JOINT QUADRATIC BASE CHANGES OF ELLIPTIC SURFACES WITH TORSION $\mathbb{Z}/4\mathbb{Z}$ AND $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$

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Abstract. We study elliptic curves over $\mathbb{Q}(t)$ obtained from an elliptic surface by imposing two quadratic conditions, each of which produces a new point. By the Kani-Rosen decomposition, the joint cover is rational only when the branch loci of the two conditions meet; for two quadratic conditions this means that they share a root. We apply this to two surfaces.

For Elkies's rank-four K3 surface with torsion $\mathbb{Z}/4\mathbb{Z}$, a catalogue of 194 quadratic conditions contains exactly two such pairs. One recovers the rank-six curve of the authors; the other gives a second, inequivalent curve with torsion $\mathbb{Z}/4\mathbb{Z}$ and rank six.

For the universal curve with torsion $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, 42 conditions give exactly 21 rational joint covers, which yield seven curves of rank two. Two more arise from a tower of conditions. Five of these nine curves are the curves of Dujella, Kazalicki and Peral; the other four are not equivalent to any of them.

In both cases we determine the Mordell-Weil groups over $\mathbb{Q}(t)$ exactly, and we prove that the curves are pairwise inequivalent under invertible changes of parameter, even over $\overline{\mathbb{Q}}$.

1. Introduction

By the Mordell-Weil theorem, $E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r$ for an elliptic curve $E/\mathbb{Q}$, and by Mazur's theorem [20] the torsion subgroup is one of fifteen groups. For each of them one asks for curves of large rank, over $\mathbb{Q}$ and over $\mathbb{Q}(t)$; the current records are collected in Dujella's tables [3, 4].

A standard way to raise the rank over $\mathbb{Q}(t)$ is quadratic base change. Suppose a surface $E \to \mathbb{P}^1_t$ acquires a new point over $\mathbb{Q}(t)(\sqrt{Q})$ for a quadratic polynomial Q. If the new point has infinite order, the rank over $\mathbb{Q}(t)(\sqrt{Q})$ is at least one larger than over $\mathbb{Q}(t)$. If moreover the conic $w^2 = Q(t)$ has a rational point, then $\mathbb{Q}(t)(\sqrt{Q})$ is again a rational function field. Imposing two such conditions Q_1, Q_2 raises the rank by at least two over $\mathbb{Q}(t)(\sqrt{Q_1}, \sqrt{Q_2})$. To obtain a curve over a rational function field again, the curve with this function field must be rational.

Section 2 develops this systematically:

- a genus criterion for joint covers (Lemma 2.1);
- the character decomposition of the rank;
- exact ranks through injective specialization, via the criteria of Gusić and Tadić [11];
- two invariants of the j-map that detect inequivalence of families under invertible changes of parameter.

We then apply it to two classical situations.

Torsion $\mathbb{Z}/4\mathbb{Z}$. Curves of positive rank with this torsion were constructed by Kihara [15] and by Khoshnam and Moody [14]; see also [5]. Elkies [10] gave a K3 surface with torsion $\mathbb{Z}/4\mathbb{Z}$ and rank four over $\mathbb{Q}(t)$, and in [8] we made a number of its quadratic sections explicit. In [9] we combined two quadratic sections of it to obtain a curve of rank six, the current record for this

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torsion group over $\mathbb{Q}(t)$ to our knowledge [4], and proved by an injective specialization that its rank is exactly six.

Theorem A. *Let E_1, E_2 be the curves over $\mathbb{Q}(q)$ of Section 3.*

(1) *$E_i(\mathbb{Q}(q)) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^6$ for $i = 1, 2$.*
(2) *The specializations of E_1 at $q = 11/7$ and of E_2 at $q = -2$ are injective, and these curves over $\mathbb{Q}$ have rank six.*
(3) *E_1 and E_2 are not equivalent (Definition 2.3), even over $\mathbb{C}$.*
(4) *In a catalogue of 194 quadratic conditions on Elkies's surface, exactly two pairs have rational joint covers: the pair used for E_1 (which is the curve of [9]) and the pair used for E_2.*

For E_1, parts (1) and (2) were proved in [9]. We include the short proof because it applies verbatim to E_2, which is new.

Torsion $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. Here the universal curve has rank zero. Curves of rank one over $\mathbb{Q}(t)$ were found by Kulesz [17], Lecacheux [18], Dujella [2] and Woo [25], among others. Rank two was first reached by Dujella and Peral [7]. Dujella, Kazalicki and Peral [6] described five curves of rank exactly two and observed that the examples of MacLeod [19] are isomorphic to some of them.

Theorem B. *Let $U_1, \dots, U_9$ be the curves over $\mathbb{Q}(t)$ of Section 4.*

(1) *$U_i(\mathbb{Q}(t)) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}^2$ for every i.*
(2) *No two of them are equivalent (Definition 2.3), even over $\overline{\mathbb{Q}}$.*
(3) *U_1, U_2, U_5, U_8, U_6 are the five curves of [6, §8], and U_1, U_5, U_8 are the three curves of [7]. The curves U_3, U_4, U_7, U_9 are not equivalent to any of the curves of [7] or [6], nor therefore, by [6, §1.3], to the examples of MacLeod [19].*
(4) *In a catalogue of 42 quadratic conditions on the universal curve, exactly 21 pairs have rational joint covers, and they give exactly the classes $U_1, \dots, U_7$.*

The paper ends with two remarks on the $\mathbb{Z}/4\mathbb{Z}$ case (Section 3.6). First, an exhaustive search shows that neither E_1 nor E_2 has a further quadratic section whose x-coordinate is a constant times a divisor of the last Weierstrass coefficient (Proposition 3.3). Second, we exhibit a specialization of E_2 whose rank is seven, provided the 2-primary part of its Tate-Shafarevich group is finite.

2. The method

2.1. **Quadratic sections.** Let $E : y^2 = x^3 + a(t)x^2 + b(t)x$ be an elliptic curve over $K = \mathbb{Q}(t)$, and let $Q \in \mathbb{Q}[t]$ be squarefree and not a constant times a square. A *quadratic section* attached to Q is an $x_Q \in K$ with

$$x_Q^3 + a\, x_Q^2 + b\, x_Q \in Q \cdot K^{\times 2}.$$

It gives a point P_Q of E over $K(\sqrt{Q})$ that the nontrivial automorphism sends to $-P_Q$, or equivalently a point of the quadratic twist $E^{(Q)}$ over K. In particular, if P_Q has infinite order it is independent of $E(K)$.

