Two-sided multiplications and phantom line bundles

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Preliminaries

C^*-algebraic formulation of Quantum Mechanics

In quantum mechanics a physical system is typically described via a unital C^*-algebra A with unit element.

- The self-adjoint elements of A are thought of as the observables; they are the measurable quantities of the system.
- A state of the system is defined as a positive functional on A (i.e. a linear map $\omega : A \to \mathbb{C}$ such that $\omega(a^*a) \geq 0$ for all $a \in A$) with $\omega(1_A) = 1$. If the system is in the state ω, then $\omega(a)$ is the expected value of the observable a.
- Automorphisms correspond to the symmetries, while one-parameter automorphism groups describe the reversible time evolution of the system (in the Heisenberg picture).

Definition

A C^*-**algebra** is a (complex) Banach $*$-algebra A whose norm $\|\cdot\|$ satisfies the C^*-identity. More precisely:

- A is a Banach algebra over the field $\mathbb{C}$.
- A is equipped with an involution, i.e. a map $*: A \to A$, $a \mapsto a^*$ satisfying the properties:

  $$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a,$$

  for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.
- Norm $\|\cdot\|$ satisfies the C^*-**identity**, i.e.

  $$\|a^* a\| = \|a\|^2$$

  for all $a \in A$.

Remark

The C^*-identity is a very strong requirement. For instance, together with the spectral radius formula, it implies that the C^*-norm is uniquely determined by the algebraic structure: For all $a \in A$ we have

$$\|a\|^2 = \|a^* a\| = \sup\{|\lambda| \ : \ \lambda \in \mathrm{spec}(a^* a)\}.$$

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Example

Let X be a LCH (locally compact Hausdorff) space and let $C_0(X)$ be the set of all continuous complex-valued functions on X that vanish at ∞. Then $C_0(X)$ becomes a commutative C^*-algebra with respect to the pointwise operations, involution $f^*(x) := \overline{f(x)}$, and max-norm $\|f\|_\infty := \sup\{|f(x)| \ : \ x \in X\}$. Obviously, $C_0(X)$ is unital if and only if X is compact.

In fact, all unital commutative C^*-algebras arise in this fashion:

Theorem (Gelfand-Naimark, 1943)

The (contravariant) functor $X \rightsquigarrow C_0(X)$ defines an equivalence of categories of LCH spaces (with proper maps as morphisms) and commutative C^*-algebras (with non-degenerate $*$-homomorphisms as morphisms).

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In other words: By passing from the space X the function algebra $C_0(X)$, no information is lost. In fact, X can be recovered from $C_0(X)$. Thus, topological properties of X can be translated into algebraic properties of $C_0(X)$, and vice versa, so the theory of C^*-algebras is often thought of as **noncommutative topology**.

Basic examples

- The set $\mathbb{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ becomes a C^*-algebra with respect to the standard operations, usual adjoint and operator norm. In particular, the complex matrix algebras $M_n(\mathbb{C})$ are C^*-algebras.
- In fact, every C^*-algebra can be isometrically embedded as a norm-closed self-adjoint subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (the noncommutative Gelfand-Naimark theorem).
- To every locally compact group G, one can associate a C^*-algebra $C^*(G)$. Everything about the representation theory of G is encoded in $C^*(G)$.
- The category of C^*-algebras is closed under the formation of direct products, direct sums, extensions, direct limits, pullbacks, pushouts, (some) tensor products, etc.

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An ideal P of A is said to be **primitive** *if P is the kernel of some irreducible representation of A. The* **primitive spectrum** *of A, which we denote by $\mathrm{Prim}(A)$, is the set of all primitive ideals of A equipped with the Jacobson topology. Hence, if S is some set of primitive ideals, its closure is*

$$\overline{S} = \left\{ P \in \mathrm{Prim}(A) \ : \ P \supseteq \bigcap_{Q \in S} Q \right\}.$$

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The good news

The primitive ideals separate the points of A. Further, $\mathrm{Prim}(A)$ is a locally compact space.

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The bad news

$\mathrm{Prim}(A)$ in general satisfies only T_0-separation axiom.

Approximations by elementary operators

Let $\mathrm{IB}(A)$ be the set of all bounded maps $\phi : A \to A$ that preserve (closed two-sided) ideals of A, i.e. $\phi(I) \subseteq I$ for all ideals I of A.

- For any ideal I of A, ϕ induces a map $\phi_I : A/I \to A/I$ which sends $a + I$ to $\phi(a) + I$.
- If S is any subset of ideals of A with zero intersection, the norm of ϕ can be computed by the formula $\|\phi\| = \sup\{\|\phi_I\| \ : \ I \in S\}$.

