BASIC TRANSFORMATION OF THE SERIES

Stationarity is needed for many statistical procedures. Real life data sets do not satisfy this assumption frequently.

Therefore: "nonstationary looking" time series are first transformed.

EXAMPLE 1

for S_t = price of financial assets we frequently apply next transformation

$$(S_t) \longrightarrow (X_t) \quad \text{where} \quad X_t = \log S_t / S_{t-1}$$

X_t = log returns,

if S_t - do not chang rapidly over time, Taylor's formula gives

$$X_t = \log\left(1 + \frac{S_t - S_{t-1}}{S_t}\right) \approx \frac{S_t - S_{t-1}}{S_{t-1}} =: X_t'$$

X_t' = (relative) returns.

Random processes $(X_t)_{t\in\mathbb{Z}}$ can be viewed as random elements in vector space $\mathbb{R}^{\mathbb{Z}}$

$$\mathbb{R}^{\mathbb{Z}} = \{ (x_i)_{i\in\mathbb{Z}} ,\ x_i \in \mathbb{R} \}$$

endowed with Borel σ-algebra $\mathcal{B}(\mathbb{R}^{\mathbb{Z}})$

On space $\mathbb{R}^{\mathbb{Z}}$ we define <u>backward</u> <u>shift</u> <u>operator</u> B by

$$BX_t := (B(X))_t = X_{t-1} \quad , \quad t \in \mathbb{Z}$$

We also need <u>differencing operator</u>

$$\nabla X_t := (1-B)X_t = X_t - X_{t-1}$$

Also recursively we define

$$B^j X_t = B(B^{j-1}X_t) = X_{t-j}$$

$$\nabla^j X_t = \nabla(\nabla^{j-1}X_t), \quad j \in \mathbb{N}$$

where $\nabla^0 = B^0 = 1 = id$

NOTE: powers of ∇ & B commute

EXAMPLE 2

Suppose (Y_t) is strictly stationary, $X_t = at + Y_t$, then X_t can't be stationary unless $a = 0$.

Still $\nabla X_t = a + Y_t - Y_{t-1}$ is strictly stationary.

EXE 1> Show $\nabla^2 X_t = X_t - 2X_{t-1} + X_{t-2}$,

& find general expression for $\nabla^k X_t$, $k \geq 1$.

EXE 2> Suppose (Y_t) is stationary with mean zero, let

$$X_t = m_t + Y_t \quad \text{where} \quad m_t = \sum_{j=0}^{k} a_j t^j$$

then $\nabla^k X_t = k!\, a_k + \nabla^k Y_t$

i.e. $(\nabla^k X_t)$ is stationary with mean $k!\, a_k$.

(differencing can cancel any polynomial trend)

For a given sequence $(h_t)_{t\in\mathbb{Z}}$ of real numbers & time series $(X_t)_{t\in\mathbb{Z}}$ if process $(Y_t)_t$ satisfies

$$Y_t = \sum_{k=-\infty}^{\infty} h_k X_{t-k}, \quad \forall t$$

we say that (Y_t) is obtained by acting of <u>linear filter</u>

$$\Psi(B) = \Psi^h(B) = \sum_{k=-\infty}^{\infty} h_k B^k \qquad \text{(just notation)}$$

on the proces (X_t).

$\boxed{\text{DEF}}$ If $(Z_t) \sim WN(0, \sigma^2)$ & (X_t) satisfies

$$X_t = \sum_{j=-\infty}^{\infty} h_j Z_{t-j} \qquad \forall t \in \mathbb{Z} \qquad (\Delta)$$

for a sequence (h_j) s.t. $\sum_{j=-\infty}^{\infty} |h_j| < \infty$, time series (X_t) is called <u>linear process</u>.

Thus

linear process = linear filter applied on WN

We write

$$X_t = \psi(B) Z_t .$$

Linear process for which $h_k = 0 \ \forall k < 0$, is also called MA(∞) process.

Condition $\sum_{j=0}^{\infty} |h_j| < \infty$ ensures that random series in (*) converges a.s.

EXE 3 > Suppose (X_t) is st. $X_t \geq 0$

then $E \sum_{t \in \mathbb{Z}} X_t = \sum_{t \in \mathbb{Z}} E X_t$ $\left(= +\infty \text{ possibly} \right)$

(L.t.m.c.)