2.2. **The genus of a joint cover.** For quadratic sections $Q_1, \dots, Q_m$ let $C_{Q_1,\dots,Q_m}$ be the curve with function field $K(\sqrt{Q_1}, \dots, \sqrt{Q_m})$. For a squarefree polynomial Q of degree 1 or 2, let $S(Q) \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ be the branch locus of $w^2 = Q(t)$:

$$S(Q) = \begin{cases} \{\text{the two roots of } Q\}, & \deg Q = 2, \\ \{\text{the root of } Q,\ \infty\}, & \deg Q = 1. \end{cases}$$

Lemma 2.1. *Let $Q_1, \dots, Q_m \in \mathbb{Q}[t]$ be squarefree of degree 1 or 2, with classes independent in $\overline{\mathbb{Q}}(t)^\times/\overline{\mathbb{Q}}(t)^{\times 2}$. Then $C = C_{Q_1,\dots,Q_m}$ is geometrically integral of genus*

$$g(C) = 2^{m-2}\left|S(Q_1) \cup \cdots \cup S(Q_m)\right| - (2^m - 1).$$

In particular:

1. *for $m = 2$, $g(C) = 0$ if and only if $S(Q_1) \cap S(Q_2) \neq \varnothing$; for two quadratics this means a common root;*
2. *for $m = 3$, $g(C) = 2\,|S(Q_1) \cup S(Q_2) \cup S(Q_3)| - 7$.*

Proof. The cover $C \to \mathbb{P}^1$ is Galois with group $(\mathbb{Z}/2)^m$, and C is geometrically integral by the independence assumption. By the Kani-Rosen decomposition [13], $J(C)$ is isogenous to the product of the Jacobians of the quotients by the subgroups of index two. These are the double covers $C_I : w^2 = \prod_{i\in I} Q_i$ for $\emptyset \neq I \subseteq \{1, \dots, m\}$. The branch locus of C_I is the symmetric difference of the sets $S(Q_i)$, $i \in I$, so $g(C_I) = \frac{1}{2}\left|\triangle_{i\in I} S(Q_i)\right| - 1$. A point lying in at least one $S(Q_i)$ belongs to the symmetric difference for exactly 2^{m-1} of the sets I, namely those meeting the set of i with that point in $S(Q_i)$ in an odd number of elements. Summing over I gives the formula. For $m = 2$ it reads $g = |S(Q_1) \cup S(Q_2)| - 3$, and the union has 3 or 4 elements. □

Two irreducible quadratics with a common root are proportional. So for families of pairwise non-proportional conditions, rational joint covers come only from split quadratics with a common root, or from conditions involving linear polynomials.

2.3. **The character decomposition.** Let $L = K(\sqrt{Q_1}, \sqrt{Q_2})$ with Q_1, Q_2 independent in $K^\times/K^{\times 2}$, and put $Q_3 = Q_1Q_2$. The isotypic decomposition of $E(L) \otimes \mathbb{Q}$ under $\mathrm{Gal}(L/K) \cong (\mathbb{Z}/2)^2$ gives

$$\operatorname{rank} E(L) = \operatorname{rank} E(K) + \operatorname{rank} E^{(Q_1)}(K) + \operatorname{rank} E^{(Q_2)}(K) + \operatorname{rank} E^{(Q_3)}(K). \tag{1}$$

If C_{Q_1,Q_2} is a conic with a rational point, then $L \cong \mathbb{Q}(s)$. Two quadratic sections of infinite order then raise the rank over $\mathbb{Q}(s)$ by at least two. When the rank over $\mathbb{Q}(s)$ is known exactly, (1) also determines the ranks of the three twists.

2.4. **Exact rank by injective specialization.** We use two criteria of Gusić and Tadić.

Theorem 2.2 ([11, Thms. 1.1 and 1.3])**.** *Let $E/\mathbb{Q}(t)$ be nonconstant and $t_0 \in \mathbb{Q}$.*

1. *If $E : y^2 = (x - e_1)(x - e_2)(x - e_3)$ with $e_i \in \mathbb{Z}[t]$, and $h(t_0)$ is not a rational square for every nonconstant squarefree divisor $h \in \mathbb{Z}[t]$ of $(e_1 - e_2)(e_1 - e_3)$, of $(e_2 - e_1)(e_2 - e_3)$, or of $(e_3 - e_1)(e_3 - e_2)$, then the specialization at t_0 is injective.*
2. *If $E : y^2 = x^3 + \mathcal{A}x^2 + \mathcal{B}x$ with $\mathcal{A}, \mathcal{B} \in \mathbb{Z}[t]$ has exactly one nontrivial rational 2-torsion point, and $h(t_0)$ is not a rational square for every nonconstant squarefree divisor $h \in \mathbb{Z}[t]$ of $\mathcal{B}$ or of $\mathcal{A}^2 - 4\mathcal{B}$, then the specialization at t_0 is injective.*

In all our applications we tested every squarefree product of the nonconstant irreducible factors, multiplied by every signed squarefree divisor of the integer contents. This is a superset of the divisors required. An injective specialization gives $\operatorname{rank} E(\mathbb{Q}(t)) \leq \operatorname{rank} E_{t_0}(\mathbb{Q})$, and the torsion of $E(\mathbb{Q}(t))$ injects into that of the fibre.

2.5. **Invariants of the j-map.**

Definition 2.3. Two elliptic curves E, E' over $k(t)$, for a field $k \supseteq \mathbb{Q}$, are *equivalent over k* if there is an invertible Möbius transformation μ of the t-line, defined over k, such that E and μ^*E' are isomorphic over $k(t)$.

Let $\varphi_1, \varphi_2 : \mathbb{P}^1 \to \mathbb{P}^1$ be base changes of an elliptic surface with j-invariant j. If $\varphi_1^* E$ and $\mu^* \varphi_2^* E$ are isomorphic over $\mathbb{C}(t)$ for some Möbius transformation μ, then $j \circ \varphi_1 = j \circ \varphi_2 \circ \mu$. Quadratic twists are included, since they do not change j. We use two consequences:

(i) $j \circ \varphi_1$ and $j \circ \varphi_2$ have the same critical values;
(ii) for each m, μ carries the poles of order m of $j \circ \varphi_1$ to those of $j \circ \varphi_2$.

