_A(X)$) and $\mathbb{K}(X)$ (or $\mathbb{K}_A(X)$). Then $\mathbb{B}(X)$ is a C^*-algebra and $\mathbb{K}(X)$ is an essential ideal of $\mathbb{B}(X)$. Moreover, $\mathbb{B}(X) = M(\mathbb{K}(X))$.

The **linking algebra** of X is defined as $\mathcal{L}(X) := \mathbb{K}(A \oplus X)$. We can write

$$\mathcal{L}(X) = \begin{bmatrix} \mathbb{K}(A) & \mathbb{K}(X, A) \\ \mathbb{K}(A, X) & \mathbb{K}(X) \end{bmatrix} = \left\{ \begin{bmatrix} T_a & l_y \\ r_x & u \end{bmatrix} : \ a \in A, \ x, y \in X, \ u \in \mathbb{K}(X) \right\},$$

where $T_a(b) = ab$ and $r_x(b) = xb$ for all $b \in A$, while $l_y(z) = \langle y, z \rangle$ for all $z \in X$. Thereby, $a \mapsto T_a$ is an isomorphism of C^*-algebras A and $\mathbb{K}(A)$, $y \mapsto l_y$ is an isometric conjugate linear isomorphism between Banach spaces X and $\mathbb{K}(X, A)$, and $x \mapsto r_x$ is an isometric linear isomorphism between Banach spaces X and $\mathbb{K}(A, X)$.

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Besides $\mathcal{L}(X)$, we need another subalgebra of $\mathbb{B}(A \oplus X)$, larger than $\mathcal{L}(X)$. We define an **extended linking algebra** of X as

$$\begin{aligned} \mathcal{L}_{\mathrm{ext}}(X) &= \begin{bmatrix} \mathbb{B}(A) & \mathbb{K}(X,A) \\ \mathbb{K}(A,X) & \mathbb{B}(X) \end{bmatrix} \\ &= \left\{ \begin{bmatrix} T_v & l_y \\ r_x & u \end{bmatrix} : v \in M(A), \ x, y \in X, \ u \in \mathbb{B}(X) \right\}, \end{aligned}$$

where, similarly as before, for $v \in M(A)$, $T_v : A \to A$ is defined by $T_v(a) = va$. It is easy to see that $\mathcal{L}_{\mathrm{ext}}(X)$ is a C^*-subalgebra of $\mathbb{B}(A \oplus X)$ which contains $\mathcal{L}(X)$ as an essential ideal.

If X is a Hilbert A-module, we can introduce the operator space structure on X via the operator space structure of its linking algebra $\mathcal{L}(X)$ (or extended linking algebra $\mathcal{L}_{\mathrm{ext}}(X)$), after identifying X as the $2-1$ corner in $\mathcal{L}(X)$ (or $\mathcal{L}_{\mathrm{ext}}(X)$), via the isometric isomorphism $X \cong \mathbb{K}(A, X)$, $x \mapsto r_x$. That is, for all $n \in \mathbb{N}$ and $\left[x_{ij}\right] \in \mathrm{M}_n(X)$ we define

$$\left\|\left[x_{ij}\right]\right\|_{\mathrm{M}_n(X)} := \left\|\left[\begin{bmatrix} 0 & 0 \\ r_{x_{ij}} & 0 \end{bmatrix}\right]\right\|_{\mathrm{M}_n(\mathcal{L}(X))} = \left\|\left[\begin{bmatrix} 0 & 0 \\ r_{x_{ij}} & 0 \end{bmatrix}\right]\right\|_{\mathrm{M}_n(\mathcal{L}_{\mathrm{ext}}(X))},$$

so that the canonical embedding

$$\iota_X : X \hookrightarrow \mathcal{L}_{\mathrm{ext}}(X), \qquad \iota_X : x \mapsto \begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}$$

becomes a complete isometry. This structure is called the **canonical operator space structure** on X.

