# Wavelet Bases of Function Spaces

*Participating Analysis Seminar*

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What we are going to discuss in this exposition is an "old school" look at wavelets and other *reproducing function systems*. This approach is motivated by the work of Strömberg [S82] (and much earlier Haar and Franklin), inaugurated by Daubechies, Lemarié, Mallat and Meyer, and dominates the literature in the last 20 years. From the viewpoint of pure harmonic analyst it is "too much concerned" with:

- *minimality and symmetry* – systems are generated from one (or finitely many) functions by applying operators such as translations, dilations and modulations,
- *rigid structure* – we prefer the system to be a sort of basis, frame, etc.,
- *computational/algorithmic aspects* – we often desire for explicit constructions, and to have an explicit reproducing formula.

However, this approach is much more appreciated from the applied viewpoint. Here we try to present possibly interesting applications to another branch of pure math – classical functional analysis. Can we use the insights and results from the "theory of wavelets" to (re)prove/simplify some classical results about function spaces?

## 1 Basic definitions

Before we formulate problems we are going to consider, let us remind ourselves of several notions related to Banach spaces, see for instance [H97] for more details.[1]

Let $X$ be a separable Banach space and $(e_n)_{n\in\mathbb{N}}$ a sequence of vectors in $X$. We say that $(e_n)_{n\in\mathbb{N}}$ is a *Schauder basis* for $X$ if for every $x \in X$ there exists a unique sequence of scalars $(c_n)_{n\in\mathbb{N}}$ such that

$$x = \sum_{n=1}^{\infty} c_n e_n = \lim_{N\to\infty} \sum_{n=1}^{N} c_n e_n,$$

where the series converges in the norm. A consequence of the uniform boundedness principle is that the coefficient functionals $f_n \colon x \mapsto c_n$ are continuous. If $X$ is reflexive, then $(f_n)_{n\in\mathbb{N}}$ is actually a Schauder basis for the dual space $X^*$.

[1] Let us also say a few words about the notation we use. For two nonnegative quantities $A(x,y)$ and $B(x,y)$ depending on $x$ and $y$, i.e. for two nonnegative functions $A$ and $B$ in (abstract) variables $x$ and $y$, we write $A(x,y) \lesssim_y B(x,y)$ if for every $y$ there exists a positive constant $C(y)$ such that for every $x$ we have $A(x,y) \le C(y)B(x,y)$. In other words, the implicit constant may depend on $y$ but not on $x$.

We write $A(x,y) \sim_y B(x,y)$ if $A(x,y) \lesssim_y B(x,y)$ and $B(x,y) \lesssim_y A(x,y)$, i.e. if for every $y$ there exist two positive constants $c(y)$ and $C(y)$ such that for every $x$ we have $c(y)B(x,y) \le A(x,y) \le C(y)B(x,y)$.

One possible characterization of a Schauder basis is that $(e_n)_{n\in\mathbb{N}}$ is complete and there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $X^*$ that is *biorthogonal* to $(e_n)_{n\in\mathbb{N}}$ (i.e. $f_m(e_n) = \delta_{m,n}$) and is such that partial sum operators $(S_N)_{N\in\mathbb{N}}$ defined by $S_N x := \sum_{n=1}^{N} f_n(x)e_n$ are uniformly bounded (i.e. $\sup_{N\in\mathbb{N}} \|S_N\|_{X\to X} < \infty$).

We say that $(e_n)_{n\in\mathbb{N}}$ is an *unconditional basis* for $X$ if for every $x \in X$ there exists a unique sequence of scalars $(c_n)_{n\in\mathbb{N}}$ such that

$$x = \sum_{n\in\mathbb{N}} c_n e_n,$$

where the series converges in the norm and as the net of sums over finite subsets of $\mathbb{N}$. In other words we require that $\sum_{n=1}^{\infty} c_{\sigma(n)} e_{\sigma(n)}$ converges for every permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$. Again, a consequence of the uniform boundedness principle is that the coefficient functionals $f_n\colon x \mapsto c_n$ are continuous. If $X$ is reflexive, then $(f_n)_{n\in\mathbb{N}}$ is an unconditional basis for the dual space $X^*$.

One convenient characterization of an unconditional basis is that $(e_n)_{n\in\mathbb{N}}$ is complete, $e_n \neq \mathbf{0}$ for every $n \in \mathbb{N}$, and

$$\Big\| \sum_{n\in F} \omega_n c_n e_n \Big\| \lesssim \Big\| \sum_{n\in F} c_n e_n \Big\|,$$

for every finite set of indices $F \subset \mathbb{N}$, every choice of signs $\omega_n \in \{-1, 1\}$, and arbitrary scalar coefficients $c_n$. The implicit constant is sometimes called the *unconditional basis constant.*

If $X = H$ is a Hilbert space, then unconditional bases with $\|e_n\| \sim 1$ coincide with *Riesz bases*, and the latter are defined to be complete systems $(e_n)_{n\in\mathbb{N}}$ satisfying

$$\Big\| \sum_{n\in\mathbb{N}} c_n e_n \Big\|_H \sim \Big( \sum_{n\in\mathbb{Z}} |c_n|^2 \Big)^{\frac{1}{2}}.$$

* * *

We will only consider function systems on the real line $\mathbb{R}$. The operators of *translation*, (dyadic) *dilation* and *modulation* are respectively defined to be

$$(\mathrm{T}_k f)(x) := f(x-k), \qquad (\mathrm{D}_j f)(x) := 2^{\frac{j}{2}} f(2^j x), \qquad (\mathrm{M}_l f)(x) := e^{2\pi i l x} f(x),$$

where $j, k, l$ are (usually) integers. Starting from only one generator function $f$, the following are the most commonly considered function systems:

- system of translates: $\{\mathrm{T}_k f : k \in \mathbb{Z}\} = \{f(\cdot - k) : k \in \mathbb{Z}\}$,
- wavelet system: $\{\mathrm{D}_j \mathrm{T}_k f : j, k \in \mathbb{Z}\} = \{2^{\frac{j}{2}} f(2^j \cdot -k) : j, k \in \mathbb{Z}\}$,
- Gabor system: $\{\mathrm{M}_l \mathrm{T}_k f : k, l \in \mathbb{Z}\} = \{e^{2\pi i l x} f(\cdot - k) : k, l \in \mathbb{Z}\}$,
- system of wave packets: $\{\mathrm{D}_j \mathrm{T}_k \mathrm{M}_l f : j, k, l \in \mathbb{Z}\}$.

For instance we say that the system $\{\mathrm{D}_j\mathrm{T}_k f : j,k \in \mathbb{Z}\}$ is an orthonormal/Riesz/Schauder wavelet system and simply say that $f$ is an orthonormal/Riesz/Schauder wavelet if $\{\mathrm{D}_j\mathrm{T}_k f : j,k \in \mathbb{Z}\}$ is an orthonormal/Riesz/Schauder basis for $\mathrm{L}^2(\mathbb{R})$. Notice once more that such system has a rigid structure (i.e. forms a basis) and that it is generated by a single function $f$, sometimes also called the *mother wavelet*.

If we open the book [T06], on the first page we will find a completely different definition of wavelets. The wavelet system is simply a family of rapidly decaying mean-zero functions indexed by (dyadic) intervals, with uniform decay estimates regarding to appropriate normalization. Symmetries of $\mathbb{R}$ only play role in the normalization, and the system need not be finitely generated. Also note that there are no orthogonality conditions imposed, nor it is required from the system to be any kind of basis or a frame. (This philosophy is especially important in applications of wave packets to "modulation invariant time-frequency analysis", because wave packets are so "overcomplete" that they cannot form any type of basis or frame anyway.) Finally, notice that usually we do not care about particular constructions, and rather prefer to have more flexible/arbitrary systems. In this exposition, however, we rather stick to the classical definition of wavelets, in the spirit of [M90].

It turns out that in $\mathbb{R}^d$ only one generating function $f$ is not enough and one needs $2^d - 1$ generators, but one can construct higher-dimensional wavelet-generators in the simplest possible way, by tensoring one-dimensional ones. Also dilations in $\mathbb{R}^d$ are defined for an expanding matrix $A \in \mathrm{M}_d(\mathbb{R})$, so one can discuss systems obtained for various matrices $A$ or even systems obtained from composite dilations determined by two matrices $A, B \in \mathrm{M}_d(\mathbb{R})$, etc. We do not discuss higher-dimensional phenomena here.

