| 1 | 2 | 3 | 4 | 5 | 6 | Σ |
|---|---|---|---|---|---|---|
| | | | | | | |

MATIČNI BROJ IME I PREZIME

# Prvi kolokvij, 5.12.2014.

## Teorijska pitanja (18 bodova)

**1.** *(2 boda)* Definirajte množenje matrica, iskažite i dokažite svojstvo distributivnosti množenja s obzirom na zbrajanje.

**2.** *(4 boda)* Definirajte pojmove linearne zavisnosti i nezavisnosti vektora. Dokažite da je skup vektora

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

linearno nezavisan. Kakav je skup ako im dodamo još jedan vektor iz $\mathbb{R}^3$. Argumentirajte!

**3.** *(3 boda)* Iskažite Kronecker–Capellijev teorem. Dokažite da ako rješenje sustava postoji da je rang proširene matrice jednak rangu matrice sustava.

**4.** *(3 boda)* Navedite sve postupke računanja determinante matrice reda 3 koje znate, te detaljno objasnite metodu koja koristi Gaussove eliminacije.

**5.** *(3 boda)* Zaokružite $T$ ako je tvrdnja točna, a $N$ ako je netočna.

| | | |
|---|---|---|
| Za kvadratne matrice vrijedi $(AB)^T = A^T B^T$. | T | N |
| Sustav 3 jednadžbe s 5 nepoznanica ne mora imati rješenja. | T | N |
| Vektori $[0\ 1\ 0]^T$, $[1\ 0\ 1]^T$ su linearno nezavisni. | T | N |
| Ako je matrica $A$ invertibilna onda je $\det A = 0$. | T | N |
| Cramerovo pravilo odnosi se samo na matrice dimenzije 3. | T | N |
| Za sve kvadratne matrice vrijedi $\det(AB) = \det B \det A$. | T | N |

**6.** *(3 boda)* Izvedite uvjet za okomitost pravaca preko koeficijenata smjerova.

**Napomena.** Ove papire predajte zajedno s papirima na kojima ste rješavali zadatke. Obavezno **odvojeno** rješavajte teorijska pitanja od zadataka.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Σ |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

MATIČNI BROJ

IME I PREZIME

## **Zadaci** (28 bodova)

**1.** *(4 boda)* Zadane su matrice $A = \begin{bmatrix} 2 & 3 & 2 \\ 2 & -1 & 2 \\ -1 & 12 & 3 \end{bmatrix}$ i $B = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. Riješite matričnu jednadžbu $AX + BA = X$.

**2.** *(4 boda)* Gaussovom metodom eliminacija riješite sustav i rješenje zapišite u vektorskom obliku.

$$\begin{array}{rcrcrcrcrcr} 4x_1 & - & 3x_2 & + & 5x_3 & - & 10x_4 & + & 11x_5 & = & -8 \\ 2x_1 & + & x_2 & + & 5x_3 & & & + & 3x_5 & = & 6 \\ & - & x_2 & - & x_3 & - & 2x_4 & + & x_5 & = & -4 \end{array}$$

**3.** *(3 boda)* Odredite rang matrice $\begin{bmatrix} 2 & 3 & -1 & 1 \\ 4 & 4 & 5 & 3 \\ -2 & 1 & -2 & 0 \\ 4 & 3 & 3 & 2 \end{bmatrix}$.

**4.** *(4 boda)* Riješite jednadžbu $\begin{vmatrix} x & 2 & 3 \\ x & 2 & 2x \\ -1 & 1 & x \end{vmatrix} = 3$.

**5.** *(3 boda)* Izračunajte $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{vmatrix}$.

**6.** *(4 boda)* Za koju vrijednost parametra $\lambda$ su vektori $\vec{a} = 3\vec{p} + \lambda\vec{q}$ i $\vec{b} = 5\vec{p} + 2\vec{q}$ međusobno okomiti ako je $|\vec{p}| = 2$, $|\vec{q}| = 3$ i $\sphericalangle(\vec{p}, \vec{q}) = \frac{2\pi}{3}$.