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The most prominent class of maps $\phi \in \mathrm{IB}(A)$ are the **elementary operators**, i.e. those that can be expressed as finite sums of two-sided multiplication maps $M_{a,b} : x \mapsto axb$, where a and b are elements of A (or more generally $M(A)$–the multiplier algebra of A).

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The important observation is that elementary operators are in fact completely bounded, i.e. $\sup_n \|\phi^{(n)}\| < \infty$, where for each n, $\phi^{(n)} : M_n(A) \to M_n(A)$ is the induced map that sends the matrix $[a_{ij}]$ to the matrix $[\phi(a_{ij})]$.

More precisely, we have the following estimate

$$\left\| \sum_i M_{a_i,b_i} \right\|_{cb} \leq \left\| \sum_i a_i \otimes b_i \right\|_h , \tag{1}$$

where $\| \cdot \|_h$ is the Haagerup tensor norm on $M(A) \otimes M(A)$, i.e.

$$\|t\|_h = \inf \left\{ \left\| \sum_i a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_i b_i^* b_i \right\|^{\frac{1}{2}} \ : \ t = \sum_i a_i \otimes b_i \right\} .$$

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Theorem (Haagerup 1980, Chatterjee-Sinclair 1992, Mathieu 2003)

The equality in (1) holds true for all elementary operators $\phi = \sum_i M_{a_i,b_i}$ if and only if A is a prime C^*-algebra.

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Remark

If the algebra A is not prime, then the map $a \otimes b \mapsto M_{a,b}$ is not even injective.

The **length** of an elementary operator $\phi \neq 0$ is the smallest positive integer $\ell = \ell(\phi)$ such that $\phi = \sum_{i=1}^{\ell} M_{a_i,b_i}$ for some $a_i, b_i \in M(A)$. We also define $\ell(0) = 0$.

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By $\mathcal{E}\ell(A)$ we denote the set of all elementary operators on A and by $\mathcal{E}\ell_k(A)$ the set of all $\phi \in \mathcal{E}\ell(A)$ with $\ell(\phi) \leq k$.

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Theorem (Timoney 2003, 2007)

For every $\phi \in \mathcal{E}\ell(A)$ of length ℓ we have

$$\|\phi\|_{cb} = \|\phi^{(\ell)}\| \leq \sqrt{\ell}\|\phi\|.$$

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Corollary

On each $\mathcal{E}\ell_k(A)$ the cb-norm is equivalent to the operator norm.

Question

Which operators $\phi \in \mathrm{IB}(A)$ can be approximated by elementary operators in the operator norm?

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Theorem (Magajna 2009)

If A is a separable C^*-algebra, then $\mathcal{E}\ell(A)$ is operator norm dense in $\mathrm{IB}(A)$ if and only if A can be decomposed as a finite direct sum $A = A_1 \oplus \cdots \oplus A_n$, where each summand A_i is homogeneous with the finite type property. In particular, in this case we have $\mathrm{IB}(A) = \mathcal{E}\ell(A)$.

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Remark

- A C^*-algebra A is called n-**homogeneous** if $A/P \cong \mathbb{M}_n$ for every $P \in \mathrm{Prim}(A)$. By a well-known theorem of Fell and Tomiyama-Takesaki, for any n-homogeneous C^*-algebra A with (primitive) spectrum X there is a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathbb{M}_n$ and structure group $\mathrm{Aut}(\mathbb{M}_n) = PU(n) = U(n)/\mathbb{S}^1$ such that A is isomorphic to the algebra $\Gamma_0(\mathcal{E})$ of sections of $\mathcal{E}$ which vanish at infinity.

Remark (continuation)

- Moreover, any two such algebras $A_i = \Gamma_0(\mathcal{E}_i)$ with spectra X_i are isomorphic if and only if there is a homeomorphism $f : X_1 \to X_2$ such that $\mathcal{E}_1 \cong f^*(\mathcal{E}_2)$ as bundles over X_1.
- An n-homogeneous C^*-algebra $\Gamma_0(\mathcal{E})$ with spectrum X is said to have the finite type property if $\mathcal{E}$ can be trivialized over some finite open cover of X.

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Theorem (G. 2011)

Let A be a separable C^*-algebra.

