THE INGRAM CONJECTURE

M. BARGE, H. BRUIN, S. ŠTIMAC

ABSTRACT. We prove the Ingram Conjecture, *i.e.,* we show that the inverse limit spaces of tent maps with different slopes in the interval $[1, 2]$ are non-homeomorphic. Based on the structure obtained from the proof, we also show that every self-homeomorphism of the inverse limit space of a tent map is pseudo-isotopic, on the core, to some power of the shift homeomorphism.

1. INTRODUCTION

Apart from their interest within continuum theory, inverse limit spaces play a key role in the description of uniformly hyperbolic attractors [35, 36], global 'Hénon-like' strange attractors [6] and the structure emerging from homoclinic tangencies in dynamical systems [5]. They find further use in the area of (substitution) tiling spaces [1] which, in some cases, are covering spaces of the type of inverse limit spaces with which we are concerned with in this paper; namely, those with a single tent map $T_s : [0, 1] \to [0, 1]$, $x \mapsto \min\{sx, s(1 - x)\}$ as bonding map. Such inverse limit spaces can be embedded in the plane as global attractors of homeomorphisms [27, 34, 13] and immersed in the plane as global attractors of skew product maps [20].

Inverse limit spaces are notoriously difficult to classify. In this paper, we solve in the affirmative the classification problem known as the Ingram Conjecture:

Theorem 1.1 (Ingram Conjecture). If $1 \le s < s' \le 2$, then the corresponding inverse limit spaces $\varprojlim([0, 1], T_s)$ and $\varprojlim([0, 1], T_{s'})$ are non-homeomorphic.

This is the main outstanding conjecture regarding dynamics on continua, dating back to at least the early nineties. In the "Continua with the Houston problem book" in 1995 [22, page 257], Ingram writes

> The [...] question was asked of the author by Stu Baldwin at the [...]
> summer meeting of the AMS at Orono, Maine, in 1991. ... There is

a related question which the author has considered to be of interest for several years. He posed it at a problem session at the 1992 Spring Topology Conference in Charlotte for the special case (*that the critical point has period*) $n = 5$.

It is clear that if two interval maps are topologically conjugate, then their inverse limit spaces are homeomorphic. Thus it may be more natural to ask the question for the 'fuller' logistic family $f_a(x) = ax(1 - x)$, $a \in [0, 4]$, which includes (infinitely) renormalizable maps (see Definition 6.1). It is well-known [26] that each logistic map with positive topological entropy is semi-conjugate to a tent map T_s with $\log s = h_{top}(f_a)$, and these semi-conjugacies collapse (pre)periodic intervals to points. The effect of renormalization on the structure of the inverse limit space is well-understood, see [4]: it produces proper subcontinua that are periodic under the shift homeomorphism and homeomorphic with the inverse limit space of the renormalized map. The solution of the Ingram Conjecture then leads to an analogous result for logistic maps.

As the parameter a for the logistic map f_a is increased, a new periodic point of period k appears when the graph of f_a^k is tangent with the diagonal. If such a tangency happens at parameter value a_* and location (x, x) then there is an $\epsilon > 0$ and a pair of continuous curves $a \mapsto x_-(a), a \mapsto x_+(a)$ of k-periodic points with $x_-(a_*) = x = x_+(a_*)$ so that $x_-(a)$ is repelling and $x_+(a)$ is attracting under f_a^k, for $a_* < a \leq a_* + \epsilon$. This is called a *saddle-node* bifurcation and $\{x_-(a), x_+(a)\}$ is called a *saddle-node pair*. For each a, let $\sim_{sn}$ be the equivalence relation that identifies saddle-node pairs and let $\Omega(f_a)$ be the non-wandering set of f_a (see Definition 6.2). The *reduced non-wandering set* of f_a is $\Omega(f_a)/\sim_{sn}$.

Theorem 1.2. In the parameter range $(1, 4]$, two logistic maps have homeomorphic inverse limit spaces if and only if they are conjugate on their reduced non-wandering sets.