EXE 4> Suppose $\sum_{t\in\mathbb{Z}} E|X_t| < \infty$. then

$\sum X_t$ converges a.s. & $E\sum X_t = \sum E X_t$.

(L.t.d.c.)

LEMMA 1

For a time series (Z_t) & a sequence (h_t) s.t. $\sum_{-\infty}^{\infty} |h_t| < \infty$ we have

i) $\sup_t E|Z_t| < \infty \Rightarrow \sum_{-\infty}^{\infty} h_k Z_{t-k}$ converges absolutely a.s. & in L_1.

(i) $\sup E Z_t^2 < \infty \Rightarrow \sum_{-\infty}^{\infty} h_k Z_{t-k}$ converges also in L_2

(ii) if (Z_t) is weakly stationary $\Rightarrow$ the same holds for $X_t = \sum_{-\infty}^{\infty} h_k Z_{t-k}$ &

$$\gamma_X(h) = \sum_i \sum_j h_j h_{j+i-h} \gamma_Z(i)$$

Recall: $Y_t \xrightarrow{L_p} Y \equiv E|Y_t - Y|^p \to 0$

Proof of Lemma

i) observe $\sum E|h_j Z_{t-j}| \le \sup E|Z_t| \sum |h_j| < \infty$

& apply exercise 4.

ii) by i) series converges a.s.

$$\Big|\sum_{|j|>k} h_j Z_{t-j}\Big|^2 \le \Big(\sum_{|j|>k} |h_j||Z_{t-j}|\Big)^2$$

$$= \sum_{|j|>k}\sum_{|i|>k} |h_j||h_i||Z_{t-i}||Z_{t-j}|$$

Cauchy - Schwarz ineq. $\Rightarrow$

$$E|Z_{t-j}||Z_{t-i}| \le \sup_t E Z_t^2 \quad \Rightarrow$$

$$E\left|\sum_{|j|\le k} h_j Z_{t-j} - \sum_j h_j Z_{t-j}\right|^2 \le \sum_{|i|,|j|>k} |h_i||h_j| \sup E Z_{t}^2$$

$$= \Big(\sum_{|i|>k} |h_i|\Big)^2 \cdot \sup_t E Z_t^2$$

$$\longrightarrow 0 \qquad n \to \infty$$

(iii) by i) X_t converges in L_1

by exerc. 4

$$E \sum h_j Z_{t-j} = \sum h_j E Z_{t-j} = \sum h_j \cdot \mu$$

Similarly

$$\gamma_X(h) = \mathrm{Cov}\left(\sum h_j Z_{t-j}, \sum h_i Z_{t-i}\right)$$

$$= \quad (\text{w.l.o.g. assume } EZ_t = 0)$$

$$= E\left(\sum_i \sum_j h_i h_j Z_{t+h-j} Z_{t-i}\right)$$

$$\overset{\text{exerc 4}}{=} \sum_i \sum_j h_i h_j \gamma_Z(h-j-i)$$

$$= \sum_j \sum_i h_j h_{i+j-h} \gamma_Z(i)$$

EXE 5 > For $Z_t \sim WN(0,\sigma^2)$ sufficient condition for convergence of $\sum \psi_j Z_{t-j}$ is $\sum \psi_j^2 < \infty$. (hint: refer to Hilb. space theory)

$\boxed{\text{DEF}}$ Linear filter $\psi(B) = \sum_{-\infty}^{\infty} h_k B^k$ is called <u>causal</u> if $h_k = 0$ $\forall k < 0$

Also linear process

$$X_t = \sum h_j Z_{t-j} \qquad t \in \mathbb{Z}$$

is causal w.r.t. to WN (Z_t) if

$h_k = 0$ $\quad \forall k < 0$.

Intuitively: causal linear process does not depend on future noise.

EXAMPLE 3

As we have seen for $|\varphi| < 1$ AR(1) stationary process exists

$$X_t = \sum_{k=0}^{\infty} \varphi^k Z_{t-k}$$

& it is causal.

EXE 6 > For linear filter $\psi(B)$ & $\Phi(B)$, convolution

$$\psi \circ \Phi(B) = \sum \vartheta_k B^k$$

$$\vartheta_k = \sum \psi_j \varphi_{k-j}$$

commutes.