(a) *If $\mathcal{E}\ell(A)$ is norm closed, then A is necessarily subhomogeneous (i.e. $\sup\{\dim(A/P) \ : P \in \mathrm{Prim}(A)\} < \infty$) and each homogeneous sub-quotient of A has the finite type property.*

(b) *The converse is also true if $\mathrm{Prim}(A)$ is Hausdorff.*

(c) *There exists a compact subset X of $\mathbb{R}$ and a unital C^*-subalgebra A of $C(X, \mathbb{M}_2)$ such that $\mathcal{E}\ell(A)$ is not closed in the operator norm.*

Approximations by two-sided multiplications

Notation

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Theorem (G.-Timoney 2016)

Suppose that a, b, c and d are norm-one elements of a C^*-algebra A. If

$$\|a \otimes b - c \otimes d\|_h < \varepsilon \leq 1/3,$$

then there exists a scalar μ of modulus one such that

$$\max\{\|a - \mu c\|, \|b - \overline{\mu} d\|\} < 6\varepsilon.$$

Consequently, $\mathrm{TM}(A)$ is norm closed if A is a prime C^*-algebra.

In the sequel $A = \Gamma_0(\mathcal{E})$ will be a homogeneous C^*-algebra with the primitive spectrum X.

Proposition

For every $\phi \in \mathrm{IB}(A)$ the norm function $x \mapsto \|\phi_x\|$ is continuous on X. In particular, for each $\phi \in \overline{\overline{\mathrm{TM}(A)}}$ we have $(x \mapsto \|\phi_x\|) \in C_0(X)$.

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The next set may seem as the most obvious candidate for the norm closure of $\mathrm{TM}(A)$:

Notation

$\mathrm{PTM}(A) := \{\phi \in \mathrm{IB}(A) : (x \mapsto \|\phi_x\|) \in C_0(X) \ \& \ \phi_x \in \mathrm{TM}(A_x) \ \forall x \in X\}.$

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Proposition

If A is a homogeneous C^*-algebra, then the set $\mathrm{PTM}(A)$ is norm closed. In particular, we have $\overline{\overline{\mathrm{TM}(A)}} \subseteq \mathrm{PTM}(A)$.

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Auxiliary notation

- $\mathrm{TM}_{\mathrm{nv}}(A) = \{\phi \in \mathrm{TM}(A) \ : \ \phi_x \neq 0 \ \forall x \in X\}$;
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Theorem (G.-Timoney 2016)

Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra. To each operator $\phi \in \mathrm{PTM}_{\mathrm{nv}}(A)$ we can (canonically) associate a complex line subbundle $\mathcal{L}_\phi$ of $\mathcal{E}$ with the property that

$$\phi \in \mathrm{TM}_{\mathrm{nv}}(A) \quad \Longleftrightarrow \quad \mathcal{L}_\phi \text{ is a trivial bundle.}$$

Further, is X is σ-compact, then for every complex line subbundle $\mathcal{L}$ of $\mathcal{E}$ we can find an operator $\phi_{\mathcal{L}} \in \mathrm{PTM}_{\mathrm{nv}}(A)$ such that $\mathcal{L}_\phi = \mathcal{L}$.

If the base space X is paracompact, then the locally trivial complex line bundles over X are classified by the homotopy classes from X to $\mathbb{C}P^{\infty}$, and/or by the elements of the second integral Čech cohomology $\check{H}^2(X;\mathbb{Z})$

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For a homogeneous C^*-algebra $A = \Gamma_0(\mathcal{E})$ we define a map

$$\theta : \mathrm{PTM}_{\mathrm{nv}}(A) \to \check{H}^2(X;\mathbb{Z})$$

which sends an operator $\phi \in \mathrm{PTM}_{\mathrm{nv}}(A)$ to the corresponding class of the bundle $\mathcal{L}_\phi$ in $\check{H}^2(X;\mathbb{Z})$. Then $\theta^{-1}(0) = \mathrm{TM}_{\mathrm{nv}}(A)$ (by the latter theorem).

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Corollary

Let A be a homogeneous C^*-algebra. If $X = \mathrm{Prim}(A)$ is paracompact with $\check{H}^2(X;\mathbb{Z}) = 0$, then $\mathrm{PTM}_{\mathrm{nv}}(A) = \mathrm{TM}_{\mathrm{nv}}(A)$.