There have been several partial results to the Ingram Conjecture, *e.g.* Barge and Diamond [3], which solved the period $n = 5$ case, and [33, 15]. The Ingram Conjecture was shown to hold when the critical point is periodic by Kailhofer [23] (see also [9]), or has finite orbit by Štimac [32]. More recently, the case where the critical point is non-recurrent was solved in [30]. Further results that classify certain features of inverse

limit spaces of tent maps with non-periodic recurrent critical orbits were obtained in *e.g.* [11, 29, 14].

Our solution to the Ingram Conjecture gives more information about the set of self-homeomorphisms on $\varprojlim([0,1], T_s)$: we show that any such homeomorphism behaves like an iterate of the shift homeomorphism σ.

The *critical point* $\frac{1}{2}$ of T_s is denoted by c, and we write $c_i = T^i(c)$. Although T_s is defined on $[0,1]$, there is a forward invariant interval $[c_2, c_1] = [s(1 - s/2), s/2]$, called the *core*, on which T_s is surjective. We call $\varprojlim([c_2, c_1], T_s)$ the *core of the inverse limit space*. The space $\varprojlim([0,1], T_s)$ is the union of the core of the inverse limit and a ray $\mathfrak{C}$ converging onto it.

Recall that the *composant* of $x \in X$ is defined as the union of all proper subcontinua of X containing x. For $1 < s < 2$, $\varprojlim([0,1], T_s)$ has only three composants: the entire inverse limit space, $\mathfrak{C}$, and $\varprojlim([0,1], T_s) \setminus \{(\ldots, 0, 0, 0)\}$. But for $s > \sqrt{2}$, $\varprojlim([c_2, c_1], T_s)$ is indecomposable and hence has uncountably many pairwise disjoint composants, each of which is dense. If $s > \sqrt{2}$ and the orbit of c is finite, the composants of $\varprojlim([c_2, c_1], T_s)$ are the same as the arc-components. Otherwise, the composants can be very complicated. For $1 < s \leq \sqrt{2}$, the core has just two proper composants that overlap in a single arc-component.

Theorem 1.3. Given $s \in [1, 2]$, for every homeomorphism $h : \varprojlim([0,1], T_s) \circlearrowleft$, there is an $R \in \mathbb{Z}$ such that h, restricted to the core $\varprojlim([c_2, c_1], T_s)$, is pseudo-isotopic to σ^R, *i.e.*, it permutes the composants of the core of the inverse limit in the same way as σ^R.

The zero-composant $\mathfrak{C}$ of $\varprojlim([0,1], T_s)$ containing the endpoint $\alpha := (\ldots, 0, 0, 0)$ is important in our proof of the Ingram Conjecture; the "core" version of the Ingram Conjecture is still outstanding. Our proof relies on the properties of so-called link-symmetric arcs in the composant $\mathfrak{C}$. Inverse limit spaces are chainable, and w.r.t. natural chains, a homeomorphism $h : \varprojlim([0,1], T_{s'}) \to \varprojlim([0,1], T_s)$ maps link-symmetric arcs to link-symmetric arcs. From this we derive that maximal link-symmetric arcs in $\varprojlim([0,1], T_{s'})$ centered at so-called salient points s_i' map to link-symmetric arcs centered at salient points $s_{i+M} \in \varprojlim([0,1], T_s)$ for some $M \in \mathbb{Z}$ and all sufficiently large $i \in \mathbb{N}$.

This in turn implies that h maps so-called q-points close to p-points, while 'translating' their levels by a fixed number M. This shows that h effectively fixes the folding

pattern of the zero-composant, with the Ingram Conjecture as an easy consequence. Additional arguments show that every self-homeomorphism of $\varprojlim([0,1], T_s)$, when restricted to the core, is *pseudo-isotopic* to a power σ^R of the shift for some $R \in \mathbb{Z}$.

We give the basic definitions in the next section. In Section 3 we investigate the lengths of maximal link-symmetric arcs, leading in Section 4 to the proof that a homeomorphism between two unimodal inverse limit spaces induces a shift of indices of salient points, and more generally, acts as a shift on the levels of q-points and p-points. This leads to the proof of the Ingram Conjecture. In Section 5, we prove the results on pseudo-isotopy. Section 6, finally, is devoted to the proof of Theorem 1.2 on logistic maps.

