# Nonlinear preserver problems on matrix domains

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# Preserver problems: from preserved data to rigid form

Throughout, $M_n(\mathbb{C})$ denotes the algebra of $n \times n$ complex matrices. For an element $a$ in a unital complex algebra $\mathcal{A}$, its **spectrum** is

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a \text{ is not invertible}\}.$$

A **preserver problem** asks which maps preserve a given quantity or property, such as

$$\det X, \qquad \operatorname{rank} X, \qquad \sigma(X), \qquad X \geq 0,$$

or a relation between matrices, such as

$$XY = 0 \quad (\textbf{zero-product preserving}), \qquad XY = YX \quad (\textbf{commutativity preserving}).$$

The aim is not only to classify such maps, but to understand when preserving a small amount of information already forces a very simple algebraic form. This rigidity is the guiding theme of the talk.

Wigner's theorem is a classical example of this phenomenon in quantum theory.

Li–Tsing, Linear Algebra Appl. 162–164 (1992), 217–235; Li–Pierce, Amer. Math. Monthly 108 (2001), 591–605;

# The classical linear model: Frobenius theorem

One of the earliest examples is the determinant preserver problem.

## Frobenius, 1897

If a linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies

$$\det \phi(X) = \det X \qquad (X \in M_n(\mathbb{C})),$$

then there are $A, B \in \mathrm{GL}(n, \mathbb{C})$ with $\det(AB) = 1$ such that

$$\phi(X) = AXB \qquad \text{or} \qquad \phi(X) = AX^t B.$$

Thus one scalar invariant, together with linearity, already determines the remaining linear symmetries. This became a basic model for linear preserver problems.

Frobenius, Sitzungsber. Königl. Preuss. Akad. Wiss. Berlin (1897), 994–1015; Li–Tsing, Linear Algebra Appl. 162–164 (1992), 217–235.

# Why Jordan homomorphisms appear

A **Jordan homomorphism** between complex algebras is a linear map $\phi$ satisfying

$$\phi(X^2) = \phi(X)^2 \qquad \iff \qquad \phi(XY + YX) = \phi(X)\phi(Y) + \phi(Y)\phi(X).$$

The basic examples are algebra homomorphisms and antihomomorphisms: for all $X, Y$,

$$\phi(XY) = \phi(X)\phi(Y) \qquad \text{or} \qquad \phi(XY) = \phi(Y)\phi(X).$$

Jordan algebras arose in the 1930s from the algebraic formulation of quantum mechanics, where the symmetric product $X \circ Y := \frac{1}{2}(XY + YX)$ naturally appears for observables.

A central problem in Jordan theory is to determine when Jordan homomorphisms must come from homomorphisms and antihomomorphisms, possibly on central summands.

For $M_n(\mathbb{C})$, every Jordan automorphism is of the form

$$X \mapsto TXT^{-1} \qquad \text{or} \qquad X \mapsto TX^tT^{-1}, \qquad T \in \mathrm{GL}(n, \mathbb{C}).$$

Jordan–von Neumann–Wigner, Ann. of Math. 35 (1934), 29–64; Jacobson–Rickart, Trans. Amer. Math. Soc. 69 (1950), 479–502; Herstein, Trans. Amer. Math. Soc. 81 (1956), 331–341.

## Two spectral conditions

Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be unital and linear.

We say that $\phi$ **preserves invertibility** if

$$a \text{ invertible} \quad \Longrightarrow \quad \phi(a) \text{ invertible},$$

and that it **preserves invertibility in both directions** if the above implication is an equivalence.

Since $\phi(a - \lambda 1_{\mathcal{A}}) = \phi(a) - \lambda 1_{\mathcal{B}}$, we have

$$\begin{aligned} \phi \text{ preserves invertibility} &\iff \sigma(\phi(a)) \subseteq \sigma(a) \quad (\textbf{spectrum shrinking}), \\ \phi \text{ preserves invertibility in both directions} &\iff \sigma(\phi(a)) = \sigma(a) \quad (\textbf{spectrum preserving}). \end{aligned}$$

The second condition is genuinely stronger. Passing from spectral equality to one-sided inclusion is not a formal step and often requires substantially different arguments.