For (ii), let $F(T, Z)$ be a binary form of degree 12 whose roots are these poles. With the transvectants [22]

$$(F, G)_r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r F}{\partial T^{r-i} \partial Z^i} \frac{\partial^r G}{\partial T^i \partial Z^{r-i}},$$

put $I_2 = (F, F)_{12}$, $H = (F, F)_{10}$, $I_4 = (H, H)_4$ and, whenever $I_2 \neq 0$,

$$\iota(F) = I_4 / I_2^2. \tag{2}$$

This quantity is unchanged by scaling F and by $\mathrm{GL}_2(\mathbb{C})$ acting on (T, Z), so equivalent base changes have equal values of ι. In all cases below, $I_2 \neq 0$.

3. Torsion $\mathbb{Z}/4\mathbb{Z}$: two rank-six curves

3.1. **Elkies's surface.** Put

$$a(t) = (8t-1)(32t+7), \qquad g(t) = (t+1)(15t-8)(31t-7),$$
$$A(t) = a^2 - 64g = 65536t^4 - 17472t^3 - 10176t^2 + 18672t - 3535, \qquad B(t) = 1024g^2,$$

and, following [9], write Elkies's surface [10] as

$$E: \quad y^2 = x^3 + A(t)x^2 + B(t)x. \tag{3}$$

Since $B = (32g)^2$ and $A + 64g = a^2$, the point $T = (32g,\, 32ag)$ satisfies $2T = (0, 0)$ and has order four. The discriminant is

$$\Delta = 2^{24}(t+1)^4(8t-1)^2(15t-8)^4(31t-7)^4(32t+7)^2(512t-113)(128t^3 - 64t^2 - 48t + 63).$$

The surface has four fibres of type I_4 (at $t = -1, 8/15, 7/31, \infty$), two of type I_2 and four of type I_1. It is K3, and the Shioda-Tate formula [24] bounds the geometric rank by $20 - 2 - 4 \cdot 3 - 2 = 4$. The sections

$$x(P_1) = -361(t+1)(31t-7), \qquad x(P_2) = -4(t+1)(15t-8)(16t-7)^2,$$
$$x(P_3) = -16(t+1)(15t-8)(8t+7)^2, \qquad x(P_4) = 4(15t-8)(31t-7)(16t+1)^2$$

are independent [9, 10], so $E(\mathbb{Q}(t)) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^4$.

3.2. **The catalogue.** Appendix A lists the split members of a catalogue of 194 rationally soluble quadratic sections $F_1, \ldots, F_{194}$ of (3). There are 50 split quadratics, 143 irreducible quadratics and one linear condition, F_{166}.

Proposition 3.1. *All* 194 *polynomials are squarefree, and no two are proportional. The only pairs with a common root are* $\{F_1, F_2\}$ *(root* $t = 4/7$*) and* $\{F_{79}, F_{124}\}$ *(root* $t = 134/111$*). Hence these are the only pairs with a joint cover of genus zero, and no triple has a joint cover of genus at most one.*

Proof. A common root can only occur among the 51 split or linear conditions, and comparing their roots gives the two pairs. The only linear condition, $F_{166} : w^2 = 18t + 7$, has branch locus $\{-7/18, \infty\}$, and no other condition vanishes at $-7/18$. So by Lemma 2.1(1) these are the only genus-zero pairs. In any triple at most one of the three pairs has intersecting branch loci, because the two pairs above are disjoint. Hence the union of the three branch loci has at least 5 points, and Lemma 2.1(2) gives genus at least 3. □

3.3. **The two curves.**

The pair F_1, F_2. Here

$$Q_1 = -(7t-4)(17t+4), \qquad Q_2 = (7t-4)(5551t-1324),$$

with

$$x_1 = \frac{64(t+1)^2 Q_1}{(4t+1)^2}, \qquad x_2 = \frac{576(15t-8)^2 Q_2}{49(28t-39)^2}.$$

The substitution $t = 4(U^2-49)/\big(7(U^2+119)\big)$ solves the first condition. The second becomes the conic $11U^2+V^2 = 1863$, which has the point $(13,2)$. With

$$K(q) = 72q^4 - 26q^3 - 271q^2 + 286q + 8712,$$

the line $V = 2 + q(U-13)$ gives

$$t_1(q) = \frac{4(3q^2-2q-110)(10q^2-2q-33)}{7K(q)}, \tag{4}$$

$$w_1 = \frac{24(13q^2-4q-143)(q^2+11)}{K(q)}, \qquad w_2 = \frac{168(11-143q-q^2)(q^2+11)}{K(q)}.$$

Since $4K(q) = (13q^2-4q-143)^2 + 119(q^2+11)^2 > 0$, t_1 is defined for all rational q. This is the curve of [9] with $r = 7q$.

The pair F_{79}, F_{124}. Here

$$Q_{79} = (111t-134)(1599t-1526), \qquad Q_{124} = -(111t-134)(5217t+1694),$$

with

$$x_{79} = \frac{28224(31t-7)^2 Q_{79}}{961(124t-273)^2}, \qquad x_{124} = \frac{64g^2 Q_{124}}{(111t-41)^2(444t^2-131t-119)^2}.$$

With $r = -37w_{79}/(111t-134)$ and $s = -w_{124}/(111t-134)$, eliminating t gives the conic $27r^2 + 1870s^2 = 444577$, and conversely

$$t = \frac{134s^2-1694}{111s^2+5217}, \qquad w_{79} = \frac{23976r}{111s^2+5217}, \qquad w_{124} = \frac{887112s}{111s^2+5217}.$$

The conic has the point $(-69,13)$. The line $s = 13 + q(r+69)$ gives

$$r(q) = \frac{1863 - 48620q - 129030q^2}{27+1870q^2}, \qquad s(q) = \frac{351+3726q-24310q^2}{27+1870q^2}.$$

With $R = 27+1870q^2$ and $S = 351+3726q-24310q^2$ this gives

$$t_2(q) = \frac{134S^2-1694R^2}{111S^2+5217R^2}. \tag{5}$$

Sections. The new points have $x = N/D^2$ and $y = Hw/D^3$, where $N(N^2+AND^2+BD^4) = H^2Q$:

	$N,\ D$	H
F_1	$64(t+1)^2Q_1,\ \ 4t+1$	$64(t+1)^2(8t-1)h_1$
F_2	$576(15t-8)^2Q_2,\ \ 7(28t-39)$	$192(8t-1)(15t-8)^2h_2$
F_{79}	$28224(31t-7)^2Q_{79},\ \ 31(124t-273)$	$1344(8t-1)(31t-7)^2h_{79}$
F_{124}	$64g^2Q_{124},\ \ (111t-41)(444t^2-131t-119)$	$64(t+1)^2(8t-1)(15t-8)^2(31t-7)^2h_{124}$

where

$$h_1 = 3448t^3 - 580t^2 - 749t - 100,$$
$$h_2 = 3756536t^3 - 6792380t^2 + 5347839t - 1050120,$$
$$h_{79} = 1093902456t^3 - 3605542420t^2 + 5214872859t - 2814810390,$$
$$h_{124} = 185997816t^5 - 919787292t^4 + 486783063t^3 + 128110240t^2 - 79350285t - 17513090.$$

