

DIFFERENCES OF SQUARES OF UPPER-TRIANGULAR 2×2 INTEGER MATRICES

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ABSTRACT. We consider the problem of characterizing upper-triangular matrices $M = \begin{pmatrix} p & r \\ 0 & q \end{pmatrix} \in M_2(\mathbb{Z})$ which can be represented in the form $A^2 - B^2$ with upper-triangular integer matrices A and B and give a complete criterion in terms of representations of p and q as differences of two squares and an additional divisibility condition on r . Also, we give a complete classification of representable matrices in terms of congruence conditions on p , q , and r .

1. INTRODUCTION AND MOTIVATION

The classical problem of deciding which integers are representable as a difference of two squares is well known: an integer n can be written in the form $n = x^2 - y^2$ with $x, y \in \mathbb{Z}$ if and only if $n \not\equiv 2 \pmod{4}$. Analogous questions can be considered in other rings. For example, a Gaussian integer $z = a + bi$ can be represented as a difference of two squares of Gaussian integers if and only if b is even and $(a, b) \not\equiv (2, 2) \pmod{4}$. In [3], the analogous problem in rings of integers of quadratic fields $\mathbb{Q}(\sqrt{d})$ was studied. More precisely, a complete characterization of elements that can be represented as a difference of two squares is obtained for those integers d that satisfy certain conditions expressed in terms of the solvability of certain Pellian equations.

One of the main motivations for studying representations as differences of squares comes from their close connection with the *problem of the existence of Diophantine quadruples*, i.e., $D(n)$ -quadruples – sets of four elements in a commutative ring such that the product of any two distinct elements, increased by n , is a perfect square.

In several papers dealing with $D(n)$ -quadruples in rings of algebraic integers, representations of elements as differences of two squares naturally arise as a preparatory step in the analysis. This approach appears, for example, in [4], [5], [6] and [11], where the structure of elements representable as differences of squares plays an important role in determining the existence of $D(n)$ -quadruples in various number fields.

2020 *Mathematics Subject Classification.* 11C20, 15A24, 11A07.

Key words and phrases. upper triangular matrices, difference of squares, Diophantine equations, congruence classification.

In some commutative rings it turns out that the existence of a $D(n)$ -quadruple is equivalent to the representability of n as a difference of two squares, while in other settings this equivalence fails. Counterexamples to such a conjectural connection were recently obtained in [1], [2] and [8].

A further connection between differences of squares and $D(n)$ -quadruples is given in [7], where it is shown that there is no polynomial $D(n)$ -quadruple in $\mathbb{Z}[X]$ (a ring of polynomials with integer coefficients) for certain polynomials $n \in \mathbb{Z}[X]$ that cannot be represented as a difference of two squares of polynomials in $\mathbb{Z}[X]$.

These results motivate further study of representations as differences of squares in various algebraic structures. An interesting setting in which to study the existence of $D(n)$ -tuples is the *ring of upper triangular 2×2 integer matrices*, i.e., matrices of the form

$$T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad a, b, c \in \mathbb{Z}.$$

We denote this set by $UT_2(\mathbb{Z})$. Note that $UT_2(\mathbb{Z})$ forms a ring with respect to the usual matrix addition and multiplication. Although this ring is not commutative, squares of upper triangular matrices are particularly easy to compute. Indeed,

$$T^2 = \begin{pmatrix} a^2 & b(a+c) \\ 0 & c^2 \end{pmatrix}.$$

So, this simple structure makes $UT_2(\mathbb{Z})$ a convenient framework for studying representations as differences of squares and their connection with $D(n)$ -tuples.