Theorem (G.-Timoney 2016)

Let X be a CH space with $\dim X \leq d < \infty$. For each $n \geq 1$ let $A_n = C(X, \mathbb{M}_n)$. If $p := \left\lceil \sqrt{(d+1)/2} \right\rceil$, then for every $n \geq p$ the mapping $\theta : \mathrm{PTM}_{\mathrm{nv}}(A) \to \check{H}^2(X;\mathbb{Z})$ is surjective. In particular, if $\check{H}^2(X;\mathbb{Z}) \neq 0$, then $\mathrm{TM}_{\mathrm{nv}}(A_n) \subsetneq \mathrm{PTM}_{\mathrm{nv}}(A_n)$ for all $n \geq p$.

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Corollary

If $X = \mathbb{S}^2$ *or* $X = \mathbb{S}^1 \times \mathbb{S}^1$, *then for* $A = C(X, \mathbb{M}_n)$ *we have* $\mathrm{TM}_{\mathrm{nv}}(A) \subsetneq \mathrm{PTM}_{\mathrm{nv}}(A)$ *for all $n \geq 2$.*

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If $X = \mathbb{S}^2$ or $X = \mathbb{S}^1 \times \mathbb{S}^1$, then for $A = C(X, \mathbb{M}_n)$ we have $\mathrm{TM}_{\mathrm{nv}}(A) \subsetneq \mathrm{PTM}_{\mathrm{nv}}(A)$ for all $n \geq 2$.

Theorem (G.-Timoney 2016)

Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra with $X = \mathrm{Prim}(A)$. Consider the following two conditions:

(a) *$\forall U \subset X$ open, each complex line subbundle of $\mathcal{E}|_U$ is trivial.*

(b) $\mathrm{PTM}(A) = \mathrm{TM}(A)$.

Then (a) $\Rightarrow$ (b). If A is separable, then (a) and (b) are equivalent.

Corollary

Let A be an n-homogeneous C^*-algebra with $n \geq 2$.

(a) *If X is second-countable with $\dim X < 2$, or if X is (homeomorphic to) a subset of a non-compact connected 2-manifold, then $\mathrm{PTM}(A) = \mathrm{TM}(A)$.*

(b) *If X is σ-compact and contains a nonempty open subset homeomorphic to (an open subset of) $\mathbb{R}^d$ for some $d \geq 3$, then $\mathrm{PTM}(A) \setminus \mathrm{TM}(A) \neq \emptyset$.*

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Let A be an n-homogeneous C^*-algebra with $n \geq 2$.

(a) *If X is second-countable with $\dim X < 2$, or if X is (homeomorphic to) a subset of a non-compact connected 2-manifold, then $\mathrm{PTM}(A) = \mathrm{TM}(A)$.*

(b) *If X is σ-compact and contains a nonempty open subset homeomorphic to (an open subset of) $\mathbb{R}^d$ for some $d \geq 3$, then $\mathrm{PTM}(A) \setminus \mathrm{TM}(A) \neq \emptyset$.*

Theorem (G.-Timoney 2016)

Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra. For an operator $\phi \in \mathrm{B}(A)$ the following two conditions are equivalent:

(a) $\phi \in \overline{\overline{\mathrm{TM}(A)}}$.

(b) $\phi \in \mathrm{PTM}(A)$ *and for* $\mathrm{coz}(\phi) := \{x \in X : \phi_x \neq 0\}$ *the bundle* $\mathcal{L}_\phi$ *is trivial on each compact subset of* $\mathrm{coz}(\phi)$.

Definition

A locally trivial fibre bundle $\mathcal{F}$ over a locally compact Hausdorff space X is said to be a **phantom bundle** *if $\mathcal{F}$ is not globally trivial, but is trivial on each compact subset of X.*

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Corollary

Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra. Then the set $\mathrm{TM}(A)$ is not norm closed if and only if there exists a σ-compact open subset U of X and a phantom subbundle of $\mathcal{E}|_U$.

Remark

Let G be a group and n a positive integer. Recall that a space X is called an **Eilenberg-MacLane** space of type $K(G,n)$, if it's n-th homotopy group $\pi_n(X)$ is isomorphic to G and all other homotopy groups trivial. If $n > 1$ then G must be abelian (since for all $n > 1$, the homotopy groups $\pi_n(X)$ are abelian). We state some basic facts about Eilenberg-MacLane spaces:

- There exists a CW-complex $K(G,n)$ for any group G at $n = 1$, and abelian group G at $n > 1$. Moreover such a CW-complex is unique up to homotopy type. Hence, by abuse of notation, it is common to denote any such space by $K(G,n)$.
- Given a CW-complex X, there is a bijection between its cohomology group $H^n(X;G)$ and the homotopy classes $[X, K(G,n)]$ of maps from X to $K(G,n)$.
- $K(\mathbb{Z},2) \cong \mathbb{C}P^\infty$. In particular, for each CW-complex X there is a bijection between $[X, K(\mathbb{Z},2)]$ and isomorphism classes of complex line bundles over X.