Kaplansky, *Algebraic and Analytic Aspects of Operator Algebras*, CBMS 1, AMS, 1970; Aupetit, J. London Math. Soc. 62 (2000), 917–924.

# From Gleason–Kahane–Żelazko to Kaplansky–Aupetit

The scalar-valued case already shows remarkable rigidity.

## Gleason–Kahane–Żelazko, 1967–68

If $\mathcal{A}$ is a unital Banach algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital and linear with $\varphi(a) \in \sigma(a)$ for every $a \in \mathcal{A}$, then $\varphi$ is multiplicative.

Kaplansky asked how far this extends to noncommutative ranges. Without semisimplicity the statement is false; Aupetit proposed the following version.

## Kaplansky–Aupetit problem

Let $\mathcal{A}$ and $\mathcal{B}$ be unital semisimple Banach algebras. If a unital, surjective, linear map $\phi : \mathcal{A} \to \mathcal{B}$ is spectrum shrinking, must it be a Jordan homomorphism?

This remains open in general, even for $C^*$-algebras.

Gleason (1967); Kahane–Żelazko (1968); Kaplansky (1970); Aupetit, J. London Math. Soc. 62 (2000), 917–924.

# Spectrum shrinking versus spectrum preserving: a genuine gap

The two spectral hypotheses have developed in parallel.

**Spectrum preserving.** Jafarian–Sourour (1986): surjective maps between algebras of bounded operators on Banach spaces. Aupetit (2000): von Neumann algebras.

**Spectrum shrinking.** Sourour (1996): bijective invertibility preservers on bounded-operator algebras. Further cases include finite von Neumann algebra ranges (Aupetit, 1998), von Neumann algebra domains (Cui–Hou, 2004), and $C^*$-algebra ranges admitting a faithful tracial state (Mathieu–Schulz, 2023).

These are separate results. Sourour's 1996 shrinking theorem, for example, uses substantially different complex-analytic arguments from the 1986 result.

Jafarian–Sourour, JFA 66 (1986), 255–261; Sourour, TAMS 348 (1996), 13–30; Aupetit, Banach Algebras '97 (1998), 55–78; Cui–Hou, Acta Math. Sinica 20 (2004), 761–768; Mathieu–Schulz, Studia Math. 272 (2023), 113–120.

# Why commutativity enters

There is a useful bridge from the Kaplansky–Aupetit problem to commutativity preservers.

**Brešar–Šemrl, 2008**

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras with $\mathcal{B}$ semisimple, and let $\phi : \mathcal{A} \to \mathcal{B}$ be unital, surjective, linear and invertibility preserving. If $\phi(a^2)$ commutes with $\phi(a)$ for every $a \in \mathcal{A}$, then $\phi$ is a Jordan homomorphism.

Thus one need not prove $\phi(a^2) = \phi(a)^2$ directly; it is enough to prove commutativity of these two elements.

This is one reason spectral and commutativity preservers belong to the same story.

Brešar–Šemrl, Expositiones Math. 26 (2008), 269–277.

# What changes when linearity is removed

Once linearity is dropped, we still need some regularity. Continuity is a natural choice: it allows genuinely nonlinear maps, while making topological arguments possible.

Even then, neither preserving condition is rigid by itself.

**Spectrum preserving.** For a continuous $S : M_n(\mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$,

$$\phi(X) = S(X)XS(X)^{-1}$$

preserves the spectrum, and for suitable $S$ is nonlinear.

**Commutativity preserving.** The continuous map

$$\psi(X) = \mathrm{tr}(X)I_n$$

has commuting range, but is not a Jordan homomorphism for $n \geq 2$.

Thus the nonlinear question is whether

$$\text{continuity} + \text{spectrum preserving} + \text{commutativity preserving}$$

can force a Jordan form.