2. Definitions

Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. Let $T_s : [0,1] \to [0, s/2]$, $T_s(x) = \min\{sx, s(1-x)\}$ be the tent map with slope $s \in [1,2]$ and critical point $c = \frac{1}{2}$. Write $c_i = c_i(s) := T_s^i(c)$, so in particular $c_1 = \frac{s}{2}$ and $c_2 = s(1 - \frac{s}{2})$.

The inverse limit space $\varprojlim([0,1], T_s)$ is the collection of backward orbits

$$\{x = (\dots, x_{-2}, x_{-1}, x_0) : T_s(x_{i-1}) = x_i \in [0, s/2] \text{ for all } i \leq 0\},$$

equipped with metric $d(x, y) = \sum_{n \leq 0} 2^n |x_n - y_n|$ and *induced (or shift) homeomorphism* $\sigma = \sigma_s$ given by

$$\sigma(\dots, x_{-2}, x_{-1}, x_0) = (\dots, x_{-2}, x_{-1}, x_0, T_s(x_0)).$$

Let $\pi_p : \varprojlim([0,1], T_s) \to [0,1]$, $\pi_p(x) = x_{-p}$, be the p-th projection map. Since T_s fixes 0, $\varprojlim([0,1], T_s)$ contains the endpoint $\alpha := (\dots, 0, 0, 0)$. The proper composant of $\varprojlim([0,1], T_s)$ containing this point is denoted by $\mathfrak{C}$; it is a ray converging from α to, but disjoint from, the core of the inverse limit space $\varprojlim([c_2, c_1], T_s)$.

Frequently, the Ingram Conjecture is posed for slopes $s, s' \in [\sqrt{2}, 2]$ only, because for $0 < s \leq \sqrt{2}$, $\varprojlim([c_2, c_1], T_s)$ is decomposable. Since $\varprojlim([0,1], T_s)$ is a single point for $s \in (0, 1)$ and an arc for $s = 1$, we will always assume that all slopes s are greater than 1. The next two lemmas show how to reduce the case $s \in (1, \sqrt{2}]$ to $s \in (\sqrt{2}, 2]$.

Lemma 2.1. For $2^{1/2^{n+1}} \leq s \leq 2^{1/2^n}$, $n \in \mathbb{N}$, the core of the inverse limit space $\varprojlim([c_2, c_1], T_s)$ is homeomorphic with two copies of $\varprojlim([0,1], T_{s^2})$ joined at their endpoints.

Proof. For this range of s, $T_s([c_2, p]) = [p, c_1]$ and $T_s([p, c_1]) = [c_2, p])$, where $p := \frac{s}{s+1}$ is the positive fixed point of T_s. It follows that $\varprojlim([c_2, c_1], T_s)$ is homeomorphic with two copies of $\varprojlim([p, c_1], T_s^2)$ joined at the endpoint $(\ldots, p, p, p)$. Direct calculation shows that, if L is the orientation preserving affine homeomorphism from $[p, c_1]$ onto $[0, c_1(s^2)]$, then $L \circ T_s^2 \circ L^{-1} = T_{s^2}$ on $[0, c_1(s^2)]$ and hence $\varprojlim([p, c_1], T_s^2)$ is homeomorphic with $\varprojlim([0, 1], T_{s^2})$. $\qquad\square$

Lemma 2.2. Suppose that $2^{1/2^n} < s \leq 2^{1/2^{n-1}}$ and $2^{1/2^{n'}} < s' \leq 2^{1/2^{n'-1}}$, $n, n' \in \mathbb{N}$, and suppose that $\varprojlim([0, 1], T_s)$ is homeomorphic with $\varprojlim([0, 1], T_{s'})$. Then $n = n'$ and assuming that the Ingram Conjecture holds for slopes $> \sqrt{2}$, then also $\varprojlim([0, 1], T_{s^{2n-1}})$ is homeomorphic with $\varprojlim([0, 1], T_{(s')^{2n-1}})$.

