GARCH processes

Bojan Basrak
University of Zagreb

ARCH(p) MODEL

Assume : $Z_t \sim \mathrm{IID}(0,1)$

$$\alpha_0 > 0,\ \alpha_p > 0,\ \alpha_1, \dots, \alpha_{p-1} \geq 0$$

$$X_t = \sigma_t \cdot Z_t$$

$$\sigma^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 \qquad t \in \mathbb{Z}$$

Define $U_t = \sigma_t^2 (Z_t^2 - 1)$, $\varphi(z) = 1 - \sum_{i=1}^{p} \alpha_i z^i$

$\Rightarrow \varphi(B) X_t^2 = \alpha_0 + U_t$

So, ARCH(p) process squared (X_t^2) can be viewed as AR(p) process with noise which is not iid.

EXE 3) If $E\sigma_0^4 < \infty$, $EZ_0^4 < \infty$, show that (V_t) is white noise.

It turned out that ARCH(p) do not fit log-returns very well unless p is large $\longrightarrow$ idea: add MA part to recursion

GARCH (p,q) MODEL

Assume : $Z_t \sim IID(0,1)$

$$X_t = \sigma_t Z_t$$

$$\alpha_0, \dots, \alpha_p, \beta_1, \dots, \beta_q \geq 0 \quad \& \quad \alpha_0, \alpha_p, \beta_q > 0$$

$$\sigma_t^2 = \alpha_0 + \sum_{1}^{p} \alpha_i X_{t-i}^2 + \sum_{1}^{q} \beta_j \sigma_{t-j}^2 \qquad t \in \mathbb{Z}$$

<u>EXAMPLE 2</u> (GARCH(1,1))

Note

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

$$= \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1) \sigma_{t-1}^2$$

SRE again

From thm 1 $\Rightarrow$ GARCH(1,1) has unique, causal, stationary solution if $E \log(\alpha_1 Z_{t-1}^2 + \beta_1) < 0$

<u>IGARCH</u> (integrated GARCH) is GARCH(p,q) process with

$$\sum_1^p \alpha_j + \sum_1^q \beta_j = 1$$

$\Rightarrow$ infinite variance of stat. solution ?!?

[estimators of α's, β's in practice often approximately satisfy this.

REMARK

- GARCH(p,q) models fit real-like financial data reasonably well over not-too long periods.
- they allow simple forecast for condit. distribution of X_{t+1}
- they are related to classical ARMA models
- statistical estimation of parameters is not too difficult

GAUSSIAN QUASI-MAXIMUM LIKELIHOOD

Assume $Z_t \overset{iid}{\sim} N(0,1)$, then

$$X_t \mid X_{t-1}, X_{t-2}, \dots \sim N(0, \sigma_t^2)$$

⇒ one can write conditional densities of X_t's given $X_1, \dots, X_p$ easily

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = f_{X_1,\dots,X_n \mid X_1,\dots,X_p}(x_1,\dots,x_n) \cdot f_{X_1,\dots,X_p}(x_1 \dots x_p)$$

we optimize this w.r.t α's, β's.

we ignore this!

to get Gaussian quasi max. likelihood estimators

Turns out

- asympt. normality with rate $\sqrt{n}$ holds for these estimators for large class of noise distribution
- sometimes "more realistic" assumptions on Z_t's can produce nonconsistent estimators
- in practice initial values of $X_0, X_{-1}, \dots$ $\sigma_0, \sigma_{-1}, \dots$ are not known & have to be initialized somehow → this can be justified theoretically.

TAILS OF S.R.E.

Assume (Y_t) is a stationary solution of s.r.e

$$Y_t = A_t Y_{t-1} + B_t \qquad t \in \mathbb{Z}$$

for some iid sequence $(A_t, B_t)_t \in \mathbb{R}_+^2$.

<u>THEOREM 2</u> (Goldie)

Suppose for some $\kappa > 0$, $\varepsilon > 0$

$$E A_t^{\kappa} = 1, \quad E B_t^{\kappa+\varepsilon} < \infty$$

then

$$P(Y_t > u) \sim c \cdot u^{-\kappa} \qquad u \to \infty \quad (*)$$

for some constant $c > 0$.

The tail of y_t in thm 2 is called power-law tail (very popular subject in contemporary statistics & probability).