# From Kaplansky–Aupetit to Šemrl

The Kaplansky–Aupetit problem retains linearity and asks whether spectral control forces Jordan form. On $M_n(\mathbb{C})$, Šemrl showed that linearity can indeed be replaced by continuity together with preservation of commutativity.

**Šemrl, 2008**

Let $n \geq 3$. Every continuous, spectrum-preserving and commutativity-preserving map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a Jordan automorphism.

Hence, there exists $T \in \mathrm{GL}(n, \mathbb{C})$ such that for all $X \in M_n(\mathbb{C})$ either

$$\phi(X) = TXT^{-1} \qquad \text{or} \qquad \phi(X) = TX^t T^{-1}.$$

Petek–Šemrl, Linear Algebra Appl. 269 (1998), 33–46; Šemrl, Operators and Matrices 2 (2008), 125–136.

# Rank-one idempotents and projective geometry

Let $V := \mathbb{C}^n$. Its projective space

$$\mathbb{P}(V) := \{\text{one-dimensional subspaces of } V\}$$

has projective lines $\mathbb{P}(W)$, where $\dim W = 2$.

## Fundamental Theorem of Projective Geometry

If $\dim V \geq 3$, every bijection of $\mathbb{P}(V)$ mapping projective lines onto projective lines is induced by an invertible semilinear map on $V$.

A rank-one idempotent $P$ determines both a projective point $\operatorname{ran} P$ and a projective hyperplane $\ker P$. Moreover,

$$PQ = 0 \quad \Longleftrightarrow \quad \operatorname{ran} Q \subseteq \ker P.$$

Thus zero products of rank-one idempotents encode projective incidence.

Faure, Geom. Dedicata 90 (2002), 145–151.

## How projective geometry enters Šemrl's proof

For rank-one idempotents, Šemrl considers the relation

$$P \sim Q \quad \Longleftrightarrow \quad \operatorname{ran} P = \operatorname{ran} Q \;\text{ or }\; \ker P = \ker Q,$$

and proves that $P \sim Q$ implies $\phi(P) \sim \phi(Q)$.
After possibly composing with transposition, families with a fixed range line are mapped to families with a fixed range line. He also proves that $PQ = 0$ implies $\phi(P)\phi(Q) = 0$.

A nonbijective form of the Fundamental Theorem of Projective Geometry then gives

$$\phi(P) = T\alpha[P]T^{-1}$$

for every rank-one idempotent $P$, where $T$ is invertible and $\alpha : \mathbb{C} \to \mathbb{C}$ is a nonzero field endomorphism.
Continuity reduces $\alpha$ to the identity or complex conjugation; the latter is excluded.
Extending to all matrices gives the two Jordan forms.

Šemrl, J. Funct. Anal. 210 (2004), 248–257; Šemrl, Operators and Matrices 2 (2008), 125–136.

## Why $n \geq 3$ is essential

The theorem fails when $n = 2$. For $A \in M_2(\mathbb{C})$ write uniquely

$$A = \lambda I_2 + \begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

For $(a, b, c) \neq (0, 0, 0)$, define

$$\Phi(A) := \lambda I_2 + \begin{bmatrix} a & e^{i\theta} b \\ e^{-i\theta} c & -a \end{bmatrix}, \qquad \theta := \frac{|b|^2}{|a|^2 + |b|^2 + |c|^2},$$

and set $\Phi(\lambda I_2) := \lambda I_2$.

The map is continuous and nonlinear. It preserves trace and determinant, hence spectrum. If two nonscalar $2 \times 2$ matrices commute, their traceless parts are proportional; the definition of $\theta$ is unchanged under nonzero scalar multiplication, so their images still commute.

Thus $n \geq 3$ cannot be removed.

Šemrl, Operators and Matrices 2 (2008), 125–136.

# Beyond $M_n(\mathbb{C})$: two natural directions

Šemrl's theorem treats the full matrix algebra. We now restrict the domain in two natural ways.