Proof. For $2^{1/2} < s < 2$, $\varprojlim([0, 1], T_s)$ consists of a ray $\mathfrak{C}$ winding onto an indecomposable continuum, namely $\varprojlim([c_2, c_1], T_s)$. It follows from Lemma 2.1 that for $2^{1/2^n} < s < 2^{1/2^{n-1}}$, $\varprojlim([0, 1], T_s)$ consists of a ray winding onto a pair of rays, each winding onto a pair of rays,..., each winding onto a pair of rays, each of which winds onto an indecomposable continuum. There are 2^{n-1} of these indecomposable continua, each homeomorphic with the core of the inverse limit space $\varprojlim([0, 1], T_{s^{2n-1}})$. Hence if $\varprojlim([0, 1], T_s)$ is homeomorphic with $\varprojlim([0, 1], T_{s'})$, then $n = n'$ and $\varprojlim([0, 1], T_{s^{2n-1}})$ is homeomorphic with $\varprojlim([0, 1], T_{(s')^{2n-1}})$. To cover the remaining cases, note that if $s = 2^{1/2^{n-1}}$, then the only alteration needed in the above description of $\varprojlim([0, 1], T_s)$ is that at the penultimate level, instead of a pair of rays winding onto a pair of indecomposable subcontinua, we just have two indecomposable subcontinua (each homeomorphic with $\varprojlim([0, 1], T_2)$) joined at their common endpoint. It is clear in this case that if $\varprojlim([0, 1], T_{s'})$ is homeomorphic with $\varprojlim([0, 1], T_s)$, then $s' = s$. $\qquad\square$

Definition 2.3. The *arc-length* or $\bar{d}$ metric on $\mathfrak{C}$ is defined as

$$\bar{d}(x, y) = s^p |x_{-p} - y_{-p}|$$

for each p so that $\pi_p : [x, y] \rightarrow [0, 1]$ is injective.

If $x, y \in \mathfrak{C}$, then we denote by $[x, y]$ the arc between x and y, and by (x, y) the interior of the arc $[x, y]$. We write $x \preceq y$ if $x \in [\alpha, y]$, *i.e.*, $\bar{d}(\alpha, x) \leq \bar{d}(\alpha, y)$.

Definition 2.4. A continuum is *chainable* if for every $\varepsilon > 0$, there is a cover $\{\ell^1, \ldots, \ell^n\}$ of open sets (called *links*) of diameter $< \varepsilon$ such that $\ell^i \cap \ell^j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Such a cover is called a *chain*. Clearly the interval $[0, s/2]$ is chainable. We call a sequence $\mathcal{C}_p$, $p \in \mathbb{N}_0$, a *natural chaining*, and each $\mathcal{C}_p$ a *natural chain*, of $\varprojlim([0,1], T_s)$, if

(1) there is a chain $\{I_p^1, I_p^2, \dots, I_p^n\}$ of $[0, s/2]$ such that $\ell_p^j := \pi_p^{-1}(I_p^j)$ are links of $\mathcal{C}_p$;

(2) each point $x \in \cup_{i=0}^p T_s^{-i}(c)$ is a boundary point of some link I_p^j;

(3) for each i there is j such that $T_s(I_{p+1}^i) \subset I_p^j$.

Let us define $\mathrm{width}(\mathcal{C}_p) := \max_j |I_p^j|$. If $\mathrm{width}(\mathcal{C}_p) < \varepsilon s^{-p}/2$ then $\mathrm{mesh}(\mathcal{C}_p) := \max\{\mathrm{diam}(\ell) : \ell \in \mathcal{C}_p\} < \varepsilon$, which shows that $\varprojlim([0,1], T_s)$ is indeed chainable.

Condition 3. ensures that $\mathcal{C}_{p+1}$ *refines* $\mathcal{C}_p$ (written $\mathcal{C}_{p+1} \preceq \mathcal{C}_p$).

