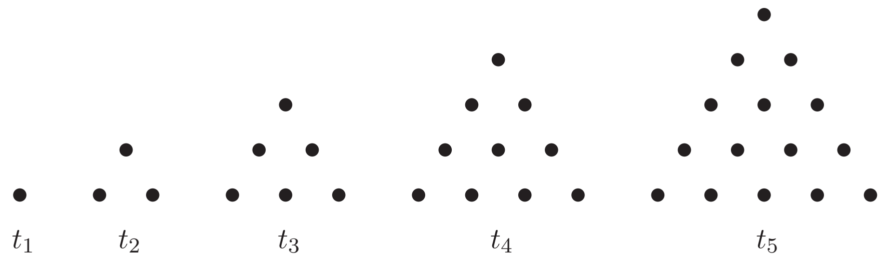

Theorem 5.21 (Gauss). *Every positive integer n can be written as a sum of three triangular numbers.*

Proof: By Theorem 5.21, the number $8n+3$ can be written as a sum of three squares

$$8n+3 = x_1^2 + x_2^2 + x_3^2,$$

where the numbers x_1, x_2, x_3 are all odd (because squares of integers are congruent to 0, 1 or 4 modulo 8). Let $x_1 = 2m_1 + 1$, $x_2 = 2m_2 + 1$, $x_3 = 2m_3 + 1$. Then

$$8n+3 = 4m_1(m_1+1) + 4m_2(m_2+1) + 4m_3(m_3+1) + 3,$$

so

$$n = \frac{m_1(m_1+1)}{2} + \frac{m_2(m_2+1)}{2} + \frac{m_3(m_3+1)}{2}. \qquad \square$$

In analogy with triangular and square numbers, we can also define pentagonal, hexagonal and, generally, m-gonal numbers (see [88, 104]). For example, the first few pentagonal numbers are $0, 1, 5, 12, 22, 35, \ldots$ If we denote by $P_m(n)$ the n-th m-gonal number, then $P_m(n) = (m-2)\frac{n(n-1)}{2} + n$. By Theorems 5.21 and 5.14, we know that for $n = 3$ and $n = 4$, every positive integer can be written as a sum of n n-gonal numbers. It can be shown that this statement holds for any integer $n \geq 3$. This statement was conjectured by Fermat and first proved by Cauchy, in the stronger form that every positive integer is a sum of n n-gonal numbers, out of which at most four are different from 0 or 1. The proof can be found in [104, Chapter 5].

5.5 Exercises

1. Which of the following numbers can be represented as a sum of two squares: 135, 343, 8450?

2. Factorize $1\,000\,009$ to prime factors, assuming that it is known that:

$$1000009 = 235^2 + 972^2.$$

3. Find all representation of n in the form of a sum of squares of two integers:

   a) $n = 85$,

   b) $n = 325$,

   c) $n = 1105$.

4. Prove that out of four consecutive positive integers at least one of them cannot be represented as a sum of two squares.

5. Let m and n be positive integers which can be represented as a sum of squares of two positive integers, and the number $m \cdot n$ cannot be represented in that form. Prove that then $m \cdot n$ is the square of an even integer.

6. Prove that for every positive integer k, the equation $x^2 + y^2 = z^k$ has a solution $(x, y, z) \in \mathbb{N}^3$.

7. Find all representations of n in the form of a difference of squares of two integers:

   a) $n = 99$,

   b) $n = 111$,

   c) $n = 200$.

8. Determine the centre of the group Γ, i.e. the set

$$C = \{c \in \Gamma \,:\, cg = gc,\ \forall g \in \Gamma\}.$$

9. Find a reduced form equivalent to:

   a) $7x^2 + 25xy + 23y^2$,

   b) $143x^2 + 120xy + 26y^2$,

   c) $117x^2 + 146xy + 46y^2$.

10. Prove that $h(d) = 1$ for $d = -7, -8, -11$.

11. Let $q = 2, 3, 5, 11, 17$ or 41 (thus, $h(1-4q) = 1$). Check that numbers $x^2 + x + q$ for $x = 0, 1, 2, \dots, q-2$ are all prime (for the explanation of this phenomenon, see [307, Chapter 4.1], [349, Chapter 3.2] and [408, Appendix]).

12. Determine $h(d)$ and find all reduced quadratic forms of discriminant d for

    a) $d = -19$,
    b) $d = -63$,
    c) $d = -151$.

13. Determine all prime numbers p which can be represented in the form $p = x^2 + 2y^2$, $x, y \in \mathbb{N}$.

14. Prove that numbers of the form $6 \cdot 4^n$ cannot be represented as a sum of squares of four positive integers.

15. Verify the identity

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 = \frac{1}{6} \sum_{1 \le i < j \le 4} (x_i + x_j)^4 + \frac{1}{6} \sum_{1 \le i < j \le 4} (x_i - x_j)^4$$

    and use it to prove the statement that every positive integer can be represented as a sum of 53 fourth powers of integers. This is a special case of Waring's problem (see [324, Chapter 1]).

16. Give an example (different from the one given in the text), which shows that the product of two positive integers, which are both representable as a sum of three squares, need not be representable as a sum of three squares.

17. Find all representations of n as a sum of squares of three integers:

    a) $n = 33$,
    b) $n = 66$,
    c) $n = 235$.

18. Let n be a triangular number. Prove that numbers $8n^2$, $8n^2 + 1$ and $8n^2 + 2$ can be written as sums of two squares.

19. If a prime number p is equal to a sum of squares of three distinct prime numbers, prove that one of those three prime numbers has to be 3.

20. Prove that every odd positive integer can be written in the form $x^2 + y^2 + 2z^2$, $x, y, z \in \mathbb{Z}$.

21. Prove that every odd positive integer can be written as a sum of squares of four integers out of which two are adjacent numbers (their difference is 1).

22. Prove that there are infinitely many prime numbers of the form $a^2 + b^2 + c^2 + 1$.

23. Let p be a prime number such that $p \equiv 7 \pmod{8}$. Prove that the equation $x^2 + y^2 + z^4 = p^2$ has no solutions in positive integers x, y, z (see [106]).