**1. Natural matrix domains.** These include the matrix Lie groups

$$\mathrm{GL}(n,\mathbb{C}), \qquad \mathrm{SL}(n,\mathbb{C}), \qquad \mathrm{U}(n),$$

the normal matrices $\mathcal{N}_n$, and the algebraic varieties $M_n^{\leq k}$ of matrices of rank at most $k$. Their geometry and topology become part of the problem, especially through the motion of eigenvalues along paths and closed loops.

**2. Proper matrix subalgebras.** Here multiplication is retained, but only certain matrix entries are allowed to be nonzero. This leads to a different, largely combinatorial structure.

## Eigenvalue selection: a first topological test

For the first direction, we ask when continuity forces a spectrum shrinker to preserve the whole spectrum.

A continuous function

$$\lambda : \mathcal{X} \to \mathbb{C}, \qquad \lambda(X) \in \sigma(X),$$

is an **eigenvalue selector**.
If such a selector exists, then for every $m \geq 1$,

$$X \longmapsto \lambda(X) I_m$$

is a continuous spectrum shrinker which may keep only one eigenvalue.

For the Hermitian matrices $H_n := \{X : X = X^*\}$, the eigenvalues can be ordered as $\lambda_1(X) \leq \cdots \leq \lambda_n(X)$, and Weyl's inequality gives

$$|\lambda_j(X) - \lambda_j(Y)| \leq \|X - Y\|.$$

Thus every ordered eigenvalue is a continuous selector.

# Why U($n$) admits no continuous eigenvalue selector

For $|z| = 1$, define

$$C_n(z)e_j := e_{j+1} \quad (1 \leq j < n), \qquad C_n(z)e_n := ze_1.$$

Then $C_n(z) \in \mathrm{U}(n)$ and

$$\det(\lambda I_n - C_n(z)) = \lambda^n - z.$$

Hence

$$\gamma(t) = C_n(e^{it}), \qquad 0 \leq t \leq 2\pi,$$

is a closed loop with distinct eigenvalues

$$\lambda_j(t) := \exp\left(\frac{i(t + 2\pi(j-1))}{n}\right).$$

When the loop closes,

$$\lambda_1 \mapsto \lambda_2 \mapsto \cdots \mapsto \lambda_n \mapsto \lambda_1.$$

A continuous selector would have to follow one branch, but no branch returns to its initial value. Hence no continuous eigenvalue selector exists on $\mathrm{U}(n)$ for $n \geq 2$.

# Why $\mathrm{SU}(n)$ behaves differently

The special unitary group

$$\mathrm{SU}(n) = \{U \in \mathrm{U}(n) : \det U = 1\}$$

does admit a continuous eigenvalue selector.
For $\mathrm{SU}(3)$, consider the closed triangle

$$\Delta := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \le x_2 \le x_3 \le x_1 + 1,\ x_1 + x_2 + x_3 = 0\right\}.$$

This is the **Weyl alcove**. Every $U \in \mathrm{SU}(3)$ is conjugate to

$$\operatorname{diag}\left(e^{2\pi i x_1}, e^{2\pi i x_2}, e^{2\pi i x_3}\right)$$

for a unique $(x_1, x_2, x_3) \in \Delta$, and this point depends continuously on the conjugacy class of $U$. Hence

$$U \longmapsto e^{2\pi i x_1(U)}$$

is a continuous eigenvalue selector. The similar construction works for every $\mathrm{SU}(n)$.

Chirvasitu, *Eigenvalue selectors for representations of compact connected groups*, arXiv:2502.08847 (2025).

## Configuration spaces: how eigenvalues move

For distinct eigenvalues, set

$$\mathrm{Conf}_n(\mathbb{C}) := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \neq z_j\}.$$

The group $S_n$ acts freely by permuting the coordinates, and

$$\mathrm{UConf}_n(\mathbb{C}) := \mathrm{Conf}_n(\mathbb{C})/S_n$$

records the same $n$ points without labels.