Definition 2.5. Let $p \in \mathbb{N}_0$. A point $x = (\dots, x_{-2}, x_{-1}, x_0) \in \mathfrak{C}$ is called a *p-point* if $x_{-p-j} = c$ for some $j \in \mathbb{N}_0$. For the largest such j, the number $L_p(x) := j$ is called the *p-level*. In particular, $x_0 = T_s^{p+j}(c)$. The ordered set of all p-points of composant $\mathfrak{C}$ is denoted by E_p, and the ordered set of all p-points of p-level l by $E_{p,l}$. Given an arc $A \subset \mathfrak{C}$ with successive p-points $x^0, \dots, x^n$, the *p-folding pattern* of A, denoted by $FP_p(A)$, is the sequence $FP_p(A) = L_p(x^0), \dots, L_p(x^n)$. The *folding pattern of composant* $\mathfrak{C}$, denoted by $FP(\mathfrak{C})$, is the sequence $L_p(z^1), L_p(z^2), \dots, L_p(z^n), \dots$, where $E_p = \{z^1, z^2, \dots, z^n, \dots\}$ and p is any nonnegative integer. Let $q \in \mathbb{N}$, $q > p$, and $E_q = \{y^1, y^2, \dots, y^n, \dots\}$. Since σ^{q-p} is an order-preserving homeomorphism of $\mathfrak{C}$, it is easy to see that, for every $i \in \mathbb{N}$, $\sigma^{q-p}(z^i) = y^i$ and $L_p(z^i) = L_q(y^i)$. Therefore, the folding pattern of $\mathfrak{C}$ does not depend on p.

For the above arc A, the projection $\pi_p : A \to [0, s/2]$ need not be injective; so the folding pattern of A can be very long and A may pass through the same link ℓ^j of the natural chain $\mathcal{C}_p$ many times. If A^j is an arc component of $A \cap \ell^j$, then we say that A^j *goes straight* through ℓ^j if $\pi_p|_{A^j}$ is injective; otherwise it *turns* in ℓ^j. If A^j turns in ℓ^j, then A^j contains at least one p-point.

Definition 2.6. Let $\ell^0, \ell^1, \dots, \ell^k$ be those links in $\mathcal{C}_p$ that are successively visited by an arc $A = [u, v] \subset \mathfrak{C}$ (hence $\ell^i \neq \ell^{i+1}$, $\ell^i \cap \ell^{i+1} \neq \emptyset$ and $\ell^i = \ell^{i+2}$ is possible if A turns in ℓ^{i+1}). Let $A^i \subset \ell^i$ be the corresponding arc components such that $\mathrm{Cl}\, A^i$ are subarcs of A. We call the arc A

- *p-link-symmetric* if $\ell^i = \ell^{k-i}$ for $i = 0, \dots, k$;

- *maximal p-link-symmetric* if it is p-link-symmetric and there is no p-link-symmetric arc $B \supset A$ and passing through more links than A;
- *p-symmetric* if $\pi_p(u) = \pi_p(v)$ and if for $A \cap E_p = \{x^0, \ldots, x^n\}$ we have $L_p(x^i) = L_p(x^{n-i})$ for every $i = 0, \ldots, n$.

In any of these cases, k is even and the p-point of $A^{k/2}$ with the highest p-level is called the *center* of A, and the link $\ell^{k/2}$ is called the *central link* of A.

It is easy to see that if A is p-symmetric, then n is even and $L_p(x^{n/2}) = \max\{L_p(x^i) : x^i \in A \cap E_p\}$. Clearly, every p-symmetric arc is p-link-symmetric as well, but the converse does not hold.

Definition 2.7. Given p, let $(s_i)_{i \in \mathbb{N}}$ be the sequence of all p-points such that $0 \leq L_p(x) < L_p(s_i)$ for every p-point $x \in (\alpha, s_i)$. We call p-points satisfying this property *salient*.

Since for every slope $s > 1$ and $p \in \mathbb{N}_0$, the sequence $FP(\mathfrak{C})$ starts as $0\ 1\ 0\ 2\ 0\ 1\ \ldots$, and since by definition $L_p(s_1) > 0$, we have $L_p(s_1) = 1$. Also, since $s_i = \sigma^{i-1}(s_1)$, $L_p(s_i) = i$, for every $i \in \mathbb{N}$. Note that the salient p-points depend on p: if $p \geq q$, then the salient p-point s_i equals the salient q-point s_{i+p-q}.

Let us extend the notion of folding pattern as follows. A sequence $e_1, \ldots, e_k$ is the *folding pattern* of $T^j|_H$ for an interval $H \subset [0,1]$ if $c_{e_1} = T^j(x_1), \ldots, c_{e_k} = T^j(x_k)$, where $x_1 < \cdots < x_k$ are the critical points of T^j on H. (If $0 \in H$, then the folding pattern starts with $*$ by convention, just as $*$ denotes the conventional p-level of α.) In this extended terminology, the p-folding pattern of $[\alpha, s_{j+1}]$ is the same as the folding pattern of T^j on $[0, c_1]$, independently of p.