## 2 Problems we want to consider

**Problem 1.** *Is there a conditional Schauder basis for a separable Hilbert space, i.e. is there a Schauder basis that is not a Riesz basis.*

Babenko [B48] gave a famous example: For any $0 < \mu < \frac{1}{2}$ the system

$$\{|t|^\mu e^{2\pi ikt} : k \in \mathbb{Z}\}$$

is a conditional Schauder basis for $\mathrm{L}^2(\mathbb{T}) \equiv \mathrm{L}^2([-\frac{1}{2}, \frac{1}{2}])$, where we well-order $\mathbb{Z}$ as $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$.

Simplicity of the example is misleading, because it is rather hard to be justified directly. Machinery developed in the next section will allow us to give relatively short proof.

Later Pelczynski and Singer showed that if an infinitely-dimensional Banach space has at least one Schauder basis, then it must actually have uncountably many (topologically) nonequivalent Schauder bases. Since it can be shown that all unconditional bases in a Hilbert space are mutually equivalent, this gives an existential proof for the affirmative answer to problem 1. However, providing explicit examples is still of greater interest.

**Problem 2.** *Give an explicit construction of an unconditional basis for the real Hardy space* $\mathrm{H}^1(\mathbb{R})$.

Maurey [M80] gave a non-constructive proof of the existence of such basis. The first explicit example was given by Carleson [C80], but Strömberg's construction [S82] was cleaner and worked for other real Hardy spaces as well (see the next "problem"). His work was not appreciated until several years later, when Lemarié and Meyer recognized the relevance of the construction and put it into the appropriate context. Actually they were inspired by much less general paper of Wojtaszczyk [W82] and proper credits to Strömberg were given in the 1990s.

Besides proving that a "reasonably nice" wavelet system forms an unconditional basis for $\mathrm{H}^1(\mathbb{R})$, we will also completely characterize the coefficient space. In such occasions we say that we give a *wavelet characterization* of the function space $\mathrm{H}^1(\mathbb{R})$.

**Problem 3.** *How to construct Schauder/unconditional bases of other interesting function spaces like:*

- $\mathrm{L}^p(\mathbb{R})$ *(Lebesgue spaces),*
- $\mathrm{W}^{p,s}(\mathbb{R})$, $\dot{\mathrm{W}}^{p,s}(\mathbb{R})$ *(inhomogenous and homogenous Sobolev spaces),*
- $\mathrm{H}^p(\mathbb{R})$ *(real Hardy spaces),*
- $\Lambda^s(\mathbb{R})$, $\dot{\Lambda}^s(\mathbb{R})$ *(inhomogenous and homogenous Lipschitz spaces),*
- $\mathrm{B}^{s,q}_p(\mathbb{R})$, $\dot{\mathrm{B}}^{s,q}_p(\mathbb{R})$ *(very general inhomogenous and homogenous Besov spaces).*

# 3 Systems of integer translates and an application to Babenko's construction

In this section we consider only systems of integer translates:

$$\{\mathrm{T}_k f : k \in \mathbb{Z}\} = \{f(\cdot - k) : k \in \mathbb{Z}\}.$$

The purpose of the section is twofold: it clarifies the situation on a single scale, which will certainly be needed later when we consider wavelet systems, and it is simple enough to allow a complete characterization of Schauder/unconditional bases of translates. The beautiful characterization that follows will be applied to justify Babenko's example and thus solve problem 1.

For $f \in \mathrm{L}^2(\mathbb{R})$ we define the *shift-invariant subspace* generated by $f$:

$$\langle f \rangle := \overline{\mathrm{span}}_{\mathrm{L}^2(\mathbb{R})}\{f(\cdot - k) : k \in \mathbb{Z}\}.$$

Also define the *periodization operator*:

$$(Pf)(\xi) := \sum_{k\in\mathbb{Z}} |\hat{f}(\xi + k)|^2; \quad \xi \in \mathbb{R}.$$

Observe that $Pf$ is 1-periodic, i.e. a function on $\mathbb{T} \equiv \mathbb{R}/\mathbb{Z}$. Also observe that $Pf \in \mathrm{L}^1(\mathbb{T})$ because $\|Pf\|_{\mathrm{L}^1(\mathbb{T})} = \|f\|^2_{\mathrm{L}^2(\mathbb{R})} < \infty$.

**Theorem 4.** *(taken from [NS07], see remark 8)*
*Take $f \in \mathrm{L}^2(\mathbb{R})$. The system $\{f(\cdot - k) : k \in \mathbb{Z}\}$ is:*

*(a) an orthonormal basis for $\langle f \rangle \iff Pf \equiv 1$ a.e.,*

*(b) an unconditional (i.e. Riesz) basis for $\langle f \rangle \iff Pf \sim_f 1$ a.e.,*

*(c) a Schauder basis for $\langle f \rangle \iff Pf \in A_2(\mathbb{T})$.*

**Remark 5.** *For those who are familiar with frames we could have also added:*

*(d) a frame for $\langle f \rangle \iff Pf \sim_f \mathbf{1}_{\mathrm{supp}(Pf)}$ a.e.*

Here a (1-periodic) function $w \colon \mathbb{T} \to [0, \infty]$ is called an $A_p(\mathbb{T})$*-weight*, $1 < p < \infty$, if for every interval $I \subseteq \mathbb{T}$

$$\Big( \fint_I w \Big) \Big( \fint_I w^{-\frac{1}{p-1}} \Big)^{p-1} \sim_{w,p} 1.$$

* * *

The first step of the proof is to transform the results to the theory of weighted spaces on the circle via the following simple auxiliary lemma.

**Lemma 6.** *Mapping $g \mapsto (g\hat{f})\check{}$ is an isometric isomorphism $\mathrm{L}^2(\mathbb{T}; Pf) \to \langle f \rangle$ between the weighted $\mathrm{L}^2$-space on the circle and the shift invariant subspace generated by $f$. Also note that under this isomorphism exponentials $e^{2\pi i k x}$ map to translates $f(\cdot + k)$.*

*Proof of lemma 6.* Observe that for any finite set $F \subset \mathbb{Z}$

$$\Big( \sum_{k \in F} c_k f(\cdot + k) \Big)\hat{}(\xi) = \Big( \sum_{k \in F} c_k e^{2\pi i k \xi} \Big) \hat{f}(\xi)$$

and that

$$\|g\|^2_{\mathrm{L}^2(\mathbb{T}; Pf)} = \int_{\mathbb{T}} |g|^2 Pf = \int_{\mathbb{T}} \sum_{k \in \mathbb{Z}} |g(\xi)|^2 |\hat{f}(\xi + k)|^2 d\xi = \|g\hat{f}\|^2_{\mathrm{L}^2(\mathbb{R})} = \|(g\hat{f})\check{}\|^2_{\mathrm{L}^2(\mathbb{R})}$$

∎

Using this lemma the main theorem becomes a statement about the exponentials being different bases of the weighted $\mathrm{L}^2$ space on the circle.

**Theorem 7.** *Let $w \colon \mathbb{T} \to [0, \infty]$ be a 1-periodic function (weight). The system $\{e^{2\pi i k t} : k \in \mathbb{Z}\}$ is:*

*(a) an orthonormal basis for $\mathrm{L}^2(\mathbb{T}; w) \iff w \equiv 1$ a.e.,*

*(b) an unconditional (i.e. Riesz) basis for $\mathrm{L}^2(\mathbb{T}; w) \iff w \sim_w 1$ a.e.,*

*(c) a Schauder basis for $\mathrm{L}^2(\mathbb{T}; w) \iff w \in A_2(\mathbb{T})$.*

**Remark 8.** *Part (c) was first explicitly formulated by Nielsen and Šikić in [NS07] and [N08], although it is actually a reformulation of the well known result by Hunt, Muckenhoupt and Wheeden [HMW73], as we will see in the proof. Also see [HP06] for a similar characterization of Gabor Schauder bases in* $\mathrm{L}^2(\mathbb{R})$*. Parts (a) and (b) are folklore.*

*Proof of theorem 7.* Before anything else notice that $\{e^{2\pi ikt} : k \in \mathbb{Z}\} \subseteq \mathrm{L}^2(\mathbb{T}; w)$ if and only if $w \in \mathrm{L}^1(\mathbb{T})$. Therefore in all that follows we assume that this is the case. Also observe that $\{e^{2\pi ikt} : k \in \mathbb{Z}\}$ is complete in $\mathrm{L}^2(\mathbb{T}; w)$ because trigonometric polynomials are dense in $\mathrm{C}(\mathbb{T}, \|\cdot\|_{\mathrm{L}^2(\mathbb{T};w)})$, which follows from Stone-Weierstrass and $\|g\|_{\mathrm{L}^2(\mathbb{T};w)} \le \|w\|_{\mathrm{L}^1(\mathbb{T})}^{1/2} \|g\|_\infty$, and $\mathrm{C}(\mathbb{T})$ is dense in $\mathrm{L}^2(\mathbb{T}; w)$, which follows from regularity of the measure $w(t)dt$ on $\mathbb{T}$.