A path of simple-spectrum matrices gives a path in $\mathrm{UConf}_n(\mathbb{C})$. After choosing labels at the starting point, the eigenvalues can be followed continuously.
For a closed path, the matrix returns to its starting point, but the labels may return permuted:

$$\lambda_j(1) = \lambda_{\sigma(j)}(0), \qquad \sigma \in S_n.$$

Thus closed loops produce permutations of the eigenvalue labels.

# When these permutations act transitively

Fix a connected component $C$ of the space of simple-spectrum matrices with labelled eigenvalues. Let

$$H_C := \{\sigma \in S_n : \sigma C = C\}.$$

Thus $H_C$ consists of the permutations of the labels that can be produced by closed paths in the corresponding simple-spectrum component.

For a continuous spectrum shrinker, the multiplicity attached to each labelled eigenvalue is constant on $C$. Therefore the multiplicity vector $(r_1, \ldots, r_n)$ is unchanged by every permutation in $H_C$.
If $H_C$ acts transitively on $\{1, \ldots, n\}$, then any label can be moved to any other, and hence

$$r_1 = \cdots = r_n.$$

So transitivity of the loop-induced permutations forces all eigenvalues to occur with the same multiplicity.

## From eigenvalue topology to spectrum preservation

For

$$\mathcal{X}_n \in \{M_n(\mathbb{C}), \mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{U}(n), \mathcal{N}_n\},$$

the simple-spectrum matrices are dense, and the possible relabellings along loops act transitively on the eigenvalue labels. For $\mathrm{U}(n)$, the explicit $n$-cycle above already provides this transitivity.

### Chirvasitu–G.–Tomašević

For positive integers $m, n$, a continuous spectrum-shrinking map $\phi : \mathcal{X}_n \to M_m(\mathbb{C})$ exists if and only if $n \mid m$. Whenever it exists,

$$k_{\phi(X)}(z) = k_X(z)^{m/n}, \qquad \text{for all } X \in \mathcal{X}_n.$$

Thus every eigenvalue survives, with all algebraic multiplicities multiplied by the same factor $m/n$. In particular, for $m = n$, spectrum shrinking already implies spectrum preservation.

Chirvasitu–Gogić–Tomašević, arXiv:2501.06840; Proc. Edinb. Math. Soc., to appear.

## Adding commutativity recovers the Jordan form

For the five domains above, the spectral part is now settled. When the codomain is $M_n(\mathbb{C})$, continuity already turns spectrum shrinking into spectrum preservation.

**Chirvasitu–G.–Tomašević**

Let $n \geq 3$. Every continuous, spectrum-shrinking and commutativity-preserving map $\phi : \mathcal{X}_n \to M_n(\mathbb{C})$ is the restriction of a Jordan automorphism of $M_n(\mathbb{C})$.

Hence, there exists $T \in \mathrm{GL}(n, \mathbb{C})$ such that for every $X \in \mathcal{X}_n$, either,

$$\phi(X) = TXT^{-1} \qquad \text{or} \qquad \phi(X) = TX^{t}T^{-1}.$$

For $\mathcal{X}_n = M_n(\mathbb{C})$ this strengthens Šemrl's theorem: spectrum preservation can be replaced by spectrum shrinking.

Chirvasitu–Gogić–Tomašević, arXiv:2501.06840; Proc. Edinb. Math. Soc., to appear.

# Rank varieties: why the topology changes

For $1 \leq k < n$, let
$$M_n^{\leq k} := \{X \in M_n(\mathbb{C}) : \operatorname{rank} X \leq k\}.$$

This is a complex algebraic variety. The rank-$k$ matrices form a smooth dense part, while the lower-rank matrices are exactly its singular points.

Every $X \in M_n^{\leq k}$ has 0 as an eigenvalue of algebraic multiplicity at least $n-k$.

- If $k \leq n-2$, no matrix has simple spectrum.
- If $k = n-1$, simple spectra do occur, but 0 is always one of the eigenvalues and remains fixed along every loop in the simple-spectrum part.

Thus the eigenvalue labels cannot be permuted transitively, and the previous topological argument no longer forces all multiplicities to be equal.

Chirvasitu–Gogić–Tomašević, Linear Algebra Appl. 724 (2025), 298–319.