A GENERAL APPROACH TO PRACTICAL T.S.A.

When we cannot justify stationarity assumption, we assume a model of the form

$$\boxed{X_t = m_t + s_t + Y_t}$$

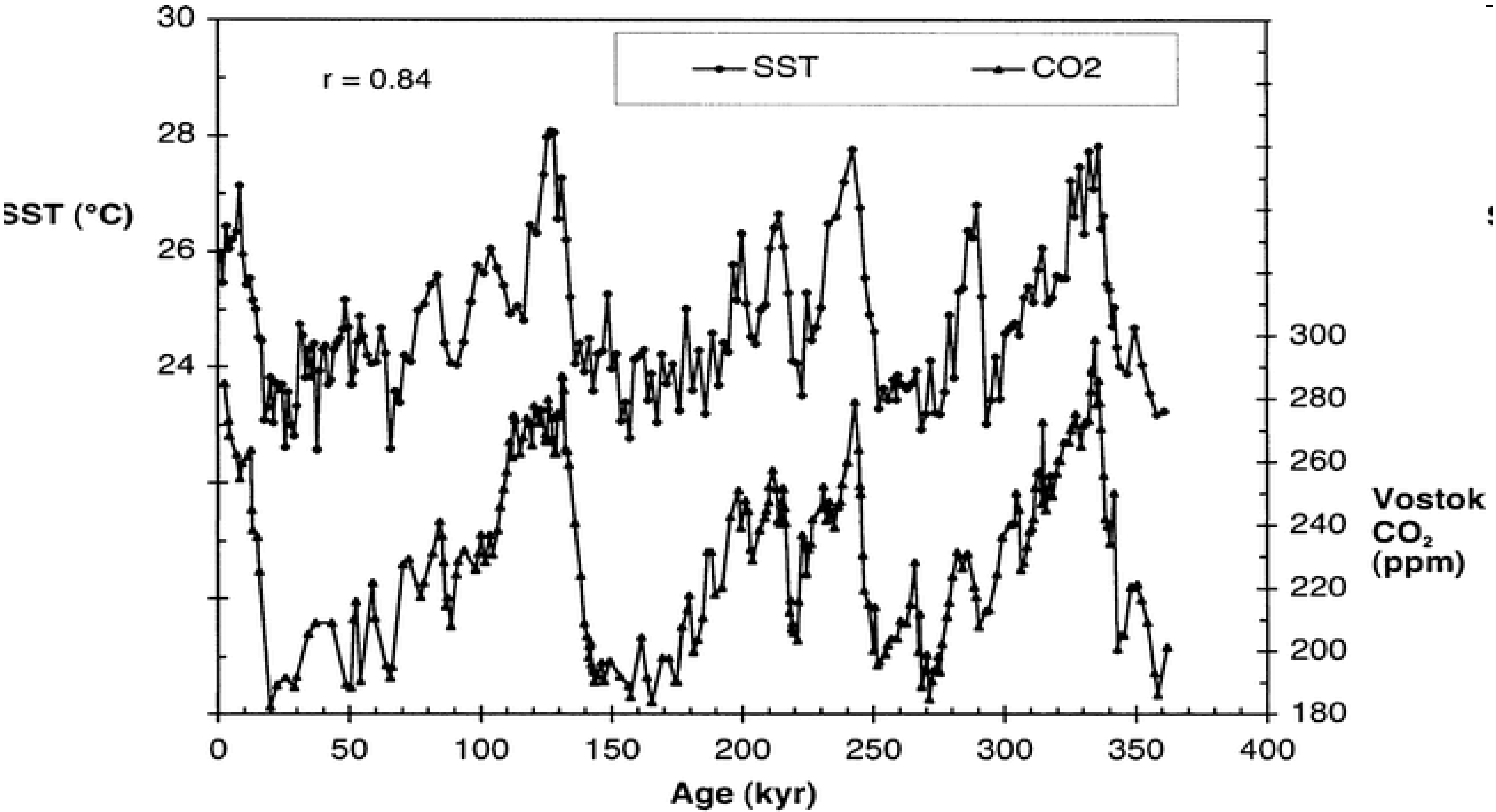
where

m_t = deterministic sequence, "not varying rapidly" → describing trend

s_t = deterministic periodic sequence of known period (e.g. 12 m)
→ describing seasonality

y_t = a stochastic process, typically a stationary time series

REM In practice, it is not easy to recognize individual components, e.g. seasonality of long period vs. trend vs. noise (→ think of climate data)