*Part (a).* Immediately follows from

$$\int_{\mathbb{T}} e^{2\pi i(k-l)t} w(t)dt = \langle e^{2\pi ikt}, e^{2\pi ilt}\rangle_{\mathrm{L}^2(\mathbb{T};w)} = \delta_{k,l}$$

by comparing Fourier coefficients of $w$ and $\mathbf{1}$.

*Part (b).* The system $\{e^{2\pi ikt} : k \in \mathbb{Z}\}$ is a Riesz basis for $\mathrm{L}^2(\mathbb{T}; w)$ if and only if for every trigonometric polynomial $Q(t) = \sum_k c_k e^{2\pi ikt}$ we have $\|Q\|_{\mathrm{L}^2(\mathbb{T};w)} \sim_w \|Q\|_{\mathrm{L}^2(\mathbb{T})}$. Since trigonometric polynomials are dense in $\mathrm{C}(\mathbb{T})$, this is equivalent to saying $\|g\|_{\mathrm{L}^2(\mathbb{T};w)} \sim_w \|g\|_{\mathrm{L}^2(\mathbb{T})}$ for all $g \in \mathrm{C}(\mathbb{T})$. Another formulation is to say that the linear operator

$$T\colon \mathrm{C}(\mathbb{T}) \to \mathrm{L}^2(\mathbb{T}), \quad T\colon g \mapsto \sqrt{w}\, g$$

extends to a bounded and invertible operator on $\mathrm{L}^2(\mathbb{T})$, but this is equivalent to boundedness of both $w$ and $\frac{1}{w}$.

*Part (c).* ($\Longrightarrow$) Suppose that $\{e_k(t) = e^{2\pi ikt} : k \in \mathbb{Z}\}$ is a Schauder basis for $\mathrm{L}^2(\mathbb{T}; w)$. Then the coefficient functionals $\{f_k : k \in \mathbb{Z}\}$ form a Schauder basis for the dual space $\mathrm{L}^2(\mathbb{T}; w)^* \equiv \mathrm{L}^2(\mathbb{T}; w)$. By the bi-orthogonality

$$\int_{\mathbb{T}} e^{2\pi ikt} f_l(t) w(t)dt = f_l(e_k) = \delta_{k,l},$$

and since $f_l w \in \mathrm{L}^1(\mathbb{T})$, we actually get $f_l(t)w(t) = e^{-2\pi ilt}$, so the Schauder basis partial sum operators are indeed the ordinary Fourier series partial sum operators

$$(S_N g)(t) = \sum_{k=-N}^{N} f_k(g)e_k(t) = \sum_{k=-N}^{N} \Big(\int_{\mathbb{T}} g(u)e^{-2\pi iku}du\Big) e^{2\pi ikt}.$$

Since Schauder basis partial sum operators are uniformly bounded we have $\sup_{N\in\mathbb{N}} \|S_N\|_{\mathrm{L}^2(\mathbb{T};w)\to\mathrm{L}^2(\mathbb{T};w)} < \infty$ and from theorem 9 below we conclude $w \in A_2(\mathbb{T})$.

($\Longleftarrow$) Suppose that $w \in A_2(\mathbb{T})$. In particular $\frac{1}{w} \in \mathrm{L}^1(\mathbb{T})$. Linear functionals $f_k(g) := \int_{\mathbb{T}} g(t)e^{-2\pi ikt}dt$ are well-defined and bounded on $\mathrm{L}^2(\mathbb{T}; w)$ because $\int_{\mathbb{T}} |g(t)|dt \le \|w^{-1}\|_{\mathrm{L}^1(\mathbb{T})}^{1/2} \|g\|_{\mathrm{L}^2(\mathbb{T};w)}$. Also observe that $f_l(e_k) = \delta_{k,l}$, so $\{e_k : k \in \mathbb{Z}\}$ and $\{f_k : k \in \mathbb{Z}\}$ are biorthogonal sequences and the corresponding partial sum operators are simply $S_N$. From theorem 9 below we conclude $\sup_{N\in\mathbb{N}} \|S_N\|_{\mathrm{L}^2(\mathbb{T};w)\to\mathrm{L}^2(\mathbb{T};w)} < \infty$, which implies that $\{e_k : k \in \mathbb{Z}\}$ is a Schauder basis for $\mathrm{L}^2(\mathbb{T}; w)$. ■

**Theorem 9.** *(Hunt, Muckenhoupt, Wheeden [HMW73])*
*Let $w\colon \mathbb{R} \to [0,\infty]$ be a 1-periodic weight and $1 < p < \infty$. If $S_N$ denote Fourier series partial sum operators, then*

$$\sup_{N\in\mathbb{N}} \|S_N\|_{\mathrm{L}^p(\mathbb{T};w)\to\mathrm{L}^p(\mathbb{T};w)} < \infty \qquad \Longleftrightarrow \qquad w \in A_p(\mathbb{T}).$$

**Remark 10.** *Part (c) of theorem 7 can be generalized to $1 < p < \infty$ without any changes in the proof, as noted in [N08].*

*(c') Exponentials form a Schauder basis for $\mathrm{L}^p(\mathbb{T};w)$ if and only if $w \in A_p(\mathbb{T})$.*

*However, the generalization of part (b) is false! Exponentials do not form an unconditional basis for $\mathrm{L}^p(\mathbb{T})$, $p \neq 2$, i.e. the basis is conditional even in the case $w \equiv 1$. This was first observed by Paley and Zygmund in [PZ32].*

*Notice that those observations for $p \neq 2$ are not relevant to theorem 4 because lemma 6 does not provide an isomorphism if $p \neq 2$.*

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Now we are ready to give a simple proof of Babenko's result. We can prove even more.

**Corollary 11.** *(taken from Nielsen [N08] but actually older, see below)*
*Let $P$ be a real polynomial of degree $d \geq 1$ such that $P(-\frac{1}{2}) = P(\frac{1}{2}) \neq 0$. For any $0 < \nu < \frac{1}{d}$ the system $\{e^{2\pi ikt} : k \in \mathbb{Z}\}$ is a Schauder basis for $\mathrm{L}^2([-\frac{1}{2},\frac{1}{2}];|P|^\nu)$.*

*Proof of corollary 11.* To verify conditions of theorem 7 we need the following simple result about polynomials.