Write $t_i = N_i/D_i$ with coprime integral polynomials:

$$N_1 = 4(3q^2 - 2q - 110)(10q^2 - 2q - 33),$$
$$D_1 = 7K(q),$$
$$N_2 = 2(339199300q^4 - 112385130q^3 - 2766301q^2 + 1622673q + 70713),$$
$$D_2 = 111(6993800q^4 - 1677390q^3 + 14477q^2 + 24219q + 1458).$$

With $a_i = (8N_i - D_i)(32N_i + 7D_i)$ and $g_i = (N_i + D_i)(15N_i - 8D_i)(31N_i - 7D_i)$, the curve $E_i = E_{t_i(q)}$ is isomorphic, via $X = D_i^4 x$, to

$$E_i : \; Y^2 = X^3 + \mathcal{A}_i X^2 + \mathcal{B}_i X, \qquad \mathcal{A}_i = a_i^2 - 64 g_i D_i, \quad \mathcal{B}_i = 1024 g_i^2 D_i^2. \tag{6}$$

3.4. **Exact rank and torsion.**

Injectivity. In (6), $\mathcal{A}_i^2 - 4\mathcal{B}_i = a_i^2(a_i^2 - 128 g_i D_i)$ is not a square, so Theorem 2.2(2) applies. $\mathcal{B}_1$ and $\mathcal{A}_1^2 - 4\mathcal{B}_1$ each have 9 distinct nonconstant irreducible factors; $\mathcal{B}_2$ has 6 and $\mathcal{A}_2^2 - 4\mathcal{B}_2$ has 9. We tested 8176 and 2044 values for E_1 at $q = 11/7$ ($t = 10676/136283$), and 1008 and 1022 values for E_2 at $q = -2$ ($t = 2104010001/2318620688$). None is a square, so both specializations are injective.

Lower bound. At a good prime p with $3 \mid M_p = \#E_{i,q_0}(\mathbb{F}_p)$, the map $P \mapsto [M_p/3]\bar{P}$ is a homomorphism to $E(\mathbb{F}_p)[3]$ that kills $3E(\mathbb{Q})$ and the torsion. At the primes used, the images of $P_1, \dots, P_4, P_Q, P_{Q'}$ lie in one cyclic subgroup, so each prime gives a row over $\mathbb{F}_3$. For E_1 we use $p = 17, 19, 41, 47, 61, 79$, and for E_2 we use $p = 7, 17, 23, 29, 41, 53$:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 & 2 & 1 \end{pmatrix}.$$

Both are invertible modulo 3. Rational 3-torsion is excluded since $\#E_{1,q_0}(\mathbb{F}_{29}) = 40$ and $\#E_{2,q_0}(\mathbb{F}_{43}) = 52$ are prime to 3. So the six points are independent on the fibres, and hence over $\mathbb{Q}(q)$.

Upper bound. For $C : y^2 = x^3 + \alpha x^2 + \beta x$ and $C' : y^2 = x^3 - 2\alpha x^2 + (\alpha^2 - 4\beta)x$ we have

$$\operatorname{rank} C(\mathbb{Q}) = \dim \kappa(C(\mathbb{Q})) + \dim \kappa'(C'(\mathbb{Q})) - 2,$$

where κ, κ' are the Kummer maps of the two-isogenies. The local images on C and C' are exact annihilators under the Hilbert pairing. So explicit local points on one side cut down the possible global classes on the other. For both fibres this leaves 16 classes on each side, giving rank at most $4 + 4 - 2 = 6$. The same bound is obtained independently by the two-isogeny descent of `ellrank` in PARI/GP [23].

Torsion. The reductions of $E_{1,11/7}$ at $p = 17, 43$ have 24 and 52 points, and those of $E_{2,-2}$ at $p = 7, 43$ have 12 and 52. Hence the fibre torsion is $\mathbb{Z}/4\mathbb{Z}$, and so is that of $E_i(\mathbb{Q}(q))$. This proves parts (1) and (2) of Theorem A.

Corollary 3.2. *For $(Q, Q') = (Q_1, Q_2)$ or (Q_{79}, Q_{124}), the twists $E^{(Q)}$ and $E^{(Q')}$ of (3) have rank one over $\mathbb{Q}(t)$, and $E^{(QQ')}$ has rank zero.*

Proof. Apply (1) with $\operatorname{rank} E(\mathbb{Q}(t)) = 4$ and total rank 6. □

3.5. **Inequivalence.** Write

$$j_E = \frac{F}{G}, \qquad F = (a^4 - 128a^2g + 1024g^2)^3, \qquad G = 4096\, g^4a^2(a^2 - 128g),$$

with $\deg F = 24$, and let $c = j_E(4/7)$. Since

$$t_1(q) - \frac{4}{7} = -\frac{24(q^2 + 11)^2}{K(q)},$$

t_1 ramifies over $4/7$, so c is a critical value of $j_E \circ t_1$. The map t_2 is branched only over $134/111$, $1526/1599$ and $-1694/5217$. Modulo 101 we find $j_E(4/7) \equiv 98$, while $j_E(134/111) \equiv 52$, $j_E(1526/1599) \equiv 38$ and $j_E(-1694/5217) \equiv 51$. Moreover, $F - 98G \in \mathbb{F}_{101}[t]$ has degree 24 and is coprime to its derivative. Hence c is neither the image of a branch point of t_2 nor a critical value of j_E, so it is not a critical value of $j_E \circ t_2$. By §2.5(i), this proves part (3) of Theorem A.

3.6. **Further remarks.**

No further quadratic sections of the simplest type.

