Formula (1.17) is also valid for negative integers. Indeed, if $n \in \mathbb{N}$, then

$$\begin{aligned}\frac{1}{\sqrt{5}}\left(\alpha^{-n}-\beta^{-n}\right) &= \frac{1}{\sqrt{5}}\Big[(-\beta)^n-(-\alpha)^n\Big] = (-1)^{n-1}\cdot\frac{1}{\sqrt{5}}\left(\alpha^n-\beta^n\right)\\ &= (-1)^{n-1}F_n = F_{-n}\,.\end{aligned}$$

It can be analogously proved that

$$L_n = \alpha^n + \beta^n. \tag{1.18}$$

Example 1.18. *Prove that*

$$L_{2n} + 2\cdot(-1)^n = L_n^2. \tag{1.19}$$

Solution: By equation (1.18), we have

$$L_{2n} + 2\cdot(-1)^n = \alpha^{2n} + \beta^{2n} + 2(\alpha\beta)^n = (\alpha^n+\beta^n)^2 = L_n^2. \qquad \diamondsuit$$

As it was already mentioned, Fibonacci numbers will often appear in the following chapters of this book. A few books have been devoted to Fibonacci numbers and especially their connection to number theory, out of which we mention [113, 220, 253, 407, 410].

1.4 Exercises

1. Using the principle of mathematical induction, prove the following formulas:

   a) $1 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$,

   b) $1 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$,

   c) $1 + 2^5 + 3^5 + \cdots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$.

2. Prove that for any positive integer n,

$$1 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}.$$

3. Prove that for any positive integer n,

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

4. Prove that for any integer $n \geq 2$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{13}{24}.$$

5. For which positive integers does inequality $2^n > n^2$ hold?

6. Prove that inequality

$$(1+x)^n > 1 + nx$$

holds for any real number $x > -1$, $x \neq 0$ and integer $n \geq 2$.

7. Prove that for all positive integers n,

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1.$$

8. Prove that for positive integers $n \geq m \geq k$,

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

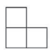
9. Prove the formula of parallel summation

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{n+m+1}{n}.$$

10. Prove Vandermonde's convolution formula

$$\sum_{i+j=k} \binom{n}{i}\binom{m}{j} = \binom{m+n}{k}.$$

11. Calculate $\displaystyle\sum_{k=0}^{n} (-1)^k \frac{1}{k+1}\binom{n}{k}$.

12. Can a board of dimensions $2^n \times 2^n$, with one square removed, be covered with tiles of the shape  ?

13. There are n lines in the plane, among which there are no parallel ones, and there are no three lines such that they intersect in one point.

    a) In how many parts is the plane divided by those lines?
    b) Prove that the parts can be coloured in two colours in such a manner that the adjacent areas are coloured in different colours.

14. Prove that there are F_{n+2} sequences of zeros and ones of length n without adjacent zeros.

15. For a subset S of $\mathbb{N}$, let us define $S+1 = \{x+1 \,:\, x \in S\}$. How many subsets S of the set $\{1,2,\ldots,n\}$ satisfy the condition $S \cup (S+1) = \{1,2,\ldots,n+1\}$?

16. Prove that a board of dimensions $n \times 2$ can be covered with dominoes (tiles with dimensions 2×1) in F_{n+1} ways.

17. Prove that $L_{n+1} + L_{n-1} = 5F_n$.

18. Find a combinatorial interpretation of the Lucas numbers.

19. Prove that for any positive integer n the following formulas hold:

    a) $\displaystyle\sum_{k=1}^{2n} F_k F_{k-1} = F_{2n}^2,$

    b) $\displaystyle\sum_{k=1}^{2n+1} F_k F_{k-1} = F_{2n+1}^2 - 1.$

20. Let x_1 and x_2 be positive integers less than $10\,000$ and let the sequence (x_n) be defined by $x_n = \min\{|x_i - x_j| \,:\, 1 \le i < j \le n-1\}$ for $n \ge 3$. Prove that $x_{21} = 0$.

21. Find all Fibonacci numbers which are also Lucas numbers. In other words, find the intersection of the sets $\{F_n : n \in \mathbb{N}\}$ and $\{L_n : n \in \mathbb{N}\}$.

22. Using Binet's formula (1.17), calculate F_{20} and F_{30}.

23. Prove formulas (1.8) and (1.9) using the method of mathematical induction.

24. *Generalized Fibonacci numbers* are numbers defined by $H_1 = p$, $H_2 = q$, $H_{n+2} = H_{n+1} + H_n$. Using the mathematical induction, prove that for any positive integer n, we have $H_{n+2} = qF_{n+1} + pF_n$.

25. Let m be an even positive integer. Prove that
$$\frac{1}{F_{2m}} = \frac{F_{m-1}}{F_m} - \frac{F_{2m-1}}{F_{2m}}\,.$$
    Using this formula and the telescoping method (Example 1.12) calculate the sum $\displaystyle\sum_{k=0}^{n} \frac{1}{F_{2^k}}$.

26. Prove the formulas

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} + 1 = F_n^4$$
$$F_nF_{n+2}F_{n+3}F_{n+5} + 1 = (F_{n+4}^2 - 2F_{n+3}^2)^2$$

(see [312]).

27. Prove the formula $F_{n-1}^4 + F_n^4 + F_{n+1}^4 = 2(2F_n^2 + (-1)^n)^2$.

28. Let n and k be arbitrary positive integers. Prove that there are at most n Fibonacci numbers F_m such that

$$n^k < F_m < n^{k+1}.$$

29. Prove that $F_{k+l} = L_lF_k + (-1)^lF_{k-l}$.

30. Calculate $\displaystyle\sum_{k=1}^{2n} \frac{1}{F_{2k+1}+1}$.

31. Note that 2×2 matrices are multiplied by the following rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

Using mathematical induction, prove that for any positive integer n,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \quad (1.20)$$

The determinant of matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, denoted by $\det A$, is the number $ad - bc$. It can easily be checked that $\det(AB) = \det A \cdot \det B$ holds for 2×2 matrices A and B (this formula also generally holds for the determinant of the matrix product and is called *Cauchy-Binet formula*). Taking determinants of both sides in (1.20), prove Cassini's identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.