**7.** *(3 boda)* Izračunajte volumen tetraedra s vrhovima $A = (1, 1, 1)$, $B = (2, 1, 2)$, $C = (3, 2, 2)$, $D = (-1, 2, 3)$.

**8.** *(3 boda)* Pravac $p$ okomit na ravninu

$$\pi \ldots \quad x - 2y + z = 2$$

prolazi točkom $T = (2, -2, 2)$. Odredite jednadžbu pravca $p$ u kanonskom obliku, probodište pravca $p$ i ravnine $\pi$ te udaljenost točke $T$ od tog probodišta.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Σ |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

MATIČNI BROJ IME I PREZIME

# Zadaci (28 bodova)

**1.** *(4 boda)* Zadane su matrice $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -2 & 1 \\ 3 & -2 & 1 \end{bmatrix}$ i $B = \begin{bmatrix} 1 & -1 & -2 \\ 5 & -2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$. Riješite matričnu jednadžbu $XAB = X + A + 2I$.

**2.** *(4 boda)* Gaussovom metodom eliminacija riješite sustav i rješenje zapišite u vektorskom obliku.

$$\begin{array}{rcrcrcrcrcr} x_1 & & & + & 2x_3 & + & x_4 & + & 4x_5 & = & 1 \\ & & x_2 & + & x_3 & - & 3x_4 & & & = & -2 \\ 4x_1 & - & 3x_2 & + & 5x_3 & + & 13x_4 & + & 16x_5 & = & 10 \\ x_1 & + & 2x_2 & + & 4x_3 & - & 5x_4 & + & 4x_5 & = & -3 \end{array}$$

**3.** *(3 boda)* Odredite rang matrice $\begin{bmatrix} 3 & 2 & 1 & 3 \\ 3 & 3 & 5 & 5 \\ 0 & 1 & 4 & 2 \\ 3 & 1 & -3 & 1 \end{bmatrix}$.

**4.** *(4 boda)* Riješite jednadžbu $\begin{vmatrix} 1 & x & -1 \\ 2 & 3x & 3 \\ x & 0 & 1 \end{vmatrix} = 5$.

**5.** *(3 boda)* Izračunajte $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{vmatrix}$.

**6.** *(4 boda)* Za koju vrijednost parametra $\lambda$ su vektori $\vec{a} = -6\vec{p} + 2\vec{q}$ i $\vec{b} = 2\vec{p} + \lambda\vec{q}$ međusobno okomiti ako je $|\vec{p}| = 1$, $|\vec{q}| = 2$ i $\sphericalangle(\vec{p}, \vec{q}) = \frac{\pi}{3}$.

**7.** *(3 boda)* Izračunajte volumen paralelepipeda s bridovima $\overrightarrow{AB} = (4, 3, 2)$, $\overrightarrow{AC} = (-2, 1, 0)$, $\overrightarrow{AD} = (1, 3, 1)$.

**8.** *(3 boda)* Dane su točke $A = (0, -1, 1)$, $B = (1, -2, 2)$, $C = (-2, -1, 0)$. Odredite jednadžbu ravnine koja prolazi točkama $A$, $B$ i $C$ te jednadžbu pravca $AB$ u kanonskom obliku.

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**(1)**

$$A=\begin{bmatrix}2&3&2\\2&-1&2\\-1&12&3\end{bmatrix},\quad B=\begin{bmatrix}-3&1&1\\0&-1&0\\-2&0&1\end{bmatrix}$$

$AX+BA=X \iff (A-I)X=-BA$

$\iff X=-(A-I)^{-1}BA$

$$\left[\begin{array}{ccc|ccc}1&3&2&1&0&0\\2&-2&2&0&1&0\\-1&12&2&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&3&2&1&0&0\\0&-8&-2&-2&1&0\\0&15&4&1&0&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&-5&0&-1&1&0\\0&-8&-2&-2&1&0\\0&-1&0&-3&2&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&14&-9&-5\\0&0&-2&22&-15&-8\\0&-1&0&-3&2&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&0&0&14&-9&-5\\0&1&0&3&-2&-1\\0&0&1&-11&\frac{15}{2}&4\end{array}\right]\qquad (A-I)^{-1}=\begin{bmatrix}14&-9&-5\\3&-2&-1\\-11&\frac{15}{2}&4\end{bmatrix}$$

$$BA=\begin{bmatrix}-5&2&-1\\-2&1&-2\\-5&6&-1\end{bmatrix}\qquad -(A-I)^{-1}=\begin{bmatrix}-14&9&5\\-3&2&1\\11&-\frac{15}{2}&-4\end{bmatrix}$$

$$X=-(A-I)^{-1}BA=\begin{bmatrix}27&11&-9\\6&2&-2\\-20&-\frac{13}{2}&8\end{bmatrix}$$