Proposition 3.3. *Let $y^2 = x^3 + ax^2 + bx$ be either (6) for $i = 1, 2$ or its two-isogenous curve $y^2 = x^3 - 2\mathcal{A}_i x^2 + (\mathcal{A}_i^2 - 4\mathcal{B}_i)x$. Let m be any product of the irreducible factors of b in $\mathbb{Z}[q]$, with exponents at most their multiplicities, and let $c \in \mathbb{Q}^\times$. If $x = c\,m$ and $x^3 + ax^2 + bx$ has squarefree part of degree at most four, then this squarefree part is constant; that is, the point is defined over a constant field extension of $\mathbb{Q}(q)$, or x is a 2-torsion point. In particular neither E_1 nor E_2 has a further quadratic section of this form, nor one whose joint cover has genus one.*

Sketch of proof. Write $P_c = c^2m^2 + cam + b$, so that $x^3 + ax^2 + bx = c\,m\,P_c$. The proof has three steps.

Modular sieve. Let p be a prime at which the factors of b keep their degrees and stay distinct and squarefree. If the squarefree part over $\mathbb{Q}$ has degree at most 4, so does that of the reduction modulo p of the primitive part. If $c \equiv 0$ or $c \equiv \infty$ modulo p, this reduction is mb, respectively m^3, up to constants. So a divisor m is excluded for all $c \in \mathbb{Q}^\times$ as soon as some such p exists at which all $p + 1$ residues of c in $\mathbb{P}^1(\mathbb{F}_p)$ fail. Over the four curves this excludes 17890 of the 32076 divisors.

Exact step. For each of the remaining 14186 divisors, let $G = \gcd(m^2, am, b)$ and $P_c = G\,P_c'$. For c outside the rational roots of the leading coefficient of P_c', of $\operatorname{Res}_q(P_c', f)$ for the irreducible factors f of mG, and of the polynomials attached to the factors of $\operatorname{Res}_c(P_c', \partial_q P_c')$, the polynomial P_c' is squarefree, coprime to mG and of full degree. The squarefree part then has degree at least 16 in every case. Among these values, every one for which the squarefree part has degree at most four gives a constant squarefree part, or a 2-torsion point.

Pencil cases. For the two divisors $m = a_i^2 - 128g_iD_i$ and $m = a_i^2$ (up to constants) on each isogenous side, the polynomial P_c' factors as $(c - c_0)(cU + V)$. These were treated in the same way with Res_c replaced by $UV' - VU'$; the only special value is $c = c_0$, for which x is a 2-torsion point.

□

By Proposition 3.1, the catalogue also contains no triple leading to a genus-one cover.

A conditional rank-seven specialization. At $q = 3/94$, i.e. $t = 252401754905/264613965267$, the fibre of E_2 is isomorphic to $y^2 = x^3 + a_2x^2 + a_4x$ with
$a_2 = 216235610937917987314078524977683476459039881217 45$,
$a_4 = 190444163880566151149219994674452704265354195701912993632780660417928957472939925276$
3025788264054784.

For this curve:

- the six sections remain independent;
- the two-isogeny descent gives rank at most 7;
- the 2-Selmer group has dimension 8 (computed with `ell2cover`);
- the root number is -1.

All of these were computed with PARI/GP [23]. As a check, the same computation at the fibre $q = -2$, whose rank is proved to be six, gives a 2-Selmer group of dimension 7. Since $E(\mathbb{Q})[2^\infty] \cong \mathbb{Z}/4\mathbb{Z}$, the 2^∞-Selmer corank s satisfies $6 \le s \le 7$. By the 2-parity theorem [21, 1], s is odd, so $s = 7$. Hence the rank is seven unless $Ш(E/\mathbb{Q})[2^\infty]$ is infinite. We have not found a seventh point.

4. Torsion $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$: nine curves of rank two

4.1. **The universal curve.** Let

$$\begin{aligned} E_k:\ y^2 &= x\big(x-(k-1)^3(k+3)\big)\big(x-(k+1)^3(k-3)\big) \\ &= x^3 - 2(k^4-6k^2-3)x^2 + (k^2-1)^3(k^2-9)\,x \end{aligned} \tag{7}$$

[16]. Besides the points of order two, it has the point $T_3 = \big((k^2-1)^2,\ 4(k^2-1)^2\big)$ of order three. (7) is Hadano's model [12] with $a = 1$, $b = (k^2-1)/4$, $x = 16X$. It is isomorphic over $\mathbb{Q}(v)$ to the model $y^2 = x^3 + A_{26}(v)x^2 + B_{26}(v)x$ of [7, 6] with $k = (3v-5)/(v+1)$.

The discriminant is $2^{12}k^2(k^2-1)^6(k^2-9)^2$, with fibres I_6 at $k = \pm1, \infty$ and I_2 at $k = 0, \pm3$. The trivial lattice of this K3 surface has rank $2 + 15 + 3 = 20$, so $\operatorname{rank} E_k(\overline{\mathbb{Q}}(k)) = 0$. The j-invariant is

$$j(k) = \frac{(k^2+3)^3(k^6-15k^4+75k^2+3)^3}{k^2(k-3)^2(k-1)^6(k+1)^6(k+3)^2}.$$

The group G of Möbius maps $k \mapsto \pm k$, $\pm(k+3)/(k-1)$, $\pm(k-3)/(k+1)$ has order six. For each $\sigma \in G$, $E_{\sigma(k)} \cong E_k$ over $\mathbb{Q}(k)$; for example, for $u = (k+3)/(k-1)$,

$$x = \tfrac{(k-1)^4}{16}x_u + (k+1)^3(k-3), \qquad y = \tfrac{(k-1)^6}{64}y_u.$$

4.2. **The catalogue and the joint covers.** Appendix B lists 42 quadratic conditions $C_1, \dots, C_{42}$, one in each orbit of $k \mapsto -k$. Each has a rational parametrization and a point of infinite order. We write C_i^+ for $Q_i(k)$ and C_i^- for $Q_i(-k)$.

Proposition 4.1. *Fourteen of the conditions are even, namely C_i for*

$$i \in \{3, 4, 5, 6, 7, 10, 11, 12, 13, 18, 28, 30, 33, 38\},$$

and for these $C_i^+ = C_i^-$. After identifying these repetitions, the signed catalogue contains 70 *distinct conditions, no two of them proportional. Among unordered pairs of distinct conditions,* 84 *have a common root. These form* 42 *classes under simultaneous reflection $k \mapsto -k$, and exactly* 21 *of the classes have rationally soluble joint covers. These* 21 *form seven orbits of three under G (Table 1).*

Proof. For $Q_1 = c_1(k-r)(k-a)$ and $Q_2 = c_2(k-r)(k-b)$, put $s = 1/(k-r)$; the joint cover becomes the conic

$$\frac{r-b}{c_1}u^2 - \frac{r-a}{c_2}w^2 = a - b.$$

We decided its solubility with `qfsolve` [23]. Two pairs are equivalent when some $\sigma \in G$ maps the subgroup $\langle Q_1, Q_2\rangle \subset \mathbb{Q}(k)^\times/\mathbb{Q}(k)^{\times 2}$ onto the other subgroup. Since $E_{\sigma(k)} \cong E_k$, equivalent covers give equivalent curves. □

TABLE 1. The 21 rational joint covers, grouped into classes.