For $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ 0 & u \end{pmatrix}$ in $UT_2(\mathbb{Z})$, a direct computation gives

$$(1) \quad A^2 - B^2 = \begin{pmatrix} a^2 - x^2 & b(a+d) - y(x+u) \\ 0 & d^2 - u^2 \end{pmatrix}.$$

In view of the classical characterization over \mathbb{Z} , i.e. of integers representable as differences of two squares, relation (1) immediately yields a necessary condition for a matrix in $UT_2(\mathbb{Z})$ to be expressible as a difference of two squares. Namely, if $T = \begin{pmatrix} p & r \\ 0 & q \end{pmatrix}$ can be written in the form $A^2 - B^2$ for some $A, B \in UT_2(\mathbb{Z})$, then both diagonal entries p and q must be representable as differences of two integer squares, i.e.

$$p \not\equiv 2 \pmod{4}, \quad q \not\equiv 2 \pmod{4}.$$

Therefore, any upper-triangular integer matrix having at least one diagonal entry congruent to 2 modulo 4 cannot be represented as a difference of two squares in $UT_2(\mathbb{Z})$.

The upper-right entry in (1) introduces an additional arithmetic condition linking the representations of the diagonal entries. Our first main result (Theorem 1) shows that this condition reduces to the solvability of a linear

Diophantine equation and can be expressed as a gcd divisibility condition. We conclude the paper with a complete classification of representable matrices in terms of congruence conditions on p , q , and r (Theorem 10).

2. THE MAIN CRITERION

Theorem 1. *The upper-triangular matrix $T = \begin{pmatrix} p & r \\ 0 & q \end{pmatrix}$ in $UT_2(\mathbb{Z})$ can be represented as a difference of two squares in $UT_2(\mathbb{Z})$ if and only if there exist integers a, x, d, u such that*

$$p = a^2 - x^2, \quad q = d^2 - u^2,$$

and either

$$(a + d, x + u) \neq (0, 0) \quad \text{and} \quad \gcd(a + d, x + u) \mid r,$$

or

$$a + d = 0, \quad x + u = 0, \quad r = 0.$$

Proof. $\boxed{\Rightarrow}$: Assume that $T = A^2 - B^2$ for upper-triangular matrices $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ 0 & u \end{pmatrix}$. By (1), we have

$$p = a^2 - x^2, \quad q = d^2 - u^2, \quad r = b(a + d) - y(x + u).$$

If $(a + d, x + u) \neq (0, 0)$, then r is an integer linear combination of $a + d$ and $x + u$, and hence

$$\gcd(a + d, x + u) \mid r.$$

If $a + d = 0$ and $x + u = 0$, then the above relation reduces to $r = 0$.

$\boxed{\Leftarrow}$: Suppose there exist integers a, x, d, u such that

$$p = a^2 - x^2, \quad q = d^2 - u^2,$$

and either $(a + d, x + u) \neq (0, 0)$ with $\gcd(a + d, x + u) \mid r$, or $a + d = x + u = 0$ and $r = 0$. If $(a + d, x + u) \neq (0, 0)$, then the linear Diophantine equation

$$b(a + d) - y(x + u) = r$$

has an integer solution (b, y) . If $a + d = x + u = 0$ and $r = 0$, then any integers b, y satisfy the equation. Defining $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ 0 & u \end{pmatrix}$, relation (1) yields $A^2 - B^2 = T$. \square

Corollary 2. *If $p \equiv 2 \pmod{4}$ or $q \equiv 2 \pmod{4}$, then $T = \begin{pmatrix} p & r \\ 0 & q \end{pmatrix}$ in $UT_2(\mathbb{Z})$ cannot be represented as a difference of two squares in $UT_2(\mathbb{Z})$.*

Remark 1. *Theorem 1 shows that the problem splits into two parts:*

- (1) *represent the diagonal entries p and q as differences of two squares;*
- (2) *choose such representations so that $\gcd(a + d, x + u)$ divides the upper-right entry r .*

Thus, the essential difficulty lies in the interaction between the chosen representations of p and q . The greatest flexibility for the entry r is obtained when this gcd is as small as possible.

Definition 1. For integers p and q representable as differences of two squares, define

$$g(p, q) := \min\{\gcd(a + d, x + u) > 0\},$$

where the minimum is taken over all quadruples $(a, x, d, u) \in \mathbb{Z}^4$ satisfying

$$p = a^2 - x^2, \quad q = d^2 - u^2.$$

If for some representation one has $a + d = 0$ and $x + u = 0$, then replacing d and u by $-d$ and $-u$ yields another representation of q for which $(a + d, x + u) \neq (0, 0)$. Hence the set in Definition 1 is nonempty.

