Time series analysis

Spring 2010 / Exercises

1. Denote by ∇ the differencing operator, i.e. $\nabla X_t = X_t - X_{t-1},\ t \in \mathbb{Z}$, and assume that (Y_t) is weakly stationary sequence with mean μ and autocovaraince function $\gamma_y(h),\ h \in \mathbb{Z}$. Show that for

$$X_t = -14 + 3t^2 + Y_t, \quad t \in \mathbb{Z},$$

time series $Z_t = \nabla^2 X_t,\ t \in \mathbb{Z}$ is again weakly stationary. Find the mean and autocovariance function of Z_t.

2. Assume that (X_t) is a causal solution of AR(1) equation

$$X_t = 0.9X_{t-1} + Z_t, \quad t \in \mathbb{Z},$$

where $(Z_t) \sim \mathrm{WN}(0, \sigma^2)$.

a) Find the best linear predictor of X_{n+2} in terms of $X_1, \ldots, X_n$ $n \in \mathbb{N}$.

b) Assume that (Z_t) is also an i.i.d. sequence find the best predictor X_{n+2} u terms of $X_1, \ldots, X_n$ $n \in \mathbb{N}$.

3. Determine which of the following ARMA equations has causal, or invertible solution if $(Z_t) \sim \mathrm{WN}(0, \sigma^2)$.

(a)

$$X_t - \frac{1}{4}X_{t-1} + \frac{1}{16}X_{t-2} = Z_t - 3Z_{t-1}, \quad t \in \mathbb{Z}$$

(b)

$$X_t - 2\sqrt{3}X_{t-1} + 4X_{t-2} = Z_t, \quad t \in \mathbb{Z}$$

(c)

$$X_t + 0.9X_{t-1} = Z_t - \frac{1}{4}Z_{t-2}, \quad t \in \mathbb{Z}.$$

(d)

$$X_t - \frac{1}{3}X_{t-1} + \frac{1}{9}X_{t-2} = Z_t + 2Z_{t-1}, \quad t \in \mathbb{Z}$$

(e)

$$X_t - \frac{\sqrt{3}}{4}X_{t-1} + \frac{1}{16}X_{t-2} = Z_t, \quad t \in \mathbb{Z}$$

(f)

$$X_t - \frac{1}{4}X_{t-1} = Z_t - \frac{5}{6}Z_{t-1} + \frac{1}{6}Z_{t-2}, \quad t \in \mathbb{Z}.$$

4. Assume that (X_t) is a weakly stationary solution of ARMA(1,1) equation

$$X_t + \varphi X_{t-1} = Z_t + \theta Z_{t-1}, \quad t \in \mathbb{Z},$$

where$(Z_t) \sim \mathrm{WN}(0, \sigma^2)$, and $\sigma = 1$. Find the linear representation of process (X_t) in terms of (Z_t) depending on constants $\varphi,\ \theta$. Give also expressions for $\mathrm{Var} X_t$ i $\mathrm{Cov}(X_t, X_{t+k}),\ k \in \mathbb{N}$.

5. Consider an MA(1) process $X_t = Z_t + \theta Z_{t-1},\ t \in \mathbb{Z}$, for an i.i.d. sequence $(Z_t) \sim \mathrm{WN}(0, \sigma^2)$ i $\theta \in (0, 1/2)$. Denote $\overline{X}_n = (X_1 + \cdots + X_n)/n$. Find a constant $K > 0$ such that for $n = 10000$

$$P(|\overline{X}_n| > K\sigma(1+\theta)) \approx 0.05.$$

6. Let (Z_t) be an i.i.d. sequence with $E \log(Z^2) < 0$. Show that

$$\sum_{j=0}^{\infty} Z_t^2 Z_{t-1}^2 \cdots Z_{t-j}^2$$

converges almost surely.

7. Find the auto-covariance function of the the volatility sequence σ^2 for a weakly stationary GARCH(1, 1) process.

8. Let φ be a polynomial without roots on the unit disc and let (X_t) be a time series that is bounded in probability. If $\varphi(B)X_t = Z_t$ for every t, show that X_t is $\sigma(Z_t, Z_{t-1}, \ldots)$-measurable.