Remark

- If B is any C^*-algebra that contains A as an ideal, then X can be also regarded as a Hilbert B-module with respect to the same inner product (which takes values in $A \subseteq B$), while the right action of B on X is defined as follows. For $x \in X$, $a \in A$ and $b \in B$, set

$$(xa)b := x(ab).$$

  Obviously, $\mathbb{B}_B(X) = \mathbb{B}_A(X)$ and $\mathbb{K}_A(X) = \mathbb{K}_B(X)$, so all $u \in \mathbb{B}_A(X)$ are also B-linear.
- In particular, by taking $B = M(A)$, any Hilbert A-module X can be regarded as a Hilbert $M(A)$-module. Now for all $u \in \mathbb{B}(X)$, $x \in X$ and $v \in M(A)$ we have $u(xv) = (ux)v$, so in this way X becomes a Banach $\mathbb{B}(X) - M(A)$-bimodule (in particular, the product uxv is unambiguously defined).
- Moreover, it is straightforward to check that each matrix space $\mathrm{M}_n(X)$ is a Banach $\mathrm{M}_n(\mathbb{B}(X)) - \mathrm{M}_n(M(A))$-bimodule in the canonical way.

Elementary operators on Hilbert C^*-modules

We now extend the notion of elementary operators to Hilbert C^*-modules. First of all, following the C^*-algebraic case, for each $u \in \mathbb{B}(X)$ and $v \in M(A)$ we define a map

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Definition

By an **elementary operator** on a Hilbert A-module X we mean a map $\phi : X \to X$ for which there exists a finite number of elements $u_1, \ldots, u_k \in \mathbb{B}(X)$ and $v_1, \ldots, v_k \in M(A)$ such that

$$\phi = \sum_{i=1}^{k} M_{u_i,v_i}.$$

Remark

If a C^*-algebra A is considered as a Hilbert A-module in the standard way, then $\mathbb{B}(A)$ and $M(A)$ coincide, so elementary operators on A, as a Hilbert A-module, agree with the usual notion of elementary operators on A.

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As in the C^*-algebraic case it is easy to see that any elementary operator ϕ on X is completely bounded. Moreover, if ϕ is represented as $\phi = \sum_i M_{u_i,v_i}$, then

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Therefore, the mapping

$$(\mathbb{B}(X) \otimes M(A), \|\cdot\|_h) \to (\mathrm{CB}(X), \|\cdot\|_{cb}) \quad \text{given by} \quad \sum_i u_i \otimes v_i \mapsto \sum_i M_{u_i,v_i},$$

is a well-defined contraction, so we can continuously extend it to the map

$$\Theta_X : (\mathbb{B}(X) \otimes_h M(A), \| \cdot \|_h) \to (\mathrm{CB}(X), \| \cdot \|_{cb}).$$

Theorem (Arambašić-G. 2020)

Let X be a non-zero Hilbert A-module. TFAE:

(i) Θ_X *is isometric.*

(ii) Θ_X *is injective.*

(iii) A *is a prime* C^**-algebra.*

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Lemma

For each map $\phi : X \to X$ *we define a map*

$$\widetilde{\phi} : \mathcal{L}_{\mathrm{ext}}(X) \to \mathcal{L}_{\mathrm{ext}}(X) \qquad \textit{by} \qquad \widetilde{\phi}\left(\begin{bmatrix} T_v & l_y \\ r_x & u \end{bmatrix}\right) := \begin{bmatrix} 0 & 0 \\ r_{\phi(x)} & 0 \end{bmatrix}.$$

(a) *If* $\phi \in \mathrm{CB}(X)$ *then* $\widetilde{\phi} \in \mathrm{CB}(\mathcal{L}_{\mathrm{ext}}(X))$ *and* $\|\widetilde{\phi}\|_{cb} = \|\phi\|_{cb}$.