Measured in arc-length, $\bar{d}(\alpha, s_1) = \frac{1}{2}s^p$, and since $\sigma(s_i) = s_{i+1}$ we obtain

$$(2.1) \qquad\qquad \bar{d}(\alpha, s_i) = \frac{1}{2}s^{p-1}s^i \quad \text{for all } i \geq 1.$$

3. MAXIMAL LINK-SYMMETRIC ARCS

In this section we establish upper bounds for the lengths of p-link-symmetric arcs. The Ingram Conjecture was previously proved for all tent maps with a (pre)periodic critical point, see [32]. So we assume from now on that the slope s is such that c is not (pre)periodic. Throughout this section we use the notation $T := T_s$, $a_k := T^k(a)$

for any point or interval (except for the closest precritical points $z_{-k} \in T^{-k}(c)$ in Definition 3.1 below), and $\hat{a} := 1 - a$ is the symmetric point around c.

Definition 3.1. We call z_{-k} a *closest precritical point* if $T^k(z_{-k}) = c$ and T^k maps $[c, z_{-k}]$ monotonically onto $[c_k, c]$. Clearly, if z_{-k} is a closest precritical points, so is $\hat{z}_{-k}$.

Lemma 3.2. There are infinitely many N and closest precritical points z_{-N} such that $\theta_N := \min\{|c_i - c| : 0 < i \leq N\} > |z_{-N} - c|$.

Proof. If c is not recurrent, then $\theta_n \not\to 0$ and the lemma is trivial. So let us assume that c is recurrent, but obviously not periodic. Let n be such that $|c_n - c| = \theta_n$.

If $x \mapsto |T^n(x) - c|$ has a local maximum at c, then $T^n([c, c_n]) \ni c$. Indeed, if this were not the case, then by the choice of n, T^n maps $[c, c_n]$ in a monotone fashion into $[c, c_n]$, which is clearly impossible for tent maps with slope > 1. So in this case, $z_{-n} \in [c_n, \hat{c}_n]$ and the lemma holds with $N = n$.

So assume now that $x \mapsto |T^n(x) - c|$ has a local minimum at c. Take $m \in \mathbb{N}$ minimal such that the closest precritical $z_{-m} \in [\hat{c}_n, c_n]$. We will show that $c_j \notin [\hat{z}_{-m}, z_{-m}]$ for $n < j \leq m$. If $j = m$, then $x \mapsto |T^j(x) - c|$ has a local maximum at c, and we can argue as above. So assume by contradiction that $c_j \in [\hat{z}_{-m}, z_{-m}]$ for some $n < j < m$. If $x \mapsto |T^j(x) - c|$ has a local maximum at c, then the closest precritical point z_{-j} satisfies $T^j([c, z_{-m}]) \subset T^j([c, z_{-j}]) = [c_j, c] \subset [\hat{z}_{-m}, c]$ or $[c, z_{-m}]$. This implies that either $[c, z_{-m}]$ or $[\hat{z}_{-m}, c]$ is mapped monotonically into itself by T^j, which is impossible. The remaining possibility is that $x \mapsto |T^j(x) - c|$ has a local minimum at c. In this case, T^{j-n} maps $[z_{-m}, c_n]$ monotonically onto $[w, c_j]$. If $c \in (w, c_j)$, then $m \in \mathbb{N}$ cannot be minimal such that $T^m([c, c_n]) \ni c$. If $c \notin (w, c_j)$, then $w \in [\hat{c}_n, c_n] \cap T^{(j-n)-m}(c)$, and since $-m < (j - n) - m < 0$, m is again not minimal such that $T^m([c, c_n]) \ni c$.