Comparison of tropical Pacific sea surface temperature (SST), derived from Mg/Ca determinations on surface-dwelling planktonic foraminifera in core TR163-19 on the southwest Cocos Ridge (Lea et al. 2000), with atmospheric CO2 levels determined from the Vostok ice core, east Antarctica (Petit et al. 1999)

Brockwell & Davis in ITSF (2002) book recommend the following general approach to modeling

1. plot the series & examine its main features — trend, seasonality, sharp changes in behavior, outliers, ...

2. remove trend & seasonality
   — it is sometimes sufficient to apply simple transformation to do this (e.g. log returns / differencing ...)

3 find a stationary model for residuals

— use sample autocorr. funct. for instance, or periodogram.

4 forecasting should be based on estimated model taking into account all performed transformations.

All 4 steps are important, still traditionally the most attention is given to step 3 which is done in several smaller steps

- model specification (AR or MA or ...)
- parameter estimation (fitting the model)
- diagnostics (goodness of fit)

This recommendation stems from seminal book Box & Jenkins (1976).

Practitioners also rely on parsimony principle or Ockham's razor in their choice of model

Still, another well known t.s. expert A. Parzen quotes A. Einstein

"everything should be made as simple as possible but not simpler"

Another frequent t.s. quote is by G. Box

"All models are wrong, some models are useful"

Introduction to R

- See http://www.r-project.org/
- R is a language and environment for statistical computing and graphics.
- It is a GNU project which is similar to the S language and environment which was developed at Bell Laboratories.
- R can be considered as a different implementation of S.

Learn R by typing

- Just type this commands after R prompt

```
help.start()
help(mean)
? solve
apropos(mean)
```

Vectors and matrices

- Vectors are easily manipulated in R

```
x <-
  c(10.2,12.3,5.6,4.7,18.0,7.9,1.4,2.3
  )
x <- c(x,6.6)
1/x
x^2
x <- 1:4
x <- seq(-5,5,by=0.2)
y <- rep(2,10)
```

- Next try

```
mean(x)
sum(x)
max(x)
median(x)
var(x)
```

- Next try

```
mean(x)
sum(x)
max(x)
median(x)
var(x)
```

- Guess what happens after

```
x <- 0:4
i <- rep(2,5)
y <- rep(x,2)
z <- rep(x,i)
rep(x,x)
x[2]
x/0
y <- rnorm(10,3,2)
y[y<0] <- -y[y<0]
```

- Special values are : NA, NaN, -Inf, Inf, 1i

```
x <- c(x,NA)
y <- x[!is.na(x)]
# x indexed by a logical vector
x[2:6]
y <- z[-(3:6)]
z[z>2]
sqrt(1+1i)->z
```

- Matrices are easy too

```
z <- rep(3,25)
dim(z) <- c(3,5,5)
z <- array(x,c(2,4,4))
z <-matrix(0,3,5)
z <- diag(c(1,2,4,8))
z[4,]
```

- Matrix operations/functions are

```
 %*% (matrix product), %o$ (outer
  product),...
solve(A,b), eigen(A), qr(A),...
```

Packages and data

- R comes with some data, but additional packages are easily added

```
library(stats)
library(tseries)
data(sunspots)
help(sunspots)
ts.plot(sunspots,lwd=2,col=2)

data(USeconomic)
help(USeconomic)
ts.plot(USeconomic)
plot.ts(USeconomic)
plot.ts(USeconomic[,2])
```

Import data

- There are many ways to do so

```
x<-scan()
x <-
  matrix(scan("filename.dat"),ncol=5,byrow=T)
```

- Or from http://www.ljse.si/

```
krka<-
  read.table("KRKAFeb09Feb10.txt",header=TRUE)
attach(krka)
ts.plot(krka)
```

Transform data

- There are many ways to do so, but it is natural to try:

```
krka<-krka[,1] #krka is data frame
m <- length(krka)
log.ret = (log(krka[2:m])-
  log(krka[1:(m-1)]) )
ts.plot(log.ret)
acf(log.ret)
acf(log.ret^2)
```