3. COMPLETE CLASSIFICATION BY CONGRUENCE TYPE

In several rings (for example \mathbb{Z} , $\mathbb{Z}[i]$, and rings of integers of certain number fields), it has proved useful to classify elements representable as differences of two squares according to their congruence classes. Such classifications often play an important role in constructions of $D(n)$ -quadruples, since in some rings the existence of $D(n)$ -quadruples is closely related to representability of n as a difference of two squares. Motivated by this, we investigate which congruence classes modulo 4 can occur for matrices representable as differences of two squares in $UT_2(\mathbb{Z})$. For this purpose, we study the corresponding problem in $UT_2(\mathbb{Z}_4)$, the ring of upper triangular 2×2 matrices with entries in \mathbb{Z}_4 .

Proposition 3. *The matrices in $UT_2(\mathbb{Z}_4)$ that cannot be represented as differences of two squares in $UT_2(\mathbb{Z}_4)$ are precisely*

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in UT_2(\mathbb{Z}_4) : a = 2 \text{ or } c = 2 \right\} \cup \left\{ \begin{pmatrix} 1 & 1, 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1, 3 \\ 0 & 3 \end{pmatrix} \right\}.$$

Proof. We determine the set $S_1 = \{A^2 - B^2 : A, B \in UT_2(\mathbb{Z}_4)\}$ using formula (1). A direct verification shows that precisely the matrices listed in the statement do not belong to S_1 . The complete lists of representable and non-representable matrices modulo 4 are given in the Appendix. \square

If $T \in UT_2(\mathbb{Z})$ and $T \bmod 4 \in S$, then T cannot be represented as a difference of two squares in $UT_2(\mathbb{Z})$. However, the converse does not hold: the condition $T \bmod 4 \notin S$ does not guarantee that T is representable as a difference of two squares in $UT_2(\mathbb{Z})$. So modulo 4 classification gives necessary but not sufficient conditions.

Example 1. *The matrix*

$$N = \begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix}$$

is not representable as a difference of two squares in $UT_2(\mathbb{Z})$, although

$$N \bmod 4 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

is representable as a difference of two squares in $UT_2(\mathbb{Z}_4)$. Indeed, suppose

$$N = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}^2 - \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}^2 = \begin{pmatrix} a_1^2 - a_2^2 & b_1(a_1 + c_1) - b_2(a_2 + c_2) \\ 0 & c_1^2 - c_2^2 \end{pmatrix}.$$

From $a_1^2 - a_2^2 = 4$ and $c_1^2 - c_2^2 = 4$, we obtain

$$(a_1, a_2) \in \{(\pm 2, 0)\}, \quad (c_1, c_2) \in \{(\pm 2, 0)\}.$$

Hence $a_1 + c_1 \in \{-4, 0, 4\}$ and $a_2 + c_2 = 0$, so

$$b_1(a_1 + c_1) - b_2(a_2 + c_2) = b_1(a_1 + c_1)$$

is divisible by 4, which contradicts the condition that the upper-right entry equals 2.

Proposition 4. *Let*

$$M = \begin{pmatrix} 2k+1 & r \\ 0 & 2n+1 \end{pmatrix} \in UT_2(\mathbb{Z}).$$

Then M is representable as a difference of two squares in $UT_2(\mathbb{Z})$ if and only if

$$\gcd(k+n, 2) \mid r.$$

(Or equivalently, if k and n have opposite parity, then every $r \in \mathbb{Z}$ is admissible, and if k and n have the same parity, then r must be even.)