## Injectivity does not prevent spectral collapse

Let $\mathrm{Nil}(n) := \{N \in M_n(\mathbb{C}) : N^n = 0\}$ denote the variety of nilpotent matrices in $M_n(\mathbb{C})$. Every element of $\mathrm{Nil}(n)$ has spectrum $\{0\}$.

**Chirvasitu–G.–Tomašević, 2025**

For every fixed $k$, and all sufficiently large $n$, there exists a real-analytic embedding of $M_n^{\leq k}$ into $\mathrm{Nil}(n)$.

The real-analyticity comes from complex geometry. For large $n$,

$$n^2 - n \geq 2k(2n-k) + 1.$$

Since $M_n^{\leq k}$ is affine, hence Stein, Narasimhan's embedding theorem embeds it holomorphically into $\mathbb{C}^{n^2-n}$; this space is real-analytically isomorphic to a ball in the smooth locus of $\mathrm{Nil}(n)$.

Since $k < n$, every $X \in M_n^{\leq k}$ has $0 \in \sigma(X)$. Thus such an embedding is an injective spectrum shrinker that discards all nonzero eigenvalues.

Chirvasitu–Gogić–Tomašević, Linear Algebra Appl. 724 (2025), 298–319; Narasimhan, Amer. J. Math. 82 (1960),

# A dichotomy for spectrum shrinkers

Although complete spectral collapse is possible, for $k > n/2$ it is the only alternative to preserving the characteristic polynomial.

**Chirvasitu–G.–Tomašević, 2025**

If $k > n/2$, every continuous spectrum-shrinking map $\phi : M_n^{\leq k} \to M_n(\mathbb{C})$ either preserves characteristic polynomials or has image contained in $\mathrm{Nil}(n)$.

There is also a stronger conclusion for injective maps, valid without the assumption $k > n/2$: the characteristic polynomial is preserved whenever

$$k > n - \sqrt{n} \qquad \text{or} \qquad \phi(M_n^{\leq k}) \subseteq M_n^{\leq k}.$$

In particular, every injective continuous spectrum-shrinking self-map of $M_n^{\leq k}$ preserves characteristic polynomials.

Chirvasitu–Gogić–Tomašević, Linear Algebra Appl. 724 (2025), 298–319.

# Why the threshold $n-\sqrt{n}$ appears

The key input is from **topological dimension theory**: a compact space cannot embed into a space of smaller covering dimension.
Here

$$\dim_{\mathbb{C}} M_n^{\leq k} = k(2n-k), \qquad \dim_{\mathbb{C}} \mathrm{Nil}(n) = n^2 - n,$$

and complex algebraic varieties have real covering dimension twice their complex dimension. Hence

$$k > n - \sqrt{n} \quad \Longleftrightarrow \quad 2k(2n-k) > 2(n^2 - n).$$

If an injective continuous spectrum shrinker had image in $\mathrm{Nil}(n)$, its restriction to a compact ball in the rank-$k$ stratum would give an embedding of dimension $2k(2n-k)$ into $\mathrm{Nil}(n)$, a contradiction.
Thus $n-\sqrt{n}$ is exactly the threshold at which dimension theory rules out the nilpotent alternative.

Chirvasitu–Gogić–Tomašević, Linear Algebra Appl. 724 (2025), 298–319; Engelking, *Dimension Theory*, 1978.

## Commutativity preserving restores the Jordan form

The nilpotent embeddings show that injectivity and spectral control are not enough. Adding preservation of commutativity changes the conclusion.

### Chirvasitu–G.–Tomašević, 2025

Let $n \geq 3$ and $1 \leq k < n$. Every injective continuous, commutativity-preserving and spectrum-shrinking map $\phi : M_n^{\leq k} \to M_n^{\leq k}$ extends to a Jordan automorphism of $M_n(\mathbb{C})$.