**Lemma 12.** *(Ricci and Stein [RS87])*
*For a real polynomial $P$ of degree $d \geq 1$, $0 < \nu < \frac{1}{d}$, and any bounded interval $I \subseteq \mathbb{R}$ we have*

$$\Big(\fint_I |P|^{-\nu}\Big)^{-\frac{1}{\nu}} \sim_{d,\nu} \Big(\fint_I |P|^{\nu}\Big)^{\frac{1}{\nu}} \sim_{d,\nu} \fint_I |P|.$$

*Proof of lemma 12.* By basic inequalities for power means we already have

$$\Big(\fint_I |P|^{-\nu}\Big)^{-\frac{1}{\nu}} \leq \Big(\fint_I |P|^{\nu}\Big)^{\frac{1}{\nu}} \leq \fint_I |P|,$$

so it remains to show

$$\fint_I |P|^{-\nu} \lesssim_{d,\nu} \Big(\fint_I |P|\Big)^{-\nu}.$$

Also observe that the class of degree $d$ polynomials is invariant under translations and dilations so it is enough to prove the statement particularly for the interval $I = [-1,1]$:

$$\int_{-1}^{1} |P(t)|^{-\nu} dt \lesssim_{d,\nu} \Big(\int_{-1}^{1} |P(t)|\, dt\Big)^{-\nu}.$$

Suppose that $P$ splits as $P(z) = \prod_{j=1}^{k}(z-\beta_j)\prod_{j=k+1}^{d}(z-\gamma_j)$, where $|\beta_j| < 2$ and $|\gamma_j| \geq 2$. Let $t_0 \in [-1,1]$ be such that $|P(t_0)| \geq \frac{1}{2}\int_{-1}^{1}|P(t)|\,dt$. For $t \in [-1,1]$ we get from $|t_0-\beta_j| < 3$ and $|\frac{t_0-\gamma_j}{t-\gamma_j}| < 3$

$$|P(t)|^{-\nu} = |P(t_0)|^{-\nu}\Big|\frac{P(t_0)}{P(t)}\Big|^{\nu} \leq 3^{2d\nu}|P(t_0)|^{-\nu}\prod_{j=1}^{k}|t-\beta_j|^{-\nu},$$

and the estimate follows (by $k$-fold Hölder's inequality) from $\int_{-1}^{1}\prod_{j=1}^{k}|t-\beta_j|^{-\nu}dt \lesssim \frac{1}{1-d\nu} < \infty$. This completes the proof of lemma 12. ■

*Proof of corollary 11 continued.* The first "inequality" in the lemma can be rewritten as

$$\Big(\fint_I |P|^{\nu}\Big)\Big(\fint_I |P|^{-\nu}\Big) \sim_{d,\nu} 1.$$

To see $|P|^{\nu} \in A_2(\mathbb{T})$ and be able to apply theorem 7 part (c), we still have to show

$$\Big(\fint_I |P|^{\nu}\Big)\Big(\fint_I |P|^{-\nu}\Big) \lesssim_{P,\nu} 1$$

for intervals $I \subseteq \mathbb{T}$ of the form $I = [-\frac{1}{2}, a) \cup [b, \frac{1}{2})$, where $-\frac{1}{2} < a < b < \frac{1}{2}$.

Let $\delta > 0$ be such that $|P(t)|^{\nu} < 2|P(\frac{1}{2})|^{\nu}$ and $|P(t)|^{-\nu} < 2|P(\frac{1}{2})|^{-\nu}$ for all $t \in [-\frac{1}{2}, -\frac{1}{2}+\delta) \cup (\frac{1}{2}-\delta, \frac{1}{2})$.

If $a < -\frac{1}{2}+\delta$ and $b > \frac{1}{2}-\delta$, then from the above we get

$$\Big(\fint_I |P|^{\nu}\Big)\Big(\fint_I |P|^{-\nu}\Big) \leq 2|P(\tfrac{1}{2})|^{\nu}2|P(\tfrac{1}{2})|^{-\nu} = 4.$$

Otherwise we have $|I| \geq \delta$ and we can estimate

$$\Big(\fint_I |P|^{\nu}\Big)\Big(\fint_I |P|^{-\nu}\Big) \leq \delta^{-2}\Big(\int_0^1 |P|^{\nu}\Big)\Big(\int_0^1 |P|^{-\nu}\Big) \lesssim_{P,\delta,\nu} 1.$$

This completes the proof. ■

**Corollary 13.** *(Babenko [B48]) For any* $0 < \mu < \frac{1}{2}$ *the system* $\{e^{2\pi ikt} : k \in \mathbb{Z}\}$ *is a conditional Schauder basis for* $\mathrm{L}^2([-\frac{1}{2},\frac{1}{2}]; |t|^{2\mu})$, *i.e. the system* $\{|t|^{\mu}e^{2\pi ikt} : k \in \mathbb{Z}\}$ *is a conditional Schauder basis for* $\mathrm{L}^2([-\frac{1}{2},\frac{1}{2}])$.

To obtain Babenko's result, we simply take $P(t) = t$ in corollary 11. To see that the system is not a Riesz basis, note that $P$ has a zero in $[-\frac{1}{2},\frac{1}{2}]$, so $|P|^{\mu}$ cannot be bounded from below by a positive number, and we use theorem 7 part (b). ■

# 4 Orthonormal wavelets in $\mathrm{L}^2(\mathbb{R})$

The main novelty of Strömberg's construction [S82] was achievement of exponentially decaying orthonormal wavelet bases for $\mathrm{L}^2(\mathbb{R})$ with arbitrarily high degree of regularity (i.e. smoothness). These were the so called *spline wavelets*, for their construction see [M80]. Lemarié and Meyer [M85], [LM86] constructed orthonormal wavelet bases for $\mathrm{L}^2(\mathbb{R})$ consisting of Schwartz functions. One of the main cornerstones of the field is the paper by Daubechies [D88], who constructed compactly supported orthonormal wavelet bases for $\mathrm{L}^2(\mathbb{R})$ with arbitrarily high regularity.

**Theorem 14.** *(Daubechies [D88])*
*For every $r \in \mathbb{N}$ there exists $\psi \in \mathrm{C}^r_{\mathrm{c}}(\mathbb{R})$ such that $\{\mathrm{D}_j \mathrm{T}_k \psi : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathrm{L}^2(\mathbb{R})$.*

Let us remark that there are no compactly supported $C^\infty$ orthonormal wavelet bases for $\mathrm{L}^2(\mathbb{R})$. More generally, there are no $C^\infty$ orthonormal wavelets with exponential decay and bounded derivatives of all orders, see [HW96] and [DH98].

For our applications in later sections, either spline wavelets (with enough regularity), or Lemarié-Meyer wavelets will be more than enough, and they are much easier to construct than compactly supported ones. We are not going to present any constructions here because it could be a topic for a separate presentation. Very nice exposition can be found in [M90] or [HW96].

It is convenient to index elements of the wavelet system by dyadic intervals $\mathcal{D}$. Therefore for any dyadic interval $I = [2^{-j}k, 2^{-j}(k+1))$, $j, k \in \mathbb{Z}$, we denote

$$\psi_I := \mathrm{D}_j \mathrm{T}_k \psi = 2^{\frac{j}{2}} \psi(2^j \cdot - k).$$

A pleasant feature is that if $\psi$ is compactly supported, then each $\psi_I$ is supported in a fixed multiple $cI$ of the corresponding interval $I$. This follows from $\mathrm{supp}(\psi(2^j \cdot - k)) = 2^{-j}(\mathrm{supp}(\psi) + k)$. However, we often do not need compact support, and even fast decay is enough. In the latter case we say that $\psi_I$ is *adapted* to $I$.

# 5 Basic facts about $\mathrm{H}^1(\mathbb{R})$

There are several equivalent definitions of the real Hardy space $\mathrm{H}^1(\mathbb{R})$. We present them here without showing their equivalence.

Stein and Weiss originally defined the *real Hardy space* $\mathrm{H}^1(\mathbb{R})$ to be the space

$$\mathrm{H}^1(\mathbb{R}) := \{f \in \mathrm{L}^1(\mathbb{R}) \,:\, \mathrm{H}f \in \mathrm{L}^1(\mathbb{R})\}, \qquad \|f\|_{\mathrm{H}^1(\mathbb{R})} := \|f\|_{\mathrm{L}^1(\mathbb{R})} + \|\mathrm{H}f\|_{\mathrm{L}^1(\mathbb{R})},$$

where H denotes the Hilbert transform $(\mathrm{H}f)(x) := \mathrm{p.v.}\int_{\mathbb{R}} \frac{f(x-t)}{\pi t}\, dt$. (In higher dimensions Riesz transforms are used.)