Proof. $\boxed{\Leftarrow}$: We use the identity

$$\begin{pmatrix} k+1 & b \\ 0 & n+1 \end{pmatrix}^2 - \begin{pmatrix} k & y \\ 0 & n \end{pmatrix}^2 = \begin{pmatrix} 2k+1 & b(k+n+2) - y(k+n) \\ 0 & 2n+1 \end{pmatrix}.$$

Therefore, it suffices to determine when the linear Diophantine equation

$$(2) \quad b(k+n+2) - y(k+n) = r$$

is solvable in integers b and y . This happens if and only if

$$\gcd(k+n+2, k+n) \mid r.$$

Since

$$\gcd(k+n+2, k+n) = \gcd(k+n, 2),$$

equation (2) is solvable if and only if

$$\gcd(k+n, 2) \mid r.$$

Hence M is representable as a difference of two squares in $UT_2(\mathbb{Z})$.

$\boxed{\Rightarrow}$: Suppose that

$$M = \begin{pmatrix} 2k+1 & r \\ 0 & 2n+1 \end{pmatrix}$$

is representable as a difference of two squares in $UT_2(\mathbb{Z})$. Reducing modulo 4, we obtain a matrix representable as a difference of two squares in $UT_2(\mathbb{Z}_4)$.

If k and n have the same parity, then the diagonal entries of $M \bmod 4$ are equal and odd, i.e. both are 1 or both are 3. By Proposition 3, such a matrix is representable modulo 4 only if its upper-right entry is even. Hence r must be even.

If k and n have opposite parity, then $\gcd(k+n, 2) = 1$, so the desired divisibility condition is automatic.

Therefore, in all cases, $\gcd(k+n, 2) \mid r$. \square

Corollary 5. *For every $k, n, r \in \mathbb{Z}$, the matrices*

$$\begin{pmatrix} 4k+1 & r \\ 0 & 4n+3 \end{pmatrix}, \begin{pmatrix} 4k+3 & r \\ 0 & 4n+1 \end{pmatrix}, \begin{pmatrix} 4k+1 & 2r \\ 0 & 4n+1 \end{pmatrix}, \begin{pmatrix} 4k+3 & 2r \\ 0 & 4n+3 \end{pmatrix}$$

are representable as differences of two squares in $UT_2(\mathbb{Z})$.

Proposition 6. *For every $k, n, r \in \mathbb{Z}$, both matrices*

$$\begin{pmatrix} 2k+1 & r \\ 0 & 4n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4k & r \\ 0 & 2n+1 \end{pmatrix}$$

are representable as differences of two squares in $UT_2(\mathbb{Z})$.

Proof. For the first matrix, we take

$$a = k+1, \quad x = k, \quad d = -(n+1), \quad u = -(n-1).$$

Then

$$a^2 - x^2 = 2k+1, \quad d^2 - u^2 = 4n,$$

and

$$a+d = k-n, \quad x+u = k-n+1.$$

Hence

$$\gcd(a+d, x+u) = \gcd(k-n, k-n+1) = 1,$$

so Theorem 1 applies for every r .

The second case is symmetric. We take

$$a = -(k+1), \quad x = -(k-1), \quad d = n+1, \quad u = n.$$

Then

$$a^2 - x^2 = 4k, \quad d^2 - u^2 = 2n+1,$$

and

$$a+d = n-k, \quad x+u = n-k+1,$$

so again $\gcd(a+d, x+u) = 1$.

Thus both matrices are representable for every r . \square

Proposition 6 shows that in the mixed parity cases, the upper-right entry imposes no additional restriction: every such matrix is representable as a difference of two squares in $UT_2(\mathbb{Z})$.

Lemma 7. *Let*

$$M = \begin{pmatrix} 4i & r \\ 0 & 4j \end{pmatrix} \in UT_2(\mathbb{Z}_{16}), \quad i, j \in \{0, 1, 2, 3\}.$$

Then M is representable as a difference of two squares in $UT_2(\mathbb{Z}_{16})$ if and only if one of the following holds:

- (1) $(i, j) \in \{(1, 1), (3, 3)\}$ and $r \equiv 0 \pmod{4}$;
- (2) $(i, j) \in \{(1, 3), (2, 2), (3, 1)\}$ and $r \equiv 0 \pmod{2}$;
- (3) $(i, j) \in \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 2), (2, 0), (2, 1), (2, 3), (3, 0), (3, 2)\}$, with arbitrary $r \in \mathbb{Z}_{16}$.