Class	Pairs		
U_1	C_4^+/C_{20}^+,	C_9^+/C_{13}^+,	C_9^+/C_{20}^-
U_2	C_4^+/C_{22}^+,	C_9^+/C_{21}^-,	C_9^+/C_{24}^-
U_3	C_7^+/C_{21}^+,	C_{19}^+/C_{22}^+,	C_{19}^+/C_{24}^-
U_4	C_8^+/C_{41}^+,	C_8^+/C_{42}^-,	C_{12}^+/C_{40}^+
U_5	C_{13}^+/C_{24}^+,	C_{20}^+/C_{21}^+,	C_{20}^+/C_{22}^+
U_6	C_{14}^+/C_{15}^+,	C_{15}^+/C_{17}^+,	C_{16}^+/C_{18}^+
U_7	C_{34}^+/C_{35}^+,	C_{35}^+/C_{37}^+,	C_{36}^+/C_{38}^+

4.3. **The nine curves.** For $U_1, \dots, U_7$ we take the first pair of each class; Table 2 gives the degree-four map $k = k_i(t)$. U_i is the pullback $E_{k_i(t)}$, with the two sections whose x-coordinates are in Appendix B.

Table 2: Representatives $U_1, \dots, U_7$.

	Conditions	$k_i(t)$
U_1	$C_4^+,\ C_{20}^+$	$\dfrac{-t^4+24t^3-152t^2+336t-196}{t^4-12t^3+62t^2-168t+196}$
U_2	$C_4^+,\ C_{22}^+$	$\dfrac{t^4+8t^3-56t^2+48t+36}{t^4-4t^3+10t^2-24t+36}$
U_3	$C_7^+,\ C_{21}^+$	$\dfrac{5(t^4+36)}{(t^2-6t+6)(t^2+6t+6)}$
U_4	$C_8^+,\ C_{41}^+$	$\dfrac{3(t^4-2t^2+4)}{2(t^2-t+2)(t^2+t+2)}$
U_5	$C_{13}^+,\ C_{24}^+$	$\dfrac{5t^4+8t^3-2t^2-8t+5}{3(t-1)(t+1)(t+3)(3t-1)}$
U_6	$C_{14}^+,\ C_{15}^+$	$\dfrac{-(3t^2-7)(7t^2-3)}{(3t^2-5t+3)(3t^2+5t+3)}$
U_7	$C_{34}^+,\ C_{35}^+$	$\dfrac{-3(3t^4+7t^2+27)}{(t^2+7)(7t^2+9)}$

The condition $C_{29}^- : v^2 = 9k^2 - 22k + 49$ has

$$x = \frac{(k-1)^2(k^2-2k+13)^2}{(k-3)^2}, \qquad k = \frac{t^2+12t-13}{t^2-9}.$$

With $d = t^2 - 9$ and $q_0 = t^2 + 6t - 11$, the pullback is isomorphic, via $x_{29} = d^4x/16$ and $y_{29} = d^6y/64$, to

$$C_{29}: \ y^2 = x^3 + a_{29}x^2 + b_{29}x,$$

where

$$a_{29} = t^8 + 12t^7 - 40t^6 - 1188t^5 - 1134t^4 + 13860t^3 - 13360t^2 - 5772t + 9157,$$
$$b_{29} = -16(t-7)(t-2)(t+1)(t+5)(3t-1)^3q_0^3.$$

It has the section with

$$x = 4(t+5)(3t-1)^3q_0, \qquad y = 4(t-1)(t+3)(t+5)(3t-1)^3(t^2+2t+17)q_0.$$

On C_{29} we impose two further conditions:

$$D_1: \ w^2 = 4t^2 + 19t - 29, \qquad x = \frac{(t+1)(t+5)(3t-1)^3q_0}{4};$$

$$D_2: \ w^2 = 3(t-3), \qquad x = -\frac{64(t-7)(t-2)(9t^3 - 5t^2 - 5t + 73)^2}{(t^2 - 5t + 22)^2}.$$

They are parametrized by

$$t = T_1(u) = \frac{5u^2 - 8u + 96}{4(u-2)(u+2)}, \qquad t = T_2(u) = 3(u^2+1).$$

We put $U_8 = C_{29}(T_1(u))$ and $U_9 = C_{29}(T_2(u))$.

Two further conditions give nothing new.

- $D_3 : w^2 = -2(t^2+8t-17)$ is parametrized by $t = T_3(u) = -(3u^2+136u+170)/\big(11(u^2+2)\big)$, and
$$k_{29}\Big(T_3\Big(\frac{16-18u}{5u+20}\Big)\Big) = \sigma\big(k_{29}(T_1(u))\big), \qquad \sigma(k) = \frac{k+3}{k-1}.$$
- $D_4 : w^2 = -6(t+3)(3t-1)$ is parametrized by $t = T_4(u) = (6-3u^2)/(u^2+18)$, and $k_{29}(T_4(4/u)) = k_{29}(T_2(u))$.

Here $k_{29}(t) = (t^2+12t-13)/(t^2-9)$. These are identities of rational functions of degree four, and $\sigma \in G$, so D_3 gives U_8 and D_4 gives U_9. The map T_2 ramifies over $t = 3, \infty$, which lie over the I_6 fibres $k = \infty, 1$, so U_9 has two fibres of type I_{12}.

4.4. **Exact rank and torsion.** For $k = N/D$ the curve has the integral model

$$y^2 = x\big(x - (N-D)^3(N+3D)\big)\big(x - (N+D)^3(N-3D)\big), \tag{8}$$

to which Theorem 2.2(1) applies. For each U_i we found the value t_0 below at which the criterion holds, testing all three products. At this t_0 the two sections are independent and `ellrank` [23] gives rank exactly two. The fibre torsion is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9
t_0	$-2/5$	$-5/7$	$-1/6$	$-3/8$	$6/5$	$-12/7$	$-1/4$	$-3/5$	$-3/5$

This proves part (1) of Theorem B. We do not claim that the given sections generate the Mordell-Weil group; by injectivity, that question reduces to the fibres above, as in [6].