Take $N = m$ and the lemma follows. $\qquad\square$

Take N_0 as in Lemma 3.2 and so large that $s^{N_0} > 100$. Let $N \geq N_0$ from Lemma 3.2 be so large that

(3.1) $$\delta := |z_{-N} - c| < |z_{-N_0} - c|/100.$$

Then $|c_n - c| \geq s^n|z_{-n} - c| \geq s^{N_0}|z_{-N} - c| > 100\delta$ for every $N_0 \leq n \leq N$ by the choice of N_0 and $|c_n - c| > |z_{-N_0} - c| > 100\delta$ for $n \leq N_0$ by the choice of N.

Definition 3.3. Given $r \in H$ with $|r - a|, |r - b| > \delta$, we say that $T^n|_H$ is ε-*symmetric around* r, if $|T^n(r + t) - T^n(r - t)| < \varepsilon$ whenever both $r - t, r + t \in H$. The point r is called the *center of symmetry*.

Mostly we will be interested in ε-symmetries around the midpoint of the interval H, but we will always specify the center of ε-symmetry, as sometimes there are multiple centers of ε-symmetry. If width$(\mathcal{C}_p) < \varepsilon$ and the arc $J \subset [\alpha, s_k]$ is p-link-symmetric, then $\pi_{p+k} : J \to H := \pi_{p+k}(J)$ is one-to-one and $T^k|_H$ is ε-symmetric.

Definition 3.4. We say that $T^n|_H$ is ε-*periodic* of period 2η if $|T^n(t) - T^n(t + 2\eta)| < \varepsilon$ for all $t, t + 2\eta \in H$.

If $T^n|_H$ is ε-symmetric around two centers r_1 and r_2 such that $|r_1 - r_2| = \eta$, then $T^n|_H$ is ε-periodic with period 2η. We will explain this fact in more detail in the proof of Proposition 3.6, where it is used several times.

Lemma 3.5. Given δ as in (3.1), there exists $r_0 = r_0(\delta)$ such that for every interval $\tilde{J}$ with $|\tilde{J}| \geq 22\delta$, there exist $l \leq r_0 N$ and an interval J with $|J| \geq 18\delta$ and concentric with $\tilde{J}$, such that $T^l|_J$ is monotone and $J_l := T^l(J) \supset [c - \delta, c + \delta]$.

Proof. Let x be the center of $\tilde{J}$ and take $m \geq 0$ minimal such that $\tilde{J}_m \ni c$; hence $T^m|_{\tilde{J}}$ is monotone.

Clearly, $m \leq (r_0 - 1)N$ for some $r_0 \geq 1$ depending only on δ. If $\partial \tilde{J}_m$ is δ-close to c, then we take $J' \subset \tilde{J}$ centered at x and slightly smaller such that $c \in \partial J'_m$ and $m' > m$ minimal such that $J'_{m'}$ contains c in its interior. Since $|J'_m| > 20\delta$, it contains z_{-N} or $\hat{z}_{-N}$ as in (3.1), and $m' - m \leq N$ and $|c_{m'-m} - c| \geq \delta$ by Lemma 3.2.

If at iterate m' the other boundary point of J' is δ-close to c, then $m' - m < N$. We take the interval $J'' \subset J'$ centered at x slightly smaller such that $c \in T^{m'}(\partial J'')$ and take $m'' > m'$ minimal such that c is an interior point of $T^{m''}(J'')$. Since $T^{m'}(z_{-N}) \in T^{m'}(J'')$, and by (3.1) again, $m \leq m' \leq m'' \leq m + N$ and $\partial J''_{m''}$ is not δ-close to c. In each case, there is $l \leq r_0 N$ and $J \in \{\tilde{J}, J', J''\}$ so that the lemma holds. $\square$

For interval $H =: [a, b]$ with center x we formulate the following property:

$$(3.2) \qquad c \in H \text{ and } \delta < \min\{|c - a|, |c - b|, |c - x|\}.$$

Proposition 3.6. Assume that $s \in [1, 2]$ is such that c is not (pre)periodic. There exists $\varepsilon > 0$ such that if H satisfies (3.2), then $T^n|_H$ is not ε-symmetric around its midpoint x for any $n \in \mathbb{N}_0$.

Proof. We will prove Proposition 3.6 using the induction hypothesis:

(IH$_n$)   if H satisfies (3.2), then $T^n|_H$ is not ε-symmetric around x.