<u>REMARK</u>

- $(*) \Rightarrow E y_t^{\alpha+\varepsilon} = +\infty \quad \forall \varepsilon > 0$
- more general are regularly varying tails

Cor. 3

If (X_t) is a stat. ARCH(1) process with stand. Gaussian noise, let

$$E(\alpha_1 Z^2)^{\varkappa} = 1 \quad \text{for } \varkappa > 0$$

$$\Rightarrow \quad P(X_t > x) \sim \frac{c}{2} \, x^{-2\varkappa} \qquad x \to \infty$$

REMARK

- for $\alpha_1 \in (0,1)$ X_t is stationary with finite variance
- for $1 \le \alpha_1 < 2e^{\gamma} \approx 3{,}56$ (γ = Euler con.) (X_t) has infinite variance
- for $\alpha_1 \ge 2e^{\gamma}$ one cannot find station. causal solution (see thm 1).

Spectral analysis of time series

Bojan Basrak
University of Zagreb

SPECTRAL ANALYSIS

OF TIME SERIES

Assume: (X_t) is weakly stationary time series with autocov. function γ_X s.t.

$$\sum_{h\in\mathbb{Z}} |\gamma_X(h)| < \infty$$

Then, the series

$$\boxed{f_X(\lambda) = \frac{1}{2\pi} \sum_{h\in\mathbb{Z}} \gamma_X(h)\, e^{-ih\lambda}} \qquad (1)$$

is absol. convergent, uniformly in $\lambda \in \mathbb{R}$

Function f_X in (1) is called the spectral density of (X_t)

It is clearly periodic 2π, so we shall only consider it on interval

$$(-\pi, \pi]$$

Moreover, unif. convergence in (1) allows us to interchange sum & integral to obtain

$$\boxed{\gamma_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda)\, d\lambda}$$

(2) INVERSION FORMULA

In analysis (1) represents Fourier series of function f_X; and by (2) $\gamma(h)$ are corresponding Fourier coefficients.

Functions $(e^{ih\cdot})_{h\in\mathbb{Z}}$ form a basis of $L_2((-\pi,\pi], \text{Leb})$.

Condition $\sum|\gamma(h)|<\infty$ is more restrictive than necessary, ($\sum\gamma^2(h)<\infty$ is enough) but sufficient for us.

LEMMA 1

Function f_X is also even & nonnegative.

Four. series in (1) will not converge unless $\gamma(h) \to \infty$ sufficiently fast, so in such cases spectral density will not exist. Still we will be able to find

<u>spectral measure</u>

TEOREM 2 (Herglotz)

For every weakly stationary sequence (X_t) there exists a unique finite measure F_X on $(-\pi, \pi]$ s.t.

$$\gamma_X(h) = \int_{(-\pi,\pi]} e^{ih\lambda} \, dF_X(\lambda) , \qquad h \in \mathbb{Z}$$

$\boxed{\text{DEF}}$ The measure F_X above is called the spectral measure of (X_t)

If it has density w.r.t the Lebesgue meas. f_X, then f_X is called spectral density.

EXAMPLE 1 (White noise)

Since $\gamma_X(h) = 0$ $h \neq 0$ for $X \sim WN$

$$\Rightarrow \quad f_X(\lambda) = \frac{1}{2\pi}\gamma_X(0) \qquad \forall \lambda$$

we'll say w. noise contains all possible frequencies in the same amount.

EXAMPLE 2 (Deterministic periodic sequence)

Let

$$X_t = A\cos\lambda t + B\sin\lambda t, \quad \lambda \in (0,\pi)$$

$$EA = EB = 0, \quad \operatorname{Var}A = \operatorname{Var}B = \sigma^2, \quad \operatorname{Corr}(A,B) = 0$$

We showed

$$\gamma_X(h) = \sigma^2 \cos h\lambda = \frac{\sigma^2}{2}\left(e^{ih\lambda} + e^{-i\lambda h}\right)$$

$\Rightarrow$ F has mass $\frac{\sigma^2}{2}$ at points λ & $-\lambda$.

REMARK : For real valued t. series spectral measure is symmetric (show it)

So we will ignore point $-\lambda$, & call only λ the frequency of this time series

EXE 1>

a) if $(X_t), (Y_t)$ are two uncorr. stationary seq., show that the spec. measure of $(X_t + Y_t)$ is the sum of corresponding spectral measures

b) Find a seq. with symmetric spectral measure concentrated at points $\pm\lambda_i$, $i = 1, .., k$ for arbitrary $\lambda_i \in (0, \pi)$

c) Find a seq s.t. F_X has all its mass at 0.

d) Find a seq s.t. F_X has all its mass at π.

e) Show that every finite measure on $(-\pi, \pi]$ is the spectral meas. of some t.series.