In other words, real functions in $\mathrm{H}^1(\mathbb{R})$ are real parts of boundary values of functions in the *complex Hardy space* $\mathbf{H}^1(\mathbb{R})$, which is the space of all holomorphic functions $F$

on the upper half-plane satisfying

$$\|F\|_{\mathbf{H}^1(\mathbb{R})} := \sup_{y>0} \int_{\mathbb{R}} |F(x+iy)| dx < \infty.$$

Later, Coifman, Stein and Weiss preferred to use the *atomic definition*, which we now state. An *atom* is any measurable function $a\colon \mathbb{R} \to \mathbb{C}$ for which there exists a bounded interval $J$ such that

$$\operatorname{supp}(a) \subseteq J, \quad \|a\|_{\mathrm{L}^2(\mathbb{R})} \le |J|^{-\frac{1}{2}}, \quad \int_J a = 0.$$

A simple consequence is $\|a\|_{\mathrm{L}^1(\mathbb{R})} \le 1$. The *atomic Hardy space* is a Banach space

$$\mathrm{H}^1(\mathbb{R}) := \Big\{ \sum_{j=1}^{\infty} \beta_j a_j \ :\ a_j \text{ are atoms},\ \beta_j \in \mathbb{C},\ \sum_{j=1}^{\infty} |\beta_j| < \infty \Big\}$$

with the norm

$$\|f\|_{\mathrm{H}^1} := \inf \Big\{ \sum_{j=1}^{\infty} |\beta_j| \ :\ a_j \text{ are atoms},\ \beta_j \in \mathbb{C},\ \sum_{j=1}^{\infty} |\beta_j| < \infty,\ \sum_{j=1}^{\infty} \beta_j a_j = f \Big\}.$$

Notice that $\mathrm{H}^1(\mathbb{R}) \subseteq \{f \in \mathrm{L}^1(\mathbb{R}) : \int_{\mathbb{R}} f = 0\}$, but the converse inclusion is false.

If in the definition of atoms we consider only dyadic intervals $J$, we obtain a “smaller” space, called the *dyadic real Hardy space* $\mathrm{H}^1_{\text{dyadic}}(\mathbb{R})$. Surprisingly Maurey [M80] showed that $\mathrm{H}^1(\mathbb{R})$ and $\mathrm{H}^1_{\text{dyadic}}(\mathbb{R})$ are (topologically) isomorphic Banach spaces. It is through this isomorphism that Maurey gave a non-constructive proof of the existence of an unconditional basis for $\mathrm{H}^1(\mathbb{R})$. Namely, the Haar system provides a “perfect” unconditional basis for $\mathrm{H}^1_{\text{dyadic}}(\mathbb{R})$, because no regularity is needed in the dyadic setting. Our approach (i.e. the approach in [S82]) is exactly the opposite: smoother wavelets will directly provide an unconditional basis for $\mathrm{H}^1(\mathbb{R})$, we will characterize the coefficient space, and then the isomorphism will be immediate, taking the wavelet basis in $\mathrm{H}^1(\mathbb{R})$ to the Haar system in $\mathrm{H}^1_{\text{dyadic}}(\mathbb{R})$.

A bit surprising result (due to Fefferman [F71]) is the duality between $\mathrm{H}^1(\mathbb{R})$ and the space $\mathrm{BMO}(\mathbb{R})$ of functions with bounded mean oscillation, i.e. $\mathrm{H}^1(\mathbb{R})$ is the pre-dual of $\mathrm{BMO}(\mathbb{R})$. More precisely, every $b \in \mathrm{BMO}(\mathbb{R})$ determines a bounded linear functional $F_b$ on $\mathrm{H}^1(\mathbb{R})$ given by the formula

$$F_b(f) = F_b\Big(\sum_{j=1}^{\infty} \beta_j a_j\Big) := \sum_{j=1}^{\infty} \beta_j \int_{\mathbb{R}} a_j b,$$

where it is important to notice that the definition of $F_b(f)$ does not depend on the representation $f = \sum_{j=1}^{\infty} \beta_j a_j$. Conversely, every continuous linear functional on $\mathrm{H}^1(\mathbb{R})$ is of that form, for instance see [M90].

* * *

A *Calderón-Zygmund operator* is a bounded linear operator $T\colon \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ with $\|T\|_{\mathrm{L}^2\to\mathrm{L}^2} \lesssim 1$ and such that there exists a $C^1$ function $K\colon \{(x,y)\in\mathbb{R}^2 : x\neq y\}\to\mathbb{C}$ satisfying:

$$|K(x,y)| \lesssim |x-y|^{-1}, \quad |\partial_1 K(x,y)| + |\partial_2 K(x,y)| \lesssim |x-y|^{-2},$$

$$(Tf)(x) = \int_{\mathbb{R}} K(x,y) f(y) dy, \quad \text{for } f\in \mathrm{L}^2_{\mathrm{c}}(\mathbb{R}) \text{ and } x\notin \mathrm{supp}(f).$$

A function $K$ with these properties is called a *Calderón-Zygmund kernel* or a *standard singular kernel.* The actual class of C-Z kernels is larger, but the above definition suffices for our purposes.

One of the basic results is the following classical theorem, for the proof see for instance the second book of [M90].

**Theorem 15.** *$T$ extends to a bounded operator on $\mathrm{L}^p(\mathbb{R})$ for every $1<p<\infty$, and more precisely $\|T\|_{\mathrm{L}^p\to\mathrm{L}^p} \lesssim_p 1$.*

*If $\int_{\mathbb{R}} f = 0 \Rightarrow \int_{\mathbb{R}} Tf = 0$, then $T$ extends to a bounded operator on $\mathrm{H}^1(\mathbb{R})$, and more precisely $\|T\|_{\mathrm{H}^1\to\mathrm{H}^1} \lesssim 1$.*

**Remark 16.** *In the statement of the theorem we wanted to emphasize that operator norms of $T$ on $\mathrm{L}^p(\mathbb{R})$ and $\mathrm{H}^1(\mathbb{R})$ depend only on implicit constants from the definition (including the operator norm of $T$ on $\mathrm{L}^2(\mathbb{R})$).*

*Also observe that the condition $\int_{\mathbb{R}} f = 0 \Rightarrow \int_{\mathbb{R}} Tf = 0$ can be replaced by a weaker one: $\int_{\mathbb{R}} Ta = 0$ for each atom $a$. Actually $T$ is bounded on $\mathrm{H}^1(\mathbb{R})$ if and only if the latter condition is satisfied. This condition is also sometimes concisely written as $T^\tau(\mathbf{1}) = \mathbf{0}$.*

# 6 Wavelets as unconditional bases for $\mathrm{L}^p(\mathbb{R})$ and $\mathrm{H}^1(\mathbb{R})$

Our goal is to prove that "reasonably nice" wavelets form unconditional bases in function spaces other than $\mathrm{L}^2(\mathbb{R})$. For the purpose of this section it will be enough to assume that $(\psi_I)_{I\in\mathcal{D}}$ is an orthonormal wavelet basis such that $\psi\in \mathrm{C}^1(\mathbb{R})$ and

$$|\psi(x)| + |\psi'(x)| \lesssim (1+|x|)^{-3}.$$

This is certainly not the weakest possible assumption on $\psi$. For somewhat less restrictive ones see [HW96] (where the class $\mathcal{R}^0$ is introduced) and [T06].

An easy consequence of the above conditions on $\psi$ is $\int_{\mathbb{R}} \psi(x)dx = 0$. To see this, take $a = 2^{-j_0}k_0$, $j_0, k_0 \in \mathbb{Z}$, such that $\psi(a)\neq 0$. By the orthogonality we have for $j > \max\{j_0, 0\}$:

$$\int_{\mathbb{R}} \psi(x)\overline{\psi(2^{-j}x + a)}\, dx = \int_{\mathbb{R}} \psi(x - 2^{j-j_0}k_0)\overline{\psi(2^{-j}x)}\, dx = 0$$

and by taking $j\to\infty$ we get what we need. One can prove that even more moments of $\psi$ have to be 0, see [HW96].

In [T06] the condition $\int_{\mathbb{R}} \psi(x)dx = 0$ is incorporated into definition of wavelets, because of lack of orthogonality, and actually any rigid structure at all. Inequalities from part (c) of theorem 17 below are still valid in that more general setting.

The main result of the whole presentation is the following theorem.