Thus, there exists $T \in \mathrm{GL}(n, \mathbb{C})$ such that for all $X \in M_n^{\leq k}$, either,

$$\phi(X) = TXT^{-1} \qquad \text{or} \qquad \phi(X) = TX^tT^{-1}.$$

When $k = n - 1$, $M_n^{\leq n-1}$ is precisely the set of singular matrices, and the same conclusion holds even when the codomain is the whole algebra $M_n(\mathbb{C})$.

Chirvasitu–Gogić–Tomašević, Linear Algebra Appl. 724 (2025), 298–319.

# The second direction: matrix subalgebras

We now keep the algebra structure, but restrict which matrix entries may be nonzero.

**Petek, 2002.** Petek described the continuous spectrum- and commutativity-preserving self-maps of the upper-triangular algebra $\mathcal{T}_n$. For $n \geq 3$, injectivity or surjectivity forces a Jordan automorphism.

**G.–Petek–Tomašević, 2025.** Let $n \geq 3$ and let $\mathcal{A} \subseteq M_n(\mathbb{C})$ contain $\mathcal{T}_n$; equivalently, $\mathcal{A}$ is a block upper-triangular (parabolic) subalgebra. Every injective continuous spectrum- and commutativity-preserving map from $\mathcal{A}$ into $M_n(\mathbb{C})$ is a Jordan embedding.

For self-maps $\mathcal{A} \to \mathcal{A}$, spectrum preservation can be weakened to spectrum shrinking: by invariance of domain, together with the spectral argument in the proof, every continuous injective spectrum-shrinking self-map of $\mathcal{A}$ is automatically spectrum preserving.

Petek, Linear Algebra Appl. 357 (2002), 107–122; Gogić–Petek–Tomašević, Linear Algebra Appl. 704 (2025), 192–217.

## Structural matrix algebras and the triangle criterion

A **structural matrix algebra** is a subalgebra $\mathcal{A} \subseteq M_n(\mathbb{C})$ containing all diagonal matrices. Equivalently, it is determined by a preorder $\preceq$ on $\{1, \ldots, n\}$:

$$\mathcal{A} = \{(a_{ij}) \in M_n(\mathbb{C}) : a_{ij} = 0 \text{ if } i \not\preceq j\}.$$

Thus $i \preceq j$ means that the $(i, j)$-entry is allowed to be nonzero.

The **comparability graph** of $\mathcal{A}$ has vertices $1, \ldots, n$; distinct vertices $i, j$ are joined when $i \preceq j$ or $j \preceq i$. An edge lies in a triangle if its endpoints, together with some third vertex, are pairwise joined.

**G.–Tomašević, 2025**

Every injective continuous spectrum- and commutativity-preserving map $\mathcal{A} \to M_n(\mathbb{C})$ is a Jordan embedding if and only if every edge of the comparability graph lies in a triangle.

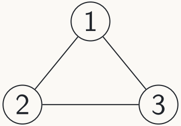
Gogić–Tomašević, J. Math. Anal. Appl. 549 (2025), 129497.

# The triangle criterion: two examples

## Upper triangular matrices

For the usual order $1 \preceq 2 \preceq 3$,

$$\mathcal{A} = \mathcal{T}_3 = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$




Every edge lies in the triangle, so the criterion holds.

## A missing comparison

Take the preorder generated by $1 \preceq 2$ and $3 \preceq 2$. Then

$$\mathcal{A} = \begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix}.$$


Here 1 and 3 are incomparable. The graph has no triangle, so the criterion fails.

## Summary

- On $M_n(\mathbb{C})$, Šemrl showed that continuity together with preservation of spectrum and commutativity already forces a Jordan automorphism. Projective geometry enters through rank-one idempotents.
- On natural matrix domains, topology enters through the way eigenvalues move along closed paths. When the induced permutations of the eigenvalues are transitive, spectrum shrinking forces spectrum preservation.
- Rank varieties are different: the persistent zero eigenvalue obstructs this transitivity and allows spectral collapse. Adding commutativity restores the Jordan form.
- For structural matrix algebras, the decisive condition is combinatorial: every edge of the comparability graph must lie in a triangle.

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**Thank you!**