SPECTRAL ANALYSIS

- inference about time series using spectrum, as opposed to the usual analysis using acf's which is called "analysis in time domain"
- also called "analysis in frequency domain".

FILTER & SPECTRUM

Recall a concept of filter $(\psi_j)_{j\in\mathbb{Z}}$ acting on a stat. time series to obtain a linear process.

In signal processing & physics by filter or transfer function we refer to

$$\psi(\lambda) = \sum \psi_h e^{ih\lambda}$$

<u>TEOREM 3</u> (effect of filtering on spectrum)

For a stationary seq (X_t) with spectral measure F_X, & filter (ψ_j) s.t. $\sum|\psi_j| < \infty$

Define $Y_t = \sum \psi_j \cdot X_{t-j}$

then $dF_Y(\lambda) = |\psi(\lambda)|^2 dF_X(\lambda)$ is sp. meas. of (Y_t).

EXAMPLE 3 (MA(1))

Since for $Z_t \sim WN(0,\sigma^2)$, $f_Z(\lambda) = \frac{\sigma^2}{2\pi}$

THM3 $\Rightarrow$ for $X_t = Z_t + \vartheta Z_{t-1}$

$$f_X(\lambda) = |1+\vartheta e^{-i\lambda}|^2 \frac{\sigma^2}{2\pi} = (1+2\vartheta\cos\lambda+\vartheta^2)\frac{\sigma^2}{2\pi}$$

Note: for $\vartheta > 0$ small frequences dominate

for $\vartheta < 0$ larger frequence dominate

EXAMPLE 4 (complex valued periodic seq.)

Assume $EA = 0$, $\operatorname{Var} A = \sigma^2 < \infty$, let $\lambda \in (0, \pi)$ &

$X_t = A e^{i\lambda t} \Rightarrow \gamma_X(h) = e^{ih\lambda} \cdot \sigma^2$

$\Rightarrow F_X$ has mass σ^2 at point λ.

$\Rightarrow Y = \sum \psi_j X_{t-j}$ has mass $|\psi(\lambda)|^2 \cdot \sigma^2$ at point λ.

EXE 2) Find the spec. measure for

$X_t = A e^{i\lambda t}$ for $\lambda \notin (-\pi, \pi]$.

EXAMPLE 5 (band pass filter)

Consider
$$\psi(\lambda) = \begin{cases} 0 & |\lambda-\lambda_0| > \delta \\ 1 & |\lambda-\lambda_0| \le \delta \end{cases}$$

for fixed frequency λ_0 & bandwidth δ.

This filter by Ex. 4 kills all frequencies $\notin [\lambda_0-\delta, \lambda_0+\delta]$. Spectral density of so filtered signal

$$Y_t = \sum \psi_j X_{t-j}$$

(if it exists) is

$$f_Y(\lambda) = |\psi(\lambda)|^2 f_X(\lambda) = \begin{cases} 0 & |\lambda-\lambda_0| > \delta \\ f_X(\lambda) & |\lambda-\lambda_0| \le \delta \end{cases}$$

For small δ

$$\mathrm{Var}\, Y_t = \gamma_Y(0) = \int_{-\pi}^{\pi} f_Y(\lambda)\, d\lambda = \int_{\lambda_0-\delta}^{\lambda_0+\delta} f_X(\lambda)\, d\lambda \approx 2\delta f_X(\lambda_0)$$

↳ $f_X(\lambda_0)$ is "proportional to variance of subsignal in (X_t) of frequency λ_0"

Note: band pass filter is theoretical filter only! In practice only smooth transfer functions can be implemented.

REMARK

- Instead of frequences λ we can use periods ($e^{i\lambda t}$ is periodic with period $2\pi/\lambda$). E.g. monthly series with period 12 months will have visible peak in the spectrum at frequency $2\pi/12$
- Frequency π is highest possible (so called Nyquist frequency), this is because we only observe t. series at integer times.

Consequence of Thm 3 is

THEOREM (Spectral density of ARMA process)

A causal ARMA process (X_t) has spectral density

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \left| \frac{\vartheta(e^{-i\lambda})}{\varphi(e^{-i\lambda})} \right|^2 \quad \lambda \in (-\pi, \pi]$$

SPECTRAL DECOMPOSITION

It turns out that:

any stationary time series can be written as a randomly weighted sum of single frequence signals $e^{i\lambda t}$.