**Theorem 17.** *Let $1 < p < \infty$. The system $(\psi_I)_{I\in\mathcal{D}}$ is an unconditional basis for $\mathrm{L}^p(\mathbb{R})$ and $\mathrm{H}^1(\mathbb{R})$. In both cases the coefficient functionals are $f \mapsto \langle f, \psi_I\rangle = \int_{\mathbb{R}} f(x)\overline{\psi_I(x)}dx$. Moreover, for $f \in \mathrm{L}^p(\mathbb{R})$ we have*

$$\|f\|_{\mathrm{L}^p(\mathbb{R})} \sim_p \Big\|\Big(\sum_{I\in\mathcal{D}} |\langle f,\psi_I\rangle|^2 |I|^{-1}\mathbf{1}_I\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^p(\mathbb{R})},$$

*and for $f \in \mathrm{H}^1(\mathbb{R})$ we have*

$$\|f\|_{\mathrm{H}^1(\mathbb{R})} \sim \Big\|\Big(\sum_{I\in\mathcal{D}} |\langle f,\psi_I\rangle|^2 |I|^{-1}\mathbf{1}_I\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^1(\mathbb{R})}.$$

*Finally, for every family of scalars $(c_I)_{I\in\mathcal{D}}$ we have*

$$\sum_{I\in\mathcal{D}} c_I\psi_I \text{ converges unconditionally in } \mathrm{L}^p(\mathbb{R}) \iff \Big\|\Big(\sum_{I\in\mathcal{D}} |c_I|^2 |I|^{-1}\mathbf{1}_I\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^p(\mathbb{R})} < \infty,$$

$$\sum_{I\in\mathcal{D}} c_I\psi_I \text{ converges unconditionally in } \mathrm{H}^1(\mathbb{R}) \iff \Big\|\Big(\sum_{I\in\mathcal{D}} |c_I|^2 |I|^{-1}\mathbf{1}_I\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^1(\mathbb{R})} < \infty.$$

**Remark 18.** *One can easily recognize the expression*

$$\mathrm{S}f := \Big(\sum_{I\in\mathcal{D}} |\langle f,\psi_I\rangle|^2 |I|^{-1}\mathbf{1}_I\Big)^{\frac{1}{2}}$$

*as the well-known Littlewood-Paley square function, and we are reproving the famous Littlewood-Paley inequality. Therefore, some aspects of the theory of wavelets are refinements of Littlewood-Paley methods.*

*Proof of theorem 17.*

For arbitrary choice of signs $\omega = (\omega_I)_{I\in\mathcal{D}} \in \{-1,1\}^{\mathcal{D}} = \Omega$ we define a unitary operator $T_\omega$ on the orthonormal basis $(\psi_I)_{I\in\mathcal{D}}$ by

$$T_\omega\colon \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}), \quad T_\omega\colon \psi_I \mapsto \omega_I\psi_I.$$

Even more generally, for arbitrary family of scalars $\lambda = (\lambda_I)_{I\in\mathcal{D}}$ such that $|\lambda_I| \le 1$, we define $T_\lambda$ on the orthonormal basis by

$$T_\lambda\colon \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}), \quad T_\lambda\colon \psi_I \mapsto \lambda_I\psi_I.$$

In the latter case $T_\lambda$ is not necessarily unitary, but it is bounded and $\|T_\lambda\|_{\mathrm{L}^2\to\mathrm{L}^2} \le 1$.

At least formally we can write

$$(T_\lambda f)(x) = \sum_{I\in\mathcal{D}} \lambda_I \langle f, \psi_I\rangle \psi_I(x) = \int_{\mathbb{R}} K_\lambda(x,y) f(y) dy,$$

where

$$K_\lambda(x,y) = \sum_{I\in\mathcal{D}} \lambda_I \, \psi_I(x)\, \overline{\psi_I(y)} = \sum_{j,k\in\mathbb{Z}} \lambda_{j,k}\, 2^j\, \psi(2^j x - k)\, \overline{\psi(2^j y - k)}.$$

We claim that $T_\lambda$ are Calderón-Zygmund operators with constants uniform in $\lambda$, and for that we have to show that $K_\lambda$ satisfy standard estimates uniformly in $\lambda$.

$$\begin{aligned} |K_\lambda(x,y)| &\le \sum_{j,k\in\mathbb{Z}} 2^j (1+|2^j x - k|)^{-3} (1+|2^j y - k|)^{-3} \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^j (1+2^j|x-y|)^{-3} \lesssim |x-y|^{-1} \end{aligned}$$

Also for derivatives:

$$\frac{\partial}{\partial x} K_\lambda(x,y) = \sum_{j,k\in\mathbb{Z}} \lambda_{j,k}\, 2^{2j}\, \psi'(2^j x - k)\, \overline{\psi(2^j y - k)}.$$

$$\begin{aligned} \Big|\frac{\partial}{\partial x} K_\lambda(x,y)\Big| &\le \sum_{j,k\in\mathbb{Z}} 2^{2j} (1+|2^j x - k|)^{-3} (1+|2^j y - k|)^{-3} \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{2j} (1+2^j|x-y|)^{-3} \lesssim |x-y|^{-2} \end{aligned}$$

The condition $\int_{\mathbb{R}} f = 0 \Rightarrow \int_{\mathbb{R}} T_\lambda f = 0$ follows from $\int_{\mathbb{R}} K_\lambda(x,y)dx = 0$, which in turn follows from $\int_{\mathbb{R}} \psi(x)dx = 0$. Therefore from theorem 15 we know that operators $T_\lambda$ extend continuously to $\mathrm{L}^p(\mathbb{R})$ and $\mathrm{H}^1(\mathbb{R})$, and we get inequalities

$$\|T_\lambda f\|_{\mathrm{L}^p} \lesssim_p \|f\|_{\mathrm{L}^p}, \quad \|T_\lambda f\|_{\mathrm{H}^1} \lesssim \|f\|_{\mathrm{H}^1}, \tag{1}$$

which particularly for $f = \sum_{I\in F} c_I\psi_I$, $F$ is a finite subset of $\mathcal{D}$, $c_I$ are scalars, can be written as

$$\Big\|\sum_{I\in F} \lambda_I c_I \psi_I\Big\|_{\mathrm{L}^p} \lesssim_p \Big\|\sum_{I\in F} c_I \psi_I\Big\|_{\mathrm{L}^p}, \quad \Big\|\sum_{I\in F} \lambda_I c_I \psi_I\Big\|_{\mathrm{H}^1} \lesssim \Big\|\sum_{I\in F} c_I \psi_I\Big\|_{\mathrm{H}^1}. \tag{2}$$

These inequalities imply that $(\psi_I)_{I\in\mathcal{D}}$ is an unconditional basis for

$$\overline{\mathrm{span}}_{\mathrm{L}^p(\mathbb{R})}(\{\psi_I : I\in\mathcal{D}\}) \qquad \text{and} \qquad \overline{\mathrm{span}}_{\mathrm{H}^1(\mathbb{R})}(\{\psi_I : I\in\mathcal{D}\}).$$

However, it is still not clear that $(\psi_I)_{I\in\mathcal{D}}$ is complete in $\mathrm{L}^p(\mathbb{R})$ and $\mathrm{H}^1(\mathbb{R})$.

* * *

If we in particular take $\omega = (\omega_I)_{I\in\mathcal{D}} \in \{-1,1\}^{\mathcal{D}}$, then (1) and $T_\omega^2 = I$ imply

$$\|T_\omega f\|_{\mathrm{L}^p} \sim_p \|f\|_{\mathrm{L}^p}, \quad \|T_\omega f\|_{\mathrm{H}^1} \sim \|f\|_{\mathrm{H}^1}. \tag{3}$$