Proof. The statement follows by direct computation. All representable matrices modulo 16 are listed in the Appendix. \square

Exactly 48 matrices in $UT_2(\mathbb{Z}_{16})$ whose diagonal entries are divisible by 4 fail to be representable as differences of two squares (see the Appendix).

Corollary 8. *Let*

$$M = \begin{pmatrix} 4i & r \\ 0 & 4j \end{pmatrix} \in UT_2(\mathbb{Z}_{16}), \quad i, j \in \{0, 1, 2, 3\}.$$

Then M cannot be represented as a difference of two squares in $UT_2(\mathbb{Z}_{16})$ if and only if one of the following holds:

- (1) $(i, j) \in \{(1, 1), (3, 3)\}$ and $r \not\equiv 0 \pmod{4}$;
- (2) $(i, j) \in \{(1, 3), (2, 2), (3, 1)\}$ and $r \not\equiv 0 \pmod{2}$.

Proposition 9. *Let*

$$M = \begin{pmatrix} 4k & r \\ 0 & 4n \end{pmatrix} \in UT_2(\mathbb{Z}).$$

Then M is representable as a difference of two squares in $UT_2(\mathbb{Z})$ if and only if one of the following conditions holds:

- (1) $k + n \equiv 1 \pmod{2}$, or $(k, n) \pmod{4} \in \{(0, 0), (0, 2), (2, 0)\}$, with arbitrary $r \in \mathbb{Z}$;
- (2) $k + n \equiv 0 \pmod{4}$, $(k, n) \pmod{4} \neq (0, 0)$, and $r \equiv 0 \pmod{2}$;
- (3) $k + n \equiv 2 \pmod{4}$, $(k, n) \pmod{4} \notin \{(0, 2), (2, 0)\}$, and $r \equiv 0 \pmod{4}$.

Proof. \Leftarrow : First, the choice

$$a = k + 1, \quad x = k - 1, \quad d = n + 1, \quad u = n - 1,$$

shows representability whenever r satisfies the divisibility condition determined by $\gcd(k + n + 2, 4)$. Indeed,

$$a^2 - x^2 = 4k, \quad d^2 - u^2 = 4n,$$

and

$$a + d = k + n + 2, \quad x + u = k + n - 2.$$

Hence

$$\gcd(a + d, x + u) = \gcd(k + n + 2, k + n - 2) = \gcd(k + n + 2, 4).$$

Therefore:

- (a) if $k + n \equiv 1 \pmod{2}$, then $\gcd(a + d, x + u) = 1$;
- (b) if $k + n \equiv 0 \pmod{4}$, then $\gcd(a + d, x + u) = 2$;
- (c) if $k + n \equiv 2 \pmod{4}$, then $\gcd(a + d, x + u) = 4$.

In each case, the assumed divisibility condition on r implies that $\gcd(a + d, x + u) \mid r$. Hence Theorem 1 shows that M is representable as a difference of two squares in $UT_2(\mathbb{Z})$ in the following cases:

- $k + n \equiv 1 \pmod{2}$,
- $k + n \equiv 0 \pmod{4}$ and $r \equiv 0 \pmod{2}$;
- $k + n \equiv 2 \pmod{4}$ and $r \equiv 0 \pmod{4}$.

For the remaining cases, we also give concrete representations. For the case $k \equiv n \equiv 0 \pmod{4}$, we put $k = 4K$ and $n = 4N$. For

$$A = \begin{pmatrix} 2(K+1) & b \\ 0 & -(4N+1) \end{pmatrix}, \quad B = \begin{pmatrix} 2(K-1) & y \\ 0 & -(4N-1) \end{pmatrix},$$

we obtain

$$A^2 - B^2 = \begin{pmatrix} 16K & b(2K - 4N + 1) - y(2K - 4N - 1) \\ 0 & 16N \end{pmatrix}.$$

Since

$$\gcd(2K - 4N + 1, 2K - 4N - 1) = 1,$$

the upper-right entry can be chosen arbitrarily. Hence every matrix

$$\begin{pmatrix} 16K & r \\ 0 & 16N \end{pmatrix}$$

is representable as a difference of two squares in $UT_2(\mathbb{Z})$.