For uncorrelated $Z_1, \dots, Z_k$ with mean 0 & arbitrary $\lambda_1, \dots, \lambda_k \in (-\pi, \pi]$,

$$X_t = \sum_{j=1}^{k} Z_j e^{i\lambda_j t}$$

has sp. meas.

$$F_X = \sum_{j=1}^{k} E|Z_j|^2 \delta_{\lambda_j}$$

Thus

$$X_t = \sum_{j=1}^{k} Z_j e^{i\lambda_j t} = \int e^{i\lambda t} \sum_{j=1}^{k} Z_j \, \delta_{\lambda_j}(dt)$$

← SPECTRAL DECOMPOSITION (REPRESENTATION) of (X_t)

⇓

X_t is the sum of uncorrelated single-frequency signals of stochastic amplitudes

Interestingly: any zero mean stationary (X_t) with discrete sp. measure has such a decomposition

More interestingly: any mean zero stationary time series has such a decomposition only the sum becomes the integral

$$X_t = \int_{(-\pi,\pi]} e^{i\lambda t}\, dZ(\lambda)$$

w.r.t. some random measure Z.

$\boxed{\text{DEF}}$ A random measure with orthogonal increments Z is a collection of r.v.'s $\{Z(B) : B \in \mathcal{B}\}$ with mean zero, on some $(\Omega, \mathcal{F}, \mathbb{P})$ s.t. for some finite Borel measure μ on $(-\pi, \pi]$

$$\text{Cov}(Z(B_1), Z(B_2)) = \mu(B_1 \cap B_2) \quad \forall B_1, B_2 \in \mathcal{B}$$

REMARK Definition of $Z \Rightarrow Z$ is σ-additive.

Also $\left(Z_\lambda = Z_{(-\pi,\lambda)} : \lambda \in (-\pi,\pi]\right)$ is a stochastic process with uncorrelated increments.

<u>Problem</u>: How to define integral w.r.t. Z.

Idea :

STEP 1 Set $\int 1_B \, dZ = Z(B)$

$$\int \sum_1^k \alpha_j 1_{B_j} \, dZ = \sum_1^k \alpha_j Z(B_j)$$

for any $\alpha_j, B_j \in \mathcal{B}$

STEP 2

- For $f \in L_2(\mu)$ take sequence of step functions f_n s.t. $f_n \to f$ in $L_2(\mu)$
- Note $f \overset{\Phi}{\longmapsto} \int f \, dZ$ is linear isometry on step functions
- Set $\int f \, dZ = \lim \int f_n \, dZ$ in $L_2(\Omega, \mathcal{F}, \mathbb{P})$

REMARK $\Phi : L_2((-\pi,\pi],\mu) \to L_2(\Omega,\mathcal{F},P)$ is a linear isometry between two Hilbert spaces

THEOREM 5

For any mean zero stationary t.s. (X_t) with spec. meas. F_X there exists a random measure Z with orth. inc. s.t

$$\boxed{X_t = \int_{(-\pi,\pi]} e^{i\lambda t}\, dZ(\lambda)} \quad \text{a.s. } \forall t \in \mathbb{Z}$$

SPECTRAL DECOMPOSITION

ESTIMATION OF THE SPECTRAL DENSITY

Recall, if $\sum |\gamma(h)| < \infty$, (X_t) stationary

$$f_X(\lambda) := \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_X(h)$$

It is natural to replace γ_X with $\hat{\gamma}_X$ to get an estimator. If we assume X_t's are centered, we could use

$$\tilde{\gamma}_X(h) = \begin{cases} \frac{1}{n} \sum^{n-|h|} X_t X_{t+|h|} & |h| < n \\ 0 & \text{otherwise} \end{cases}$$

The natural estimator is thus

$$I_{n,X}(\lambda) = \frac{1}{2\pi} \sum_{|h|<n} e^{-ih\lambda} \tilde{\gamma}_X(h)$$

PERIODOGRAM

$$= \frac{1}{2\pi} \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} X_t X_s e^{-i\lambda(t-s)}$$

$$= \frac{1}{2\pi} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{-i\lambda t} X_t \right|^2$$

We usually evaluate $I_{n,X}$ only at

Fourier frequencies: $\lambda_j = \frac{2\pi \cdot j}{n}$, $0<j<\lfloor n/2 \rfloor$

i.e $\lambda_j \in (0,\pi]$

Since $\sum_{t=1}^{n} e^{i\lambda_j t} = 0$ at Fourier frequencies

$\Downarrow$

Periodogram is the same whether we center X_t or not

$\Downarrow$

$$I_{n,X}(\lambda_j) = \frac{1}{2\pi} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{-i\lambda_j t}(X_t - \overline{X}_n) \right|^2$$

$$= \frac{1}{2\pi} \sum_{|h|<n} e^{-ih\lambda_j} \hat{\gamma}(h)$$

EXAMPLE 6 (periodogram of Gaussian w. noise)