Let us endow $\Omega = \{-1,1\}^{\mathcal{D}}$ with the product probability measure. At first we consider only functions of the form $f = \sum_{I\in F} c_I \psi_I$, for some finite $F \subset \mathcal{D}$ and some scalars $c_I$. Integrating (3) over $\Omega$, interchanging the order of integration and using Khintchine's inequality we get:

$$\|f\|_{\mathrm{L}^p(\mathbb{R})} \sim_p \Big( \int_\Omega \|T_\omega f\|^p_{\mathrm{L}^p(\mathbb{R})} d\omega \Big)^{\frac{1}{p}} = \Big\| \Big( \int_\Omega \Big| \sum_{I\in F} \omega_I c_I \psi_I \Big|^p d\omega \Big)^{\frac{1}{p}} \Big\|_{\mathrm{L}^p(\mathbb{R})} \sim_p$$

$$\sim_p \Big\| \Big( \sum_{I\in F} |c_I|^2 |\psi_I|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(\mathbb{R})}, \tag{4}$$

and similarly

$$\begin{aligned}
\|f\|_{\mathrm{H}^1(\mathbb{R})} &\sim \int_\Omega \|T_\omega f\|_{\mathrm{H}^1(\mathbb{R})} d\omega = \int_\Omega \|T_\omega f\|_{\mathrm{L}^1(\mathbb{R})} d\omega + \int_\Omega \|\mathrm{H} T_\omega f\|_{\mathrm{L}^1(\mathbb{R})} d\omega \\
&= \Big\| \int_\Omega \Big| \sum_{I\in F} \omega_I c_I \psi_I \Big| d\omega \Big\|_{\mathrm{L}^1(\mathbb{R})} + \Big\| \int_\Omega \Big| \sum_{I\in F} \omega_I c_I \mathrm{H}\psi_I \Big| d\omega \Big\|_{\mathrm{L}^1(\mathbb{R})} \\
&\sim \Big\| \Big( \sum_{I\in F} |c_I|^2 |\psi_I|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^1(\mathbb{R})} + \Big\| \Big( \sum_{I\in F} |c_I|^2 |\mathrm{H}\psi_I|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^1(\mathbb{R})}.
\end{aligned} \tag{5}$$

* * *

In the next step take $f \in \mathrm{L}^p(\mathbb{R})$. For some $F$ finite subset of $\mathcal{D}$ define $\lambda = (\lambda_I)_{I\in\mathcal{D}}$ by $\lambda_I = 1$ if $I \in F$, and $\lambda_I = 0$ if $I \notin F$, so that $T_\lambda f = \sum_{I\in F} \langle f, \psi_I \rangle \psi_I$. Applying estimate (4) to $T_\lambda f$ and using (1) we get

$$\Big\| \Big( \sum_{I\in F} |\langle f, \psi_I \rangle|^2 |\psi_I|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(\mathbb{R})} \sim_p \|T_\lambda f\|_{\mathrm{L}^p(\mathbb{R})} \lesssim_p \|f\|_{\mathrm{L}^p(\mathbb{R})}.$$

Finally, using the monotone convergence theorem as finite sets $F$ exhaust $\mathcal{D}$ we obtain

$$\Big\| \Big( \sum_{I\in\mathcal{D}} |\langle f, \psi_I \rangle|^2 |\psi_I|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(\mathbb{R})} \lesssim_p \|f\|_{\mathrm{L}^p(\mathbb{R})}. \tag{6}$$

Now we want to prove the opposite inequality, but before that we return to the proof of completeness. Order the set of dyadic intervals arbitrarily as $\mathcal{D} = \{I_1, I_2, I_3, \ldots\}$. For any $f \in \mathrm{L}^p(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$ we have $f = \sum_{l=1}^\infty \langle f, \psi_{I_l} \rangle \psi_{I_l}$, with convergence in $\mathrm{L}^2(\mathbb{R})$, and there exists a subsequence of partial sums $(\sum_{l=1}^{N_m} \langle f, \psi_{I_l} \rangle \psi_{I_l})_{m\in\mathbb{N}}$ that converges a.e. on $\mathbb{R}$. Using Fatou's lemma and (4) we get

$$\Big\| f - \sum_{l=1}^N \langle f, \psi_{I_l} \rangle \psi_{I_l} \Big\|_{\mathrm{L}^p(\mathbb{R})} \le \liminf_{m\to\infty} \Big\| \sum_{l=N+1}^{N_m} \langle f, \psi_{I_l} \rangle \psi_{I_l} \Big\|_{\mathrm{L}^p(\mathbb{R})} \lesssim_p \Big\| \Big( \sum_{l=N+1}^{\infty} |\langle f, \psi_{I_l} \rangle|^2 |\psi_{I_l}|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(\mathbb{R})}.$$

Since (6) gives $\|(\sum_{I\in\mathcal{D}}|\langle f,\psi_I\rangle|^2|\psi_I|^2)^{\frac{1}{2}}\|_{\mathrm{L}^p(\mathbb{R})}<\infty$, we can use the dominated convergence theorem to obtain $f=\sum_{l=1}^{\infty}\langle f,\psi_{I_l}\rangle\psi_{I_l}$, with convergence in $\mathrm{L}^p(\mathbb{R})$. Therefore $\overline{\mathrm{span}}_{\mathrm{L}^p(\mathbb{R})}(\{\psi_I : I\in\mathcal{D}\})\supseteq\mathrm{L}^p(\mathbb{R})\cap\mathrm{L}^2(\mathbb{R})$, and since $\mathrm{L}^p(\mathbb{R})\cap\mathrm{L}^2(\mathbb{R})$ is dense in $\mathrm{L}^p(\mathbb{R})$, we conclude that $(\psi_I)_{I\in\mathcal{D}}$ is complete in $\mathrm{L}^p(\mathbb{R})$. Together with (2) this finally proves that $(\psi_I)_{I\in\mathcal{D}}$ is an unconditional basis for $\mathrm{L}^p(\mathbb{R})$.

Similarly we handle $\mathrm{H}^1(\mathbb{R})$ using (5).

* * *

From inequalities

$$|\langle f,\psi_I\rangle|\le\|f\|_{\mathrm{L}^p}\|\psi_I\|_{\mathrm{L}^{p'}},\quad |\langle f,\psi_I\rangle|\le\|f\|_{\mathrm{H}^1}\|\psi_I\|_{\mathrm{BMO}}$$

we see that $\{\langle\cdot,\psi_I\rangle : I\in\mathcal{D}\}$ define continuous linear functionals on $\mathrm{L}^p(\mathbb{R})$, $1<p<\infty$, and $\mathrm{H}^1(\mathbb{R})$. If $f=\sum_{I\in\mathcal{D}}c_I\psi_I$ unconditionally in $\mathrm{L}^p(\mathbb{R})$ or $\mathrm{H}^1(\mathbb{R})$, then for every $I'\in\mathcal{D}$ by continuity of $\langle\cdot,\psi_{I'}\rangle$:

$$\langle f,\psi_{I'}\rangle=\sum_{I\in\mathcal{D}}c_I\langle\psi_I,\psi_{I'}\rangle=\sum_{I\in\mathcal{D}}c_I\delta_{I,I'}=c_{I'},$$

which proves the claim about coefficient functionals.

For every $f\in\mathrm{L}^p(\mathbb{R})$ we now have $f=\sum_{l=1}^{\infty}\langle f,\psi_{I_l}\rangle\psi_{I_l}$, with convergence in $\mathrm{L}^p(\mathbb{R})$. Using (4) we get

$$\|f\|_{\mathrm{L}^p(\mathbb{R})}=\lim_{N\to\infty}\Big\|\sum_{l=1}^{N}\langle f,\psi_{I_l}\rangle\psi_{I_l}\Big\|_{\mathrm{L}^p(\mathbb{R})}\lesssim_p\Big\|\Big(\sum_{I\in\mathcal{D}}|\langle f,\psi_I\rangle|^2|\psi_I|^2\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^p(\mathbb{R})},$$

which together with (6) finally proves

$$\|f\|_{\mathrm{L}^p(\mathbb{R})}\sim_p\Big\|\Big(\sum_{I\in\mathcal{D}}|\langle f,\psi_I\rangle|^2|\psi_I|^2\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^p(\mathbb{R})}\tag{7}$$

for **every** $f\in\mathrm{L}^p(\mathbb{R})$.

For the purpose of the last theorem statement we make the following observation. Suppose that $(c_I)_{I\in\mathcal{D}}$ is a family of scalars such that $\|(\sum_{I\in\mathcal{D}}|c_I|^2|\psi_I|^2)^{1/2}\|_{\mathrm{L}^p(\mathbb{R})}<\infty$. For every $\varepsilon>0$ we can find $F_0$ finite subset of $\mathcal{D}$ such that $\|(\sum_{I\in\mathcal{D}\setminus F_0}|c_I|^2|\psi_I|^2)^{1/2}\|_{\mathrm{L}^p(\mathbb{R})}\le\varepsilon$. Then for every $F$ finite subset of $\mathcal{D}\setminus F_0$ by (4) we have $\|\sum_{I\in F}c_I\psi_I\|_{\mathrm{L}^p(\mathbb{R})}\lesssim_p\varepsilon$. This shows that the series $\sum_{I\in\mathcal{D}}c_I\psi_I$ converges unconditionally in $\mathrm{L}^p(\mathbb{R})$ to some $\mathrm{L}^p$-function. In short, we have proved:

$$\sum_{I\in\mathcal{D}}c_I\psi_I \text{ converges unconditionally in } \mathrm{L}^p(\mathbb{R}) \iff \Big\|\Big(\sum_{I\in\mathcal{D}}|c_I|^2|\psi_I|^2\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^p(\mathbb{R})}<\infty.\tag{8}$$

This characterizes convergence of wavelet series in $\mathrm{L}^p(\mathbb{R})$. Similarly we would characterize convergence in $\mathrm{H}^1(\mathbb{R})$.