Assume $k \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$, say $k = 4K$ and $n = 4N + 2$. Consider

$$A = \begin{pmatrix} 2(K+1) & b \\ 0 & -(4N+3) \end{pmatrix}, \quad B = \begin{pmatrix} 2(K-1) & y \\ 0 & 4N+1 \end{pmatrix}.$$

Then

$$A^2 - B^2 = \begin{pmatrix} 16K & b(2K - 4N - 1) + y(2K - 4N - 3) \\ 0 & 8(2N + 1) \end{pmatrix}.$$

Since

$$\gcd(2K - 4N - 1, 2K - 4N - 3) = 1,$$

the upper-right entry can be chosen arbitrarily. Hence every matrix of the form

$$\begin{pmatrix} 16K & r \\ 0 & 8(2N + 1) \end{pmatrix}$$

is representable as a difference of two squares in $UT_2(\mathbb{Z})$. The case $(k, n) \equiv (2, 0) \pmod{4}$ is analogous.

\square : The claim follows from Corollary 8, by contraposition of the fact that representability over \mathbb{Z} implies representability modulo 16. \square

Remark 2. *Example 1 can now be explained in terms of Proposition 9. Indeed, case (3) of Proposition 9 requires $r \equiv 0 \pmod{4}$.*

4. FINAL CLASSIFICATION AND COMMENTS

We may now summarize the representation problem completely.

Theorem 10. *Let*

$$M = \begin{pmatrix} p & r \\ 0 & q \end{pmatrix} \in UT_2(\mathbb{Z}).$$

Then M is representable as a difference of two squares in $UT_2(\mathbb{Z})$ if and only if one of the following holds:

- (1) *p and q are odd, and*
 - *$p \equiv q \pmod{4}$ and $r \equiv 0 \pmod{2}$;*
 - *$p \not\equiv q \pmod{4}$ and r is arbitrary;*
- (2) *one of p, q is odd and the other is divisible by 4, and r is arbitrary;*
- (3) *$p \equiv q \equiv 0 \pmod{4}$, and*
 - *$(p, q) \pmod{16} \in \{(4, 4), (12, 12)\}$ and $r \equiv 0 \pmod{4}$;*
 - *$(p, q) \pmod{16} \in \{(4, 12), (8, 8), (12, 4)\}$ and $r \equiv 0 \pmod{2}$;*
 - *otherwise, r is arbitrary.*

Proof. The statement follows by combining the classification in the odd-diagonal case (Proposition 4), the mixed cases (Proposition 6), and the case of diagonal entries divisible by 4 (Proposition 9), together with the classical criterion for integers to be representable as differences of two squares. \square

Theorem 11. *The matrix $T = \begin{pmatrix} p & r \\ 0 & q \end{pmatrix} \in UT_2(\mathbb{Z})$ is representable as a difference of two squares in $UT_2(\mathbb{Z})$ if and only if*

- (1) *p and q are representable as differences of two squares in \mathbb{Z} , and*
- (2) *$g(p, q) \mid r$.*

Proof. Sufficiency follows immediately from Theorem 1.

For necessity, Theorem 10 shows that for every pair (p, q) representable as differences of two squares, there exists a number $m(p, q) \in \{1, 2, 4\}$ such that $\begin{pmatrix} p & r \\ 0 & q \end{pmatrix}$ is representable in $UT_2(\mathbb{Z})$ if and only if $m(p, q) \mid r$.

On the other hand, the explicit constructions in the proofs of Propositions 4, 6, and 9 show that this value $m(p, q)$ is attained as $\gcd(a + d, x + u)$ for suitable representations $p = a^2 - x^2$, $q = d^2 - u^2$. Hence,

$$g(p, q) \leq m(p, q).$$

If $g(p, q) < m(p, q)$, then by Theorem 1 the matrix

$$\begin{pmatrix} p & g(p, q) \\ 0 & q \end{pmatrix}$$

would be representable, contradicting Theorem 10. Therefore $g(p, q) = m(p, q)$, and the claim follows. \square

APPENDIX: TABLES OF REPRESENTATIONS MODULO 4 AND 16

Notation. We write (a, b, c) for the matrix $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$.