**(2)**

$$\left[\begin{array}{ccccc|c}4&-3&5&-10&11&-8\\2&1&5&0&3&6\\0&-1&-1&-2&1&-4\end{array}\right]\sim\left[\begin{array}{ccccc|c}0&-5&-5&-10&5&-20\\2&1&5&0&3&6\\0&-1&-1&-2&1&-4\end{array}\right]$$

$$\sim\left[\begin{array}{ccccc|c}0&0&0&0&0&0\\2&0&4&-2&4&2\\0&-1&-1&-2&1&-4\end{array}\right]\begin{matrix}\\ /\frac{1}{2}\\ /(-1)\end{matrix}\sim\left[\begin{array}{ccccc|c}0&0&0&0&0&0\\1&0&2&-1&2&1\\0&1&1&2&-1&4\end{array}\right]$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1-2x_3+x_4-2x_5\\4-x_3-2x_4+x_5\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1\\4\\0\\0\\0\end{bmatrix}+x_3\begin{bmatrix}-2\\-1\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}1\\-2\\0\\1\\0\end{bmatrix}+x_5\begin{bmatrix}-2\\1\\0\\0\\1\end{bmatrix}$$

$x_3, x_4, x_5 \in \mathbb{R}$

**(3)**

$$\begin{bmatrix}2&3&-1&1\\4&4&5&3\\-2&1&-2&0\\4&3&3&2\end{bmatrix}\sim\begin{bmatrix}2&3&-1&1\\0&-2&7&1\\0&4&-3&1\\0&-3&5&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&-2&7&1\\0&4&-3&1\\0&-3&5&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&-2&7&1\\0&6&-10&0\\0&-3&5&0\end{bmatrix}/\cdot\frac{1}{2}$$

$$\sim\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&3&-5&0\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&0&0\end{bmatrix}\quad r=3$$

**(4)**

$$3=\begin{vmatrix}x&2&3\\x&2&2x\\-1&1&x\end{vmatrix}\overset{-I}{=}\begin{vmatrix}x&2&3\\0&0&2x-3\\-1&1&x\end{vmatrix}\ \overset{\text{Laplaceov razvoj po 2. retku}}{=}$$

$=-(2x-3)(x+2)=-2x^2+3x-4x+6$

$2x^2+x-3=0$

$(2x+3)(x-1)=0 \iff x=1,\ x=-\frac{3}{2}$

**(5)**

$$\begin{vmatrix}1&1&1&1&1\\1&2&2&2&2\\1&2&3&3&3\\1&2&3&4&4\\1&2&3&4&5\end{vmatrix}/(-2)\ =\ \begin{vmatrix}1&1&1&1&1\\-1&0&0&0&0\\-2&-1&0&0&0\\-3&-2&-1&0&0\\-4&-3&-2&-1&0\end{vmatrix}$$

$$=\begin{vmatrix}0&0&0&0&1\\-1&0&0&0&0\\-2&-1&0&0&0\\-3&-2&-1&0&0\\-4&-3&-2&-1&0\end{vmatrix}\overset{L.r.}{=}\begin{vmatrix}-1&0&0&0\\-2&-1&0&0\\-3&-2&-1&0\\-4&-3&-2&-1\end{vmatrix}\ \text{direktno}$$

$=(-1)^4=1$

**(6)** $\vec{a}=3\vec{p}+\lambda\vec{q},\ \vec{b}=5\vec{p}+2\vec{q}$

$|\vec{p}|=2,\ |\vec{q}|=3,\ \angle(\vec{p},\vec{q})=\frac{2\pi}{3}\Rightarrow \vec{p}\vec{q}=-3$

