(Example 2.5), the series $(\frac{1}{3}+\frac{1}{5})+(\frac{1}{5}+\frac{1}{7})+(\frac{1}{11}+\frac{1}{13})+(\frac{1}{17}+\frac{1}{19})+(\frac{1}{29}+\frac{1}{31})+\cdots$ converges. Chen proved in 1966 that there are infinitely many prime numbers p for which $p+2$ is either prime or a product of two prime numbers. In 1849 De Polignac proposed a more general conjecture that for any positive integer k, there are infinitely many prime numbers p such that $p+2k$ is also a prime. Zhang proved in 2013 that there is a positive integer $N \leq 70000000$, such that there are infinitely many pairs of prime numbers which differ by N. By a joint effort of a group of mathematicians, that result was improved to $N \leq 246$.

- Is it true that for any positive integer n, there is a prime number between n^2 and $(n+1)^2$?

- (Goldbach's conjecture) Is it possible to express any even number ≥ 4 as a sum of two prime numbers (for example $4=2+2$, $6=3+3$, $8=3+5$, $10=3+7$, $12=5+7$)? It is known that the following so-called ternary (or weak) Goldbach's conjecture holds: every odd number $n \geq 7$ can be expressed as a sum of three prime numbers (the conjecture was proved for numbers n large enough by Vinogradov in 1937, and it was completely proved by Helfgott in 2013).

2.4 Exercises

1. Find positive integers x and y such that $x(x+1) \mid y(y+1)$, but $x \nmid y$, $x \nmid y+1$, $x+1 \nmid y$ and $x+1 \nmid y+1$.

2. Find positive integers x and y such that $x^x \mid y^y$, but $x \nmid y$.

3. Prove Proposition 2.1.

4. Prove that the sum of cubes of three consecutive positive integers is divisible by 9.

5. Prove using the mathematical induction that for every integer $n \geq 0$ the number $11^{n+1}+12^{2n-1}$ is divisible by 133.

6. Prove using the mathematical induction that for every integer $n \geq 0$ the number $7^{2n+1}+2\cdot 13^{2n+1}+17^{2n+1}$ is divisible by 50.

7. Which is the smallest positive rational number that can be expressed in the form
$$\frac{x}{77}+\frac{y}{91},$$
where x and y are integers?

8. If $\gcd(a,b)=1$, which values can $\gcd(a+b,a-b)$ take?

9. If $\gcd(a,b)=10$, which values can $\gcd(a^3,b^4)$ take?

10. Find the greatest common divisor of the numbers
$$1+2+2^2+\cdots+2^{5n-1}\quad \text{for } n=1,2,\ldots,100\,.$$

11. Determine $g=\gcd(a,b)$ and find integers x,y such that $ax+by=g$ if

    a) $a=423$, $b=198$,
    b) $a=679$, $b=777$,
    c) $a=987$, $b=610$.

12. Determine integers x,y such that

    a) $50x+71y=1$,
    b) $93x+81y=3$,
    c) $105x+55y=5$.

13. Prove that for $i=0,1,\ldots,j+1$ we have $x_{i-1}y_i-x_iy_{i-1}=(-1)^i$ and $\gcd(x_i,y_i)=1$, where x_i,y_i are the coefficients appearing in the extended Euclid's algorithm.

14. Find the smallest positive integer n such that the set $\{n,n+1,\ldots,n+6\}$ does not contain any prime numbers.

15. Find the smallest positive integer n such that n^2-1 is a product of four distinct prime numbers.

16. Find all prime numbers p such that numbers $p+2$ and $p+4$ are also prime.

17. Find all prime numbers p such that $21p+1$ is a perfect square.

18. Prove that $a^3\mid b^2$ implies that $a\mid b$. Show by an example that $a^2\mid b^3$ does not imply that $a\mid b$.

19. Find a positive integer n such that $n/2$ is a square, $n/3$ a cube and $n/5$ a fifth power of an integer.

20. Let a, b, c be positive integers such that ab, ac and bc are cubes (third powers) of positive integers. Prove that then also a, b, c have to be cubes. Is this statement valid if "cubes" is replaced by "squares"?

21. Prove that $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is not an integer for $n > 1$.

22. Let $p(x) = x^2 - x + 41$. Check that the numbers $p(0), p(1), \ldots, p(40)$ are all prime, but $p(41)$ is composite.

23. Let $f_n = 2^{2^n} + 1$. Prove that $\gcd(f_m, f_n) = 1$ for $m \neq n$. Show that this fact implies that there are infinitely many prime numbers.

24. Let $p = 6n + 1$ be a prime number. Prove that the numerator of the rational number $\sum_{k=1}^{4n} \frac{(-1)^{k-1}}{k}$ is divisible by p.

25. Prove that all palindromic prime numbers, except the number 11, have an odd number of digits.