Matrices representable as a difference of two squares in $UT_2(\mathbb{Z}_4)$:

(0,0,0)	(0,0,1)	(0,0,3)	(0,1,0)	(0,1,1)	(0,1,3)	(0,2,0)	(0,2,1)	(0,2,3)	(0,3,0)	(0,3,1)
(0,3,3)	(1,0,0)	(1,0,1)	(1,0,3)	(1,1,0)	(1,1,3)	(1,2,0)	(1,2,1)	(1,2,3)	(1,3,0)	(1,3,3)
(3,0,0)	(3,0,1)	(3,0,3)	(3,1,0)	(3,1,1)	(3,2,0)	(3,2,1)	(3,2,3)	(3,3,0)	(3,3,1)	

Matrices not representable as a difference of two squares in $UT_2(\mathbb{Z}_4)$:

(0,0,2)	(0,1,2)	(0,2,2)	(0,3,2)	(1,0,2)	(1,1,1)	(1,1,2)	(1,2,2)	(1,3,1)	(1,3,2)	(2,0,0)
(2,0,1)	(2,0,2)	(2,0,3)	(2,1,0)	(2,1,1)	(2,1,2)	(2,1,3)	(2,2,0)	(2,2,1)	(2,2,2)	(2,2,3)
(2,3,0)	(2,3,1)	(2,3,2)	(2,3,3)	(3,0,2)	(3,1,2)	(3,1,3)	(3,2,2)	(3,3,2)	(3,3,3)	

Matrices representable as a difference of two squares in $UT_2(\mathbb{Z}_{16})$, and $4 \mid a, c$:

(0,0,0)	(0,0,4)	(0,0,8)	(0,0,12)	(0,1,0)	(0,1,4)	(0,1,8)	(0,1,12)	(0,2,0)		
(0,2,4)	(0,2,8)	(0,2,12)	(0,3,0)	(0,3,4)	(0,3,8)	(0,3,12)	(0,4,0)	(0,4,4)		
(0,4,8)	(0,4,12)	(0,5,0)	(0,5,4)	(0,5,8)	(0,5,12)	(0,6,0)	(0,6,4)	(0,6,8)		
(0,6,12)	(0,7,0)	(0,7,4)	(0,7,8)	(0,7,12)	(0,8,0)	(0,8,4)	(0,8,8)	(0,8,12)		
(0,9,0)	(0,9,4)	(0,9,8)	(0,9,12)	(0,10,0)	(0,10,4)	(0,10,8)	(0,10,12)	(0,11,0)		
(0,11,4)	(0,11,8)	(0,11,12)	(0,12,0)	(0,12,4)	(0,12,8)	(0,12,12)	(0,13,0)	(0,13,4)		
(0,13,8)	(0,13,12)	(0,14,0)	(0,14,4)	(0,14,8)	(0,14,12)	(0,15,0)	(0,15,4)	(0,15,8)		
(0,15,12)	(4,0,0)	(4,0,4)	(4,0,8)	(4,0,12)	(4,1,0)	(4,1,8)	(4,2,0)	(4,2,8)		
(4,2,12)	(4,3,0)	(4,3,8)	(4,4,0)	(4,4,4)	(4,4,8)	(4,4,12)	(4,5,0)	(4,5,8)		
(4,6,0)	(4,6,8)	(4,6,12)	(4,7,0)	(4,7,8)	(4,8,0)	(4,8,4)	(4,8,8)	(4,8,12)		
(4,9,0)	(4,9,8)	(4,10,0)	(4,10,8)	(4,10,12)	(4,11,0)	(4,11,8)	(4,12,0)	(4,12,4)		
(4,12,8)	(4,12,12)	(4,13,0)	(4,13,8)	(4,14,0)	(4,14,8)	(4,14,12)	(4,15,0)	(4,15,8)		
(8,0,0)	(8,0,4)	(8,0,8)	(8,0,12)	(8,1,0)	(8,1,4)	(8,1,8)	(8,2,0)	(8,2,4)		
(8,2,8)	(8,2,12)	(8,3,0)	(8,3,4)	(8,3,8)	(8,4,0)	(8,4,4)	(8,4,8)	(8,4,12)		
(8,5,0)	(8,5,4)	(8,5,8)	(8,6,0)	(8,6,4)	(8,6,8)	(8,6,12)	(8,7,0)	(8,7,4)		
(8,7,12)	(8,8,0)	(8,8,4)	(8,8,8)	(8,8,12)	(8,9,0)	(8,9,4)	(8,9,8)	(8,9,12)		
(8,10,4)	(8,10,8)	(8,10,12)	(8,11,0)	(8,11,4)	(8,11,8)	(8,12,0)	(8,12,4)	(8,12,8)		
(8,12,12)	(8,13,0)	(8,13,4)	(8,13,8)	(8,14,0)	(8,14,4)	(8,14,8)	(8,14,12)	(8,15,0)		
(8,15,4)	(8,15,8)	(12,0,0)	(12,0,4)	(12,0,8)	(12,0,12)	(12,1,0)	(12,1,8)	(12,2,0)		
(12,2,4)	(12,2,8)	(12,3,0)	(12,3,8)	(12,4,0)	(12,4,4)	(12,4,8)	(12,4,12)	(12,5,0)		
(12,5,8)	(12,6,0)	(12,6,4)	(12,6,8)	(12,7,0)	(12,7,8)	(12,8,0)	(12,8,4)	(12,8,8)		
(12,8,12)	(12,9,0)	(12,9,8)	(12,10,0)	(12,10,4)	(12,10,8)	(12,11,0)	(12,11,8)	(12,12,0)		
(12,12,4)	(12,12,8)	(12,12,12)	(12,13,0)	(12,13,8)	(12,14,0)	(12,14,4)	(12,14,8)	(12,15,0)		
(12,15,8)										