$0=\vec{a}\cdot\vec{b}=(3\vec{p}+\lambda\vec{q})(5\vec{p}+2\vec{q})=$

$=15|\vec{p}|^2+2\lambda|\vec{q}|^2+(5\lambda+6)\vec{p}\vec{q}$

$=15\cdot4+2\lambda\cdot9+(5\lambda+6)(-3)=42+3\lambda$

$\iff \lambda=-14$

**(7)** $A=(1,1,1),\ B=(2,1,2),\ C=(3,2,2),\ D=(-1,2,$

$\vec{AB}=(1,0,1),\ \vec{AC}=(2,1,1),\ \vec{AD}=(-2,1,2)$

Volumen tetraedra $=\frac{1}{6}|(\vec{AB}\times\vec{AC})\cdot\vec{AD}|=$

$$=\frac{1}{6}\left|\begin{vmatrix}1&0&1\\2&1&1\\-2&1&2\end{vmatrix}\right|=\frac{1}{6}\left|\begin{vmatrix}1&0&0\\2&1&-1\\-2&1&4\end{vmatrix}\right|=\frac{1}{6}|5|=\frac{5}{6}$$

$(-1)$

**(8)** $p\perp\pi \ldots\ x-2y+z=2$

$p$ prolazi kroz $T=(2,-2,2)$

$$\frac{x-2}{1}=\frac{y+2}{-2}=\frac{z-2}{1}$$

$x=2+t \qquad x-2y+z=2$

$y=-2-2t \qquad (2+t)-2(-2-2t)+(2+t)=2$

$z=2+t \qquad 6t+8=2 \Rightarrow t=-1$

Probodište je $(x,y,z)=(1,0,1)$, udaljenost je

$\|(2,-2,2)-(1,0,1)\|=\|(1,-2,1)\|=\sqrt{6}$

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(1)
$$A=\begin{bmatrix}1&-1&2\\3&-2&1\\3&-2&1\end{bmatrix},\quad B=\begin{bmatrix}1&-1&-2\\5&-2&0\\2&1&-1\end{bmatrix}$$

$$XAB = X + A + 2I \iff X(AB-I) = A+2I$$
$$\iff X = (A+2I)(AB-I)^{-1}$$

$$AB=\begin{bmatrix}0&3&-4\\-5&2&-7\\-5&2&-7\end{bmatrix}\qquad AB-I=\begin{bmatrix}-1&3&-4\\-5&1&-7\\-5&2&-8\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}-1&3&-4&1&0&0\\-5&1&-7&0&1&0\\-5&2&-8&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}-1&3&-4&1&0&0\\0&-14&13&-5&1&0\\0&-13&12&-5&0&1\end{array}\right]/(-1)$$

$$\sim\left[\begin{array}{ccc|ccc}-1&3&-4&1&0&0\\0&-14&13&-5&1&0\\0&1&-1&0&-1&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}-1&0&-1&1&3&-3\\0&0&-1&-5&-13&14\\0&1&-1&0&-1&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}-1&0&0&6&16&-17\\0&0&-1&-5&-13&14\\0&1&0&5&12&-13\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&-6&-16&17\\0&1&0&5&12&-13\\0&0&1&5&13&-14\end{array}\right]$$

$$A+2I=\begin{bmatrix}3&-1&2\\3&0&1\\3&-2&3\end{bmatrix}\qquad (AB-I)^{-1}=\begin{bmatrix}-6&-16&17\\5&12&-13\\5&13&-14\end{bmatrix},$$

$$X=(A+2I)(AB-I)^{-1}=\begin{bmatrix}-13&-34&36\\-13&-35&37\\-13&-33&35\end{bmatrix}$$