Assume

$$Z_t \sim WN(0,\sigma^2) \text{ Gaussian},$$

Since $I_{n,Z}(\lambda_j) = \frac{1}{2\pi} \left| \frac{1}{\sqrt{n}} \sum_1^n e^{i\lambda_j t} Z_t \right|^2$

observe inner product of two complex Gaussian r.v.'s

$$E\left(\frac{1}{\sqrt{n}} \sum_{t=1}^n e^{-\lambda_j t} Z_t\right)\overline{\left(\frac{1}{\sqrt{n}} \sum_{s=1}^n e^{i\lambda_k s} Z_s\right)} =: \langle X_j, X_k \rangle$$

$$= \frac{1}{n} \sum_{t=1}^n E Z_1^2 e^{-i(\lambda_j - \lambda_k)t}$$

$$= \dots = \begin{cases} \sigma^2 & j = k \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow X_j, X_k$ are mean zero, uncorrelated Gaussian r.v.'s $\Rightarrow X_i \overset{iid}{\sim} N(0, \sigma^2)$

Observe $I_{n,Z}(\lambda_j) = \frac{1}{2\pi} \qquad |X_j|^2$

$$= \frac{1}{2\pi}\Big(\underbrace{\frac{1}{\sqrt{n}}\sum_1^n Z_t \cos\lambda_j t}\Big)^2 + \frac{1}{2\pi}\Big(\underbrace{\frac{1}{\sqrt{n}}\sum_1^n Z_t \sin\lambda_j t}\Big)^2$$

independent

$\Downarrow$ & both $N(0, \sigma^2/2)$ distributed

$2\pi I_{n,Z}$ has $\frac{\sigma^2}{2}\chi_2^2$ distribution $= \sigma^2 \cdot Exp(1)$

Thus the periodogram of iid Gaussian seq. at the Fourier frequences has values which are iid exponential with mean $\sigma^2/2\pi$!?!?

Recall the spectral density was $\dfrac{\sigma^2}{2\pi}$.

→ Periodogram is not consistent estmator

→ Fisher derived g-test for Gaussian white noise from this.

Still periodogram is not far from consistency

PROPOSITION 6

(X_t) mean zero, stationary, $\sum|\gamma(h)| < \infty$

$$\Rightarrow \quad E I_{n,X}(\lambda) \longrightarrow f_X(\lambda) \quad \forall\, \lambda \in (0,\pi]$$

EXE 3) Prove Proposition 6.

Example 6 really points the asymptotic properties of periodogram

THEOREM 7

Assume (X_t) is a linear process with abs. summable coefficients (ψ_j) driven by white noise with variance σ^2.

For fixed frequencies $0 < \omega_1 < \cdots < \omega_m < \pi$

$$\left(I_{n,X}(\omega_j) \right)_j \xrightarrow{d} \left(\frac{\sigma^2}{2\pi} \cdot |\psi(e^{-i\omega_j})| \cdot E_j \right)$$

$$= \left(f_X(\omega_j) \cdot E_j \right)$$

for $E_j \overset{iid}{\sim} \mathrm{Exp}(1)$ $\quad j = 1, \ldots, n.$

Although, it is inconsistent, smoothed periodogram can be used to get consistent estimator.

For some <u>weights</u>: $(W_n(k))_{|k|\le m}$

$$W_n(k) = W_n(-k), \quad \sum_{|k|\le m} W_n(k) = 1, \quad \sum_{|k|\le m} W_n^2(k) \to 0$$

Use

$$\hat{f}_n(\lambda) = \frac{1}{2\pi} \sum W_n(j)\, I_n\left(g_n(\lambda) + 2\pi \frac{j}{n}\right)$$

DISCRETE SPECTRAL AVERAGE ESTIMATOR

where $g_n(\lambda)$ is closest Fourier freq $\frac{2\pi}{n}$ to the point λ.

In theory we need to let

$$m = m_n \to \infty \quad \& \quad m_n/n \to 0 \quad \text{as } n \to \infty$$

The simplest weights

$$W_n(k) = \frac{1}{2m+1}$$

are called Daniell weights.