Matrices not representable as a diff. of two squares in $UT_2(\mathbb{Z}_{16})$, and $4 \mid a, c$:

(4,1,4)	(4,1,12)	(4,2,4)	(4,3,4)	(4,3,12)	(4,5,4)	(4,5,12)	(4,6,4)
(4,7,4)	(4,7,12)	(4,9,4)	(4,9,12)	(4,10,4)	(4,11,4)	(4,11,12)	(4,13,4)
(4,13,12)	(4,14,4)	(4,15,4)	(4,15,12)	(8,1,8)	(8,3,8)	(8,5,8)	(8,7,8)
(8,9,8)	(8,11,8)	(8,13,8)	(8,15,8)	(12,1,4)	(12,1,12)	(12,2,12)	(12,3,4)
(12,3,12)	(12,5,4)	(12,5,12)	(12,6,12)	(12,7,4)	(12,7,12)	(12,9,4)	(12,9,12)
(12,10,12)	(12,11,4)	(12,11,12)	(12,13,4)	(12,13,12)	(12,14,12)	(12,15,4)	(12,15,12)

The previous tables support Proposition 3, Lemma 7 and Corollary 8. In the case modulo 16, we restrict to matrices whose diagonal entries are divisible by 4, since only in this case do new obstructions arise that are not visible in the modulo 4 classification.

Acknowledgements. The authors were supported by the Croatian Science Foundation under the project no. IP-2022-10-5008 (TEBAG). The authors acknowledge support from the project “Implementation of cutting-edge research and its application as part of the Scientific Center of Excellence for Quantum and Complex Systems, and Representations of Lie Algebras”, Grant No. PK.1.1.10.0004, co-financed by the European Union through the European Regional Development Fund – Competitiveness and Cohesion Programme 2021-2027. This research was funded by the European union: NextGenerationEU through the National Recovery and Resilience

Plan 2021-2026. Institutional grant of University of Zagreb Faculty of Science (IK IA 1.1.3. Impact4Math). The authors used an AI-based language model (ChatGPT, OpenAI) for minor assistance with wording and presentation.

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