(b) *For each* $t \in \mathbb{B}(X) \otimes_h M(A)$ *we have*

$$\widetilde{\Theta_X(t)} = \Theta_{\mathcal{L}_{\mathrm{ext}}(X)}((\iota_{\mathbb{B}(X)} \otimes \iota_{M(A)})(t)).$$

We shall also need the following characterisations of Hilbert C^*-modules over prime C^*-algebras:

Proposition

Let X be a non-zero Hilbert A-module. TFAE:

- **(i)** *A is prime.*
- **(ii)** *X is essentially full and $\mathbb{K}(X)$ is prime.*
- **(iii)** *The linking algebra $\mathcal{L}(X)$ is prime.*
- **(iv)** *The extended linking algebra $\mathcal{L}_{\mathrm{ext}}(X)$ is prime.*
- **(v)** *If $a \in A$ and $u \in \mathbb{K}(X)$ are such that $uxa = 0$ for all $x \in X$, then $a = 0$ or $u = 0$.*
- **(vi)** *X is essentially full and if $x_1, x_2 \in X$ are such that $x_1\langle x, x_2\rangle = 0$ for all $x \in X$, then $x_1 = 0$ or $x_2 = 0$.*

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Corollary

The primeness is an invariant property under Morita equivalence.

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(ii)$\Longrightarrow$ (iii). Assume that A is not prime. Then there are non-zero elements $u \in \mathbb{K}(X)$ and $a \in A$ such that $uxa = 0$ for all $x \in X$. Then $u \otimes a$ is a non-zero tensor in $\mathbb{K}(X) \otimes A \subseteq \mathbb{B}(X) \otimes M(A)$ but $\Theta_X(u \otimes a)(x) = uxa = 0$ for all $x \in X$.

Proof of Theorem. (i) $\Longrightarrow$ (ii). This is trivial.

(ii)$\Longrightarrow$ (iii). Assume that A is not prime. Then there are non-zero elements $u \in \mathbb{K}(X)$ and $a \in A$ such that $uxa = 0$ for all $x \in X$. Then $u \otimes a$ is a non-zero tensor in $\mathbb{K}(X) \otimes A \subseteq \mathbb{B}(X) \otimes M(A)$ but $\Theta_X(u \otimes a)(x) = uxa = 0$ for all $x \in X$.

(iii) $\Longrightarrow$ (i). Since the canonical embeddings $\iota_{\mathbb{B}(X)} : \mathbb{B}(X) \hookrightarrow \mathcal{L}_{\text{ext}}(X)$ and $\iota_{M(A)} : M(A) \hookrightarrow \mathcal{L}_{\text{ext}}(X)$ are completely isometric, the injectivity of the Haagerup tensor product implies

$$\|(\iota_{\mathbb{B}(X)} \otimes \iota_{M(A)})(t)\|_h = \|t\|_h \qquad \forall t \in \mathbb{B}(X) \otimes_h M(A).$$

If A is a prime C^*-algebra, then $\mathcal{L}_{\text{ext}}(X)$ is also prime, so Mathieu's theorem implies

$$\|\Theta_{\mathcal{L}_{\text{ext}}(X)}(t')\|_{cb} = \|t'\|_h \qquad \forall t' \in \mathcal{L}_{\text{ext}}(X) \otimes_h \mathcal{L}_{\text{ext}}(X).$$

Hence, for all $t \in \mathbb{B}(X) \otimes_h M(A)$ we have

$$\begin{aligned} \|\Theta_X(t)\|_{cb} &= \|\widetilde{\Theta_X(t)}\|_{cb} = \|\Theta_{\mathcal{L}_{\text{ext}}(X)}((\iota_{\mathbb{B}(X)} \otimes \iota_{M(A)})(t))\|_{cb} \\ &= \|(\iota_{\mathbb{B}(X)} \otimes \iota_{M(A)})(t)\|_h = \|t\|_h. \end{aligned}$$

Thus, Θ_X is isometric.

Some open problems

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If the underlying C^*-algebra A of X is not prime (so that Θ_X is non-injective), can we describe the kernel of Θ_X?

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By a beautiful result due to Archbold, Mathieu and Somerset from 1999 we know that for any elementary operator ϕ on a C^*-algebra A we have $\|\phi\|_{cb} = \|\phi\|$ if and only if A is an extension of an antiliminal C^*-algebra by an abelian one. Can we generalize this result in the context of Hilbert C^*-modules?