(2)
$$\left[\begin{array}{ccccc|c}1&0&2&1&4&1\\0&1&1&-3&0&-2\\4&-3&5&13&16&10\\1&2&4&-5&4&-3\end{array}\right]\sim\left[\begin{array}{ccccc|c}1&0&2&1&4&1\\0&1&1&-3&0&-2\\0&-3&-3&9&0&6\\0&2&2&-6&0&-4\end{array}\right]\sim$$

$$\sim\left[\begin{array}{ccccc|c}1&0&2&1&4&1\\0&1&1&-3&0&-2\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right]$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1-2x_3-x_4-4x_5\\-2-x_3+3x_4\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1\\-2\\0\\0\\0\end{bmatrix}+x_3\begin{bmatrix}-2\\-1\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-1\\3\\0\\1\\0\end{bmatrix}+x_5\begin{bmatrix}-4\\0\\0\\0\\1\end{bmatrix}$$

$x_3, x_4, x_5 \in \mathbb{R}$

(3)
$$\begin{bmatrix}3&2&1&3\\3&3&5&5\\0&1&4&2\\3&1&-3&1\end{bmatrix}\sim\begin{bmatrix}3&2&1&3\\0&1&4&2\\0&1&4&2\\0&-1&-4&-2\end{bmatrix}$$

$$\sim\begin{bmatrix}3&0&-7&-1\\0&1&4&2\\0&0&0&0\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

rang = 2

(4)
$$5=\begin{vmatrix}1&x&-1\\2&3x&3\\x&0&1\end{vmatrix}=\begin{vmatrix}0&0&-1\\5&6x&3\\x+1&x&1\end{vmatrix}$$
$$=(-1)\left(5x-6x(x+1)\right)$$
$$=-5x+6x^2+6x$$
$$6x^2+x-5=0$$
$$(6x-5)(x+1)=0 \iff x=-1,\ x=\frac{5}{6}$$

(5)
$$\begin{vmatrix}1&2&3&4&5\\2&2&3&4&5\\3&3&3&4&5\\4&4&4&4&5\\5&5&5&5&5\end{vmatrix}/(-1)=\begin{vmatrix}1&2&3&4&5\\1&0&0&0&0\\2&1&0&0&0\\3&2&1&0&0\\4&3&2&1&0\end{vmatrix}\ \text{L. razvoj} = \text{po 5. stupcu}$$

$$=5\begin{vmatrix}1&0&0&0\\2&1&0&0\\3&2&1&0\\4&3&2&1\end{vmatrix}=5\cdot1^4=5.$$

(6)
$$\vec a=-6\vec p+2\vec q,\quad \vec b=2\vec p+\lambda\vec q$$
$$|\vec p|=1,\ |\vec q|=2,\ \angle(\vec p,\vec q)=\frac{\pi}{3}\implies \vec p\,\vec q=1$$
$$0=\vec a\cdot\vec b=(-6\vec p+2\vec q)(2\vec p+\lambda\vec q)$$
$$=-12|\vec p|^2+2\lambda|\vec q|^2+(4-6\lambda)\vec p\,\vec q$$
$$=-12+8\lambda+(4-6\lambda)=2\lambda-8$$
$$\iff \lambda=4$$

(7)
$$\overrightarrow{AB}=(4,3,2),\ \overrightarrow{AC}=(-2,1,0),\ \overrightarrow{AD}=(1,3,1)$$

Volumen paralelepipeda $=|(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}|$

$$=\left|\begin{vmatrix}4&3&2\\-2&1&0\\1&3&1\end{vmatrix}\right|=\left|\begin{vmatrix}0&-9&-2\\0&7&2\\1&3&1\end{vmatrix}\right|=|1\cdot(-18+14)|=4.$$

(8)
$$A=(0,-1,1),\ B=(1,-2,2),\ C=(-2,-1,0)$$

ravnina kroz A, B, C

$$0=\begin{vmatrix}x-0&y+1&z-1\\1-0&-2+1&2-1\\-2-0&-1+1&0-1\end{vmatrix}=\begin{vmatrix}x&y+1&z-1\\1&-1&1\\-2&0&-1\end{vmatrix}\ \overset{\text{L. razvoj po 3. retku}}{=}$$

$$=-2(y+1+z-1)-1(-x-y-1)$$
$$=-2y-2z+x+y+1=x-y-2z+1$$

pravac AB

$$\frac{x}{1}=\frac{y+1}{-1}=\frac{z-1}{1}$$