Corollary 4.2. *Let Q, Q' be the conditions defining one of $U_1, \ldots, U_7$. Over $\mathbb{Q}(k)$, $E_k^{(Q)}$ and $E_k^{(Q')}$ have rank one, and $E_k^{(QQ')}$ has rank zero. C_{29} has rank one over $\mathbb{Q}(t)$, and so do its twists by $4t^2 + 19t - 29$ and $3(t-3)$.*

Proof. For $U_1, \dots, U_7$, apply (1) with $\operatorname{rank} E_k(\mathbb{Q}(k)) = 0$ and total rank 2. For U_8, U_9, apply the same argument to the quadratic extension $\mathbb{Q}(u)/\mathbb{Q}(t)$. □

4.5. **Inequivalence.** For $U_1, \dots, U_8$ the map $k(t)$ is unramified over $k = \pm 1, \infty$. So the order-six poles of $j \circ k$ are the twelve roots of $F = D(N^2 - D^2)$, with N, D homogenized. The invariant (2) takes the values

Class	ι
U_1	3290567866923750/1250489503032121
U_2	5728687791846/2550450522169
U_3	213643125/49441568
U_4	201327525/72673568
U_5	7026563975136/1786026434929
U_6	2810533160401893654/1145523935643327361
U_7	2515760648404915433712541931745702135015783747 61
U_8	704606945786056950/881192578069527889

These eight values are distinct. The pole divisor of $j \circ k$ for U_9 consists of two poles of order 12, eight of order 6 and twelve of order 2; for $U_1, \dots, U_8$ it consists of twelve poles of order 6 and twelve of order 2. By §2.5(ii), this proves part (2) of Theorem B. Together with Proposition 4.1, it also proves part (4).

4.6. **Comparison with earlier constructions.** The three curves of [7] are obtained as follows.

- The first two come from $v = 3/(5-w^2)$ followed by $2w^2-7 = \square$, respectively $5w^2-4 = \square$. Under $k = (3v-5)/(v+1)$ these are the pairs C_9^+/C_{20}^- (class U_1) and C_{20}^-/C_{21}^-, the reflection of C_{20}^+/C_{21}^+ (class U_5).
- The third, built from Hadano's model, is the curve of [6, §8.4], and hence equivalent to U_8 by the table below. Directly, with $k_{DP}(u)$ the corresponding map to the k-line, $k_{DP}\big(-(9u+12)/(2u+16)\big) = \sigma\big(k_{29}(T_1(u))\big)$ with $\sigma(k) = -(k-3)/(k+1)$.

For the five curves of [6, §8], with u their parameter:

[6]	Class	Change of parameter	$\sigma \in G$
§8.1	U_1	$t = u$	$-(k-3)/(k+1)$
§8.2	U_2	$t = u$	$-(k-3)/(k+1)$
§8.3	U_5	$t = (u-6)/(u-18)$	$-(k+3)/(k-1)$
§8.4	U_8	$u_{U_8} = 4(7u+24)/(19u+18)$	$(k+3)/(k-1)$
§8.5	U_6	$t = (u-1)/(u+1)$	k

In each line, write $v(u)$ for the parametrization of [6] and k_{U_i} for our map. Then $(3v(u) - 5)/(v(u)+1) = \sigma\big(k_{U_i}(t(u))\big)$ is an identity of rational functions of degree four. Since $E_{(3v-5)/(v+1)}$ is isomorphic over $\mathbb{Q}(v)$ to the model of [6], and $E_{\sigma(k)} \cong E_k$ over $\mathbb{Q}(k)$, the curves are isomorphic over $\mathbb{Q}(u)$, not merely twists of each other. By [6, §1.3], MacLeod's examples [19] are isomorphic to the first three of these curves. Hence U_3, U_4, U_7, U_9 are not equivalent to any of these curves, which proves part (3) of Theorem B.

Appendix A. Split quadratic sections of Elkies's surface

The table lists the 50 split conditions and the linear condition F_{166} of the $\mathbb{Z}/4\mathbb{Z}$ catalogue, in its original numbering.

Label	$Q_i(t)$
F_1	$-(7t-4)(17t+4)$
F_2	$(7t-4)(5551t-1324)$
F_{47}	$(50209t+18034)(553554225t+1143732242)$
F_{48}	$(87t-56)(346262175t-569042072)$
F_{49}	$(2431t-382)(26801775t-28655582)$
F_{50}	$(95t-22)(855t-623)$
F_{51}	$(928t-1241)(14848t-12881)$
F_{52}	$(3289t+967)(3289t+4792)$
F_{53}	$(399t-128)(9975t+1543)$
F_{54}	$(104t-113)(6656t-1793)$
F_{55}	$(1408t+13049)(90112t+22961)$
F_{56}	$(5880t+5041)(7680t+4081)$
F_{57}	$(552t-553)(317952t-372673)$
F_{58}	$(11968t-3017)(765952t-274673)$
F_{59}	$1793(4352t-3019)(69632t-314843)$
F_{67}	$(3615t+238)(5135t-1138)$
F_{71}	$(7089t+2942)(19185t+15038)$
F_{75}	$(6041t+5573)(44849t+12349)$
F_{79}	$(111t-134)(1599t-1526)$
F_{95}	$(4t-421)(4t-1)$
F_{109}	$(2279t-2563)(13039t-13483)$
F_{111}	$(113152t-32369)(12864512t+2563111)$
F_{112}	$-(409088t+89489)(1024095232t-953498393)$
F_{116}	$(69719t-26293)(3242879t-568621)$
F_{122}	$(t+1)(1537t+889)$
F_{124}	$-(111t-134)(5217t+1694)$
F_{125}	$(2644575t-748727)(140839575t-17847503)$
F_{129}	$-(181185t+40187)(757335t-187763)$
F_{131}	$-(407745t+89747)(4286295t-881483)$
F_{132}	$-(266305t+1385482)(927095t-4164322)$
F_{141}	$2(1432t-1943)(18431t-17119)$
F_{142}	$(8t+1175)(392t+167)$
F_{143}	$-(5888t+2513)(13919968t-71114057)$
F_{144}	$(5888t-1961)(1849568t-559721)$
F_{145}	$2(24t-7)(57615t-20288)$
F_{146}	$-(9728t+2753)(558837120t-385598929)$
F_{147}	$(9728t-1841)(2881920t-783889)$
F_{149}	$2(2072t-359)(595999t-356953)$
F_{150}	$(7688t-2921)(13448t-2681)$
F_{151}	$(36608t+17033)(1032450848t+816636473)$
F_{152}	$(36608t-13601)(500326112t-387820889)$
F_{153}	$2(9t+77)(680t+239)$
F_{154}	$2(49t+22)(872t+791)$
F_{155}	$-2(25t+7)(5592t-3199)$
F_{166}	$18t+7$
F_{189}	$(356168t+322679)(1417928t+945959)$
F_{190}	$(6083072t+5817847)(11727872t+7052647)$