Example

Let us consider the Eilenberg-MacLane space $K(\mathbb{Q},1)$.

- The standard model of $K(\mathbb{Q},1)$ is the mapping telescope Δ of the sequence
$$\mathbb{S}^1 \xrightarrow{z} \mathbb{S}^1 \xrightarrow{z^2} \mathbb{S}^1 \xrightarrow{z^3} \cdots .$$
- Applying $H_1(-;\mathbb{Z})$ to the levels of this mapping telescope gives the system
$$\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 3} \cdots .$$
The colimit of this system is $H_1(\Delta;\mathbb{Z}) = \mathbb{Q}$ and all other integral homology groups are trivial. By the universal coefficient theorem for cohomology each integral cohomology group of Δ is trivial, except for $H^2(\Delta;\mathbb{Z})$ which is isomorphic to $\mathrm{Ext}(\mathbb{Q};\mathbb{Z}) \cong \mathbb{R}$.
- Hence, there exist uncountably many mutually nontrivial complex line bundles over X. Each such bundle $\mathcal{L}$ is a phantom bundle, since all restrictions of $\mathcal{L}$ over finite subcomplexes of Δ are trivial.

Conclusion

Since Δ is a 2-complex, $\mathcal{L}$ is a direct summand of a trivial bundle $\Delta \times \mathbb{C}^2$. In particular, if $A = C_0(\Delta, \mathbb{M}_2)$, then $\mathrm{TM}(A)$ is not norm closed.

Conclusion

Since Δ is a 2-complex, $\mathcal{L}$ is a direct summand of a trivial bundle $\Delta \times \mathbb{C}^2$. In particular, if $A = C_0(\Delta, \mathbb{M}_2)$, then $\mathrm{TM}(A)$ is not norm closed.

In private correspondence Prof. Mladen Bestvina (University of Utah) informed us that even inside $\mathbb{R}^3$ there are open subsets of type $K(\mathbb{Q}, 1)$. Using this observation we can show the following fact:

Theorem (G.-Timoney 2016)

Let A be an n-homogeneous C^*-algebra with $n \geq 2$.

(a) *If X is second-countable with $\dim X < 2$ or if X is (homeomorphic to) a subset of a non-compact connected 2-manifold, then $\mathrm{TM}(A)$ is not norm closed.*

(b) *If there is a nonempty open subset of X homeomorphic to (an open subset of) $\mathbb{R}^d$ for some $d \geq 3$, then $\mathrm{TM}(A)$ fails to be norm closed.*

Summary

Let A be a separable n-homogeneous C^*-algebra with $n \geq 2$ such that $\dim X = d < \infty$. If X is a CW-complex or a subset of a d-manifold, the following relations between $\mathrm{TM}(A)$, $\overline{\overline{\mathrm{TM}(A)}}$ and $\mathrm{PTM}(A)$ occur:

(a) If $d < 2$ we always have $\mathrm{TM}(A) = \overline{\overline{\mathrm{TM}(A)}} = \mathrm{PTM}(A)$.

(b) If $d = 2$ we have four possibilities:

- **(i)** $\mathrm{TM}(A) = \overline{\overline{\mathrm{TM}(A)}} = \mathrm{PTM}(A)$: e.g. if X is a subset of a non-compact connected 2-manifold
- **(ii)** $\mathrm{TM}(A) = \overline{\overline{\mathrm{TM}(A)}} \subsetneq \mathrm{PTM}(A)$: e.g. if $A = C(X, \mathbb{M}_n)$, where $X = \mathbb{S}^2$.
- **(iii)** $\mathrm{TM}(A) \subsetneq \overline{\overline{\mathrm{TM}(A)}} = \mathrm{PTM}(A)$: e.g. if $A = C_0(X, \mathbb{M}_n)$, where $X = \Delta$ is the standard model for $K(\mathbb{Q}, 1)$.
- **(iv)** $\mathrm{TM}(A) \subsetneq \overline{\overline{\mathrm{TM}(A)}} \subsetneq \mathrm{PTM}(A)$: e.g. for $A = C_0(X, \mathbb{M}_n)$, where X is the topological disjoint union of $\mathbb{S}^2$ and Δ.

(c) If $d > 2$ we always have $\mathrm{TM}(A) \subsetneq \overline{\overline{\mathrm{TM}(A)}} \subsetneq \mathrm{PTM}(A)$.