* * *

In the last stage of the proof we have to find a way to replace $|\psi_I|^2$ by $|I|^{-1}\mathbf{1}_I$ in (7) and (8). For the Haar system $\psi_I = |I|^{-\frac{1}{2}}(\mathbf{1}_{I_{\text{left half}}} - \mathbf{1}_{I_{\text{right half}}})$ we even have $|\psi_I|^2 = |I|^{-1}\mathbf{1}_I$, but in general those two functions are not even comparable. For $\mathrm{H}^1(\mathbb{R})$ there is one more complication that we also have to replace $|\mathrm{H}\psi_I|^2$ by $|I|^{-1}\mathbf{1}_I$. However, notice that the Hilbert transform H is a unitary operator that commutes with translations and dilations, so $(\mathrm{H}\psi_I)_{I\in\mathcal{D}}$ is again an orthonormal wavelet basis for $\mathrm{L}^2(\mathbb{R})$.

To end the proof we would have to show

$$\Big\|\Big(\sum_{I\in\mathcal{D}}|c_I|^2|\psi_I|^2\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^p(\mathbb{R})} \sim_{p,\psi} \Big\|\Big(\sum_{I\in\mathcal{D}}|c_I|^2|I|^{-1}\mathbf{1}_I\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^p(\mathbb{R})}$$

for $1 \le p < \infty$ and for every family of scalars $(c_I)_{I\in\mathcal{D}}$. This is somewhat easier when $\psi \in \mathrm{C_c}(\mathbb{R})$, as commented in [M80], or when $\hat{\psi}$ has a compact support, as assumed in [HW96], but is not obvious at all in the general case. Also, it would be better to obtain constants depending only on $p$ and decay/regularity of $\psi$, but on no other information about $\psi$. For that reason, a better approach is to prove square function estimates "from scratch", using Calderón-Zygmund decomposition and real interpolation, as is done in [T06]. ■

**Remark 19.** *The density argument (i.e. completeness of the system) in the proof might seem unnecessary complicated, but the matter is surprisingly subtle. For $p > 2$ it is possible to construct a wavelet system consisting of Schwartz functions and such that it forms a Riesz basis for* $\mathrm{L}^2(\mathbb{R})$ *but is not complete in* $\mathrm{L}^p(\mathbb{R})$.

# 7 A few words on wavelet characterizations of other function spaces

Whenever we have an unconditional/Schauder basis $(e_n)_{n\in\mathbb{N}}$ for a Banach space $X$ we can consider the *coefficient space*

$$\mathcal{C} := \Big\{(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \ : \ \sum_{n\in\mathbb{N}} c_n e_n \text{ converges unconditionally/conditionally in } X\Big\},$$

with the norm

$$\|(c_n)_{n\in\mathbb{N}}\|_{\mathcal{C}} := \Big\|\sum_{n\in\mathbb{N}} c_n e_n\Big\|_X.$$

It is a Banach space (isometrically isomorphic to $X$) and its unconditional/Schauder basis is the usual "canonical basis" in the space of sequences, $((\delta_{n,m})_{n\in\mathbb{N}} : m \in \mathbb{N})$.

When we are able to describe the coefficient space $\mathcal{C}$ explicitly and find a convenient explicit norm equivalent to $\|\cdot\|_{\mathcal{C}}$, then we say that we did a complete *characterization* of $X$ by $(e_n)_{n\in\mathbb{N}}$. Therefore the last statement of theorem 17 gives *wavelet characterization* of function spaces $\mathrm{L}^p(\mathbb{R})$, $1 < p < \infty$, and $\mathrm{H}^1(\mathbb{R})$. Indeed, it can be shown [Y80] that if two coefficient spaces are equal (as subsets of $\mathbb{C}^{\mathbb{N}}$), then their norms are equivalent and corresponding Banach spaces are (topologically) isomorphic through an isomorphism sending one basis to another.

Wavelet systems provide explicit unconditional bases for many other function spaces. We list some more characterizations in the table. Precise formulations and proofs can be found in [M90] or [HW96].

| **function space** | **convenient unconditional wavelet basis** | **coefficient space and equivalent norm $\|(c_I)_{I\in\mathcal{D}}\|$ on it** |
|---|---|---|
| $\mathrm{H}^1(\mathbb{R})$ | orthonormal $r$-regular MRA wavelets, $r \geq 1$ | $\|(\sum_I \|c_I\|^2 \|I\|^{-1}\mathbf{1}_I)^{1/2}\|_{\mathrm{L}^1}$ |
| $\mathrm{H}^1_{\text{dyadic}}(\mathbb{R})$ | Haar system | $\|(\sum_I \|c_I\|^2 \|I\|^{-1}\mathbf{1}_I)^{1/2}\|_{\mathrm{L}^1}$ |
| $\mathrm{L}^p(\mathbb{R})$, $1<p<\infty$ | orthonormal $r$-regular MRA wavelets, $r \geq 0$ | $\|(\sum_I \|c_I\|^2 \|I\|^{-1}\mathbf{1}_I)^{1/2}\|_{\mathrm{L}^p}$ |
| $\mathrm{W}^{p,s}(\mathbb{R})$, $1<p<\infty$, $s \geq 0$ | orthonormal $r$-regular MRA wavelets, $r > s$ | $\|(\sum_I \|c_I\|^2 (1+\|I\|^{-2s})\|I\|^{-1}\mathbf{1}_I)^{1/2}\|_{\mathrm{L}^p}$ |
| $\mathrm{W}^{p,s}(\mathbb{R})$, $1<p<\infty$, $s \leq 0$ | orthonormal $r$-regular MRA wavelets, $r > -s$ | $\|(\sum_I \|c_I\|^2 (1+\|I\|^{2s})^{-1}\|I\|^{-1}\mathbf{1}_I)^{1/2}\|_{\mathrm{L}^p}$ |
| $\dot{\mathrm{W}}^{p,s}(\mathbb{R})$, $1<p<\infty$ | orthonormal $r$-regular MRA wavelets, $r > \|s\|$ | $\|(\sum_I \|c_I\|^2 \|I\|^{-2s-1}\mathbf{1}_I)^{1/2}\|_{\mathrm{L}^p}$ |
| $\dot{\Lambda}^s(\mathbb{R})$, $s > 0$ | orthonormal $r$-regular MRA wavelets, $r > s$ | $\sup_I(\|c_I\|\,\|I\|^{-\frac{1}{2}-s})$ |

The conditions on the "convenient wavelet basis" are not always optimal — sometimes a larger class of wavelets might provide a basis, but the proof might require some changes or new ingredients. For instance, any orthonormal wavelet $\psi$ such that $\psi$ and $\psi'$ have a common radial decreasing $\mathrm{L}^1$-majorant $w$ with $\int_0^\infty t\,w(t)dt < \infty$ (as in [HW96]) also generates an unconditional basis for $\mathrm{L}^p(\mathbb{R})$, $1 < p < \infty$, and $\mathrm{H}^1(\mathbb{R})$, although it has less regularity. Gripenberg [G93] has shown that wavelets with enough decay but no regularity/smoothness at all still provide an unconditional basis for $\mathrm{L}^p(\mathbb{R})$, if we additionally assume that the wavelet comes from the multiresolution analysis construction. Such wavelets are called *MRA wavelets.*

Exact characterization of all wavelets that provide a sort of basis for a given function space seems to be a **very** hard problem, and is resolved to a large extent only in $\mathrm{L}^2(\mathbb{R})$, see [HW96]. In other function spaces all kinds of pathologies are possible, see [M90] for references.

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