Label	$Q_i(t)$
F_{191}	$(763466888t - 133116361)(1167217928t - 92109241)$
F_{192}	$(1169538248t + 535858319)(3207311048t + 281136719)$
F_{193}	$(7120209613952t - 1606644901673)(7202855130752t - 1588566194873)$
F_{194}	$(533496016772168t + 414269992621679)(563331048336968t + 410540613676079)$

Appendix B. Quadratic conditions on the universal $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ curve

For each i the conic $v^2 = Q_i(k)$ has a rational point, and $x_i(k)$ is the x-coordinate of a point of infinite order on E_k over $\mathbb{Q}(k)(\sqrt{Q_i})$.

	$Q_i(k)$	$x_i(k)$
C_1	$3(k^2 - 4k + 7)$	$\frac{(k-3)(k+1)^2(k+3)}{3}$
C_2	$k^2 - 4k - 1$	$(k-3)(k+1)^2(k+3)$
C_3	$3(3k^2 + 1)$	$(k+1)^3(k+3)$
C_4	$-3(k-2)(k+2)$	$-\frac{(k-3)(k-1)(k+1)(k+3)}{3}$
C_5	$5 - k^2$	$-(k-3)(k-1)(k+1)^2$
C_6	$k^2 + 3$	$(k-3)(k-1)(k+1)^2$
C_7	$3(k-5)(k+5)$	$27(k-3)(k+3)$
C_8	$2(k-9)(2k-3)$	$\frac{(k-1)(k+1)^2(2k-3)}{2}$
C_9	$3(k+5)(3k-1)$	$(k-3)(k-1)^2(k+5)$
C_{10}	$3(3k^2 - 2)$	$\frac{k^2(3k^2-2)}{3}$
C_{11}	$5k^2 + 4$	$\frac{(k-1)^2(k+1)^2(5k^2+4)}{k^2}$
C_{12}	$(5k-3)(5k+3)$	$\frac{(k-3)(k-1)^2(k+1)^2(k+3)}{k^2}$
C_{13}	$(3k-1)(3k+1)$	$\frac{(k-1)^3(k+1)^3}{k^2}$
C_{14}	$-3(3k+7)(11k-16)$	$\frac{(k+1)(3k-5)^2(3k+7)}{27}$
C_{15}	$3(k+4)(3k+7)$	$\frac{(k+1)(k+9)^2(3k+7)}{3}$
C_{16}	$6(5k-49)(5k+1)$	$\frac{27(k-3)(k+1)^2(k+3)}{32}$
C_{17}	$3(k+4)(27k-17)$	$\frac{(k+1)^3(27k-17)}{27}$
C_{18}	$6(5k-1)(5k+1)$	$\frac{3(k-3)(k-1)^2(k+1)^2(k+3)}{128k^2}$
C_{19}	$-6(k-2)(3k-1)$	$\frac{(k+1)^3(3k-7)^2(3k-1)}{27(k-1)^2}$
C_{20}	$2(k-5)(k-2)$	$\frac{(k-5)(k-1)^3(k+1)^2}{(k-3)^2}$
C_{21}	$3(k-5)(3k-5)$	$\frac{(k-3)(k-1)^3(k+1)^2}{(k-2)^2}$
C_{22}	$6(k-7)(k-2)$	$\frac{3(k-2)(k-1)^2(k+1)}{2}$
C_{23}	$3(3k^2 - 16k + 25)$	$(k-7)^2(k-3)(k-1)$
C_{24}	$6(2k-1)(3k+1)$	$\frac{(k+1)^2(k+3)(2k-1)}{2}$
C_{25}	$(k-9)(5k+3)$	$\frac{(k-1)^3(5k+3)}{4}$
C_{26}	$9k^2 - 12k - 121$	$(k-9)^2(k-3)(k+3)$

	$Q_i(k)$	$x_i(k)$
C_{27}	$3(k^2-22k+25)$	$\frac{(k+1)^2(7k-9)^2}{48}$
C_{28}	$-(2k-3)(2k+3)$	$(k^2-3)^2$
C_{29}	$9k^2+22k+49$	$\frac{(k+1)^2(k^2+2k+13)^2}{(k+3)^2}$
C_{30}	$3(11k^2+1)$	$3(k-1)(k+1)(11k^2+1)$
C_{31}	$-31k^2+68k-1$	$-(k-3)(k-1)(31k^2-68k+1)$
C_{32}	$-25k^2+80k-19$	$-(k-3)(k-1)(7k-1)^2$
C_{33}	$3(k-9)(k+9)$	$\frac{(k-3)(k+3)(k^2+63)^2}{243}$
C_{34}	$-(7k+9)(16k-33)$	$-\frac{(k+3)(5k-9)^2(7k+9)(17k-81)^2}{729(37k-81)^2}$
C_{35}	$-(4k+3)(7k+9)$	$-\frac{(k+3)(3k+1)^2(7k+9)(15k^2+33k+16)^2}{(47k^2+93k+48)^2}$
C_{36}	$2(k+15)(49k-15)$	$\frac{729(k-1)^3(k+1)^3(59k-15)^2}{32(275k^2-192k+45)^2}$
C_{37}	$(4k+3)(17k-81)$	$\frac{(k+3)(17k-81)(65k^2-177k-144)^2}{729(11k-48)^2}$
C_{38}	$2(k-15)(k+15)$	$\frac{(k-1)^3(k+1)^3(k^2+375)^2}{128(3k^2+125)^2}$
C_{39}	$-6(2k-3)(5k-3)$	$-\frac{(k-3)(5k-3)(19k^2-36k+9)^2}{243(k-1)^2}$
C_{40}	$-(5k-9)(5k-3)$	$-\frac{(k-3)(k-1)^3(11k-21)^2}{(2k-3)^2}$
C_{41}	$-2(2k-3)(7k-3)$	$\frac{(k+3)(2k-3)(8k^2-9k+3)^2}{2(8k-3)^2}$
C_{42}	$2(k-6)(k+9)$	$\frac{(k-6)(k+1)^3(7k-27)^2}{2(5k-27)^2}$

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AI systems were used to optimise the programs and convert the resulting data into a readable format. The authors are responsible for the final manuscript.

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