Take N_0, N and δ as in (3.1), r_0 as in Lemma 3.5 and H that satisfies (3.2).
     Let $\varepsilon \in (0, \delta)$ be so small that

$$(3.3) \qquad\qquad \varepsilon < \min\{|c_i - c_j| : 0 \le i < j \le (2 + r_0)N\}.$$

Since c lies off-center in H by at least δ, by the choice of ε, (IH$_k$) holds for all $k \le (2 + r_0)N$. Assume now that (IH$_j$) holds for all $j < n$. We will prove (IH$_n$), but first, continuing with the interval $\tilde{J}$ of Lemma 3.5, we prove the following lemma.

Lemma 3.7. Let $\tilde{J}$ be an interval of length $|\tilde{J}| \ge 22\delta$ centered at c_k for some $1 \le k \le 2N$. If $T^j|_{\tilde{J}}$ is ε-symmetric around c_k for some $0 \le j \le n$, then the interval $J_l := T^l(J)$ from Lemma 3.5 satisfies condition (3.2).

Proof. We know already from Lemma 3.5 that $J_l \supset [c - \delta, c + \delta]$. Hence if (3.2) fails, then $\eta := |c_{k+l} - c| \le \delta$. Since $T^l|_J$ is monotone, $j > l$. Therefore $T^{j-l}|_{J_l}$ is ε-symmetric around c_{k+l} and symmetric around c, and it follows that $T^{j-l}|_{J_l}$ is ε-periodic with period 2η. Indeed, by symmetry around c, $T^{j-l}|_{J_l}$ is ε-symmetric around the symmetric point $\hat{c}_{k+l}$ (see Figure 1). Hence $T^{j-l}|_{J_l}$ must also be ε-symmetric around the points $c \pm 2\eta$, which are the reflections of c in c_{k+l} and $\hat{c}_{k+l}$, etc. Extending these symmetries, we see that $|T^{j-l}(t) - T^{j-l}(t + 2\eta)| < \varepsilon$ for all $t, t + 2\eta \in J_l$, so $T^{j-l}|_{J_l}$ is ε-periodic with period 2η. Even more, $T^{j-l}|_{J_l}$ is ε-symmetric around $c + 2i\eta$ on every separate subarc $P_i := [c + (2i - 1)\eta, c + (2i + 1)\eta] \subset J_l$.
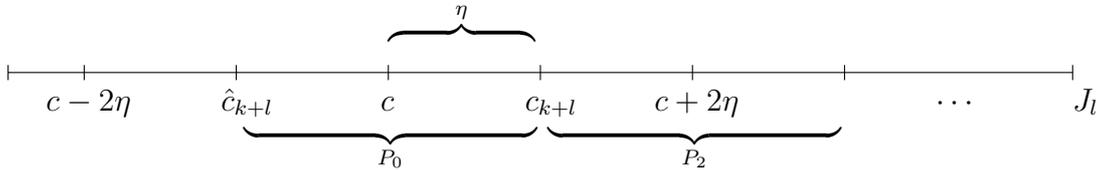

FIGURE 1. $T^{j-l}|_{J_l}$ is ε-symmetric around c_{k+l} and symmetric around c, implying that $T^{j-l}|_{J_l}$ is ε-periodic with period 2η.

Recall that $1 \leq k \leq 2N$ and $l \leq r_0 N$, so we have $\eta > \varepsilon$ by the choice of ε in (3.3). Since $|J_l| \geq 18\delta = 18|z_{-N} - c|$, one of the components of $J_l \setminus \{c\}$, say the one containing $z_{-N} > c$, has length $\geq 9\delta$. We can take $r \leq N$ minimal such that $z_{-r} \in [c + \delta, c + 8.9\delta]$. Take $i \in \mathbb{Z}$ such that if

$$(3.4) \qquad z_{-r} \in \begin{cases} (c, c + 4.3\delta], & \text{then } c + 2i\eta \in (z_{-r} + 0.1\delta, z_{-r} + 2.1\delta), \\ (c + 4.3\delta, c + 8.9\delta], & \text{then } c + 2i\eta \in (z_{-r} - 2.1\delta, z_{-r} - 0.1\delta). \end{cases}$$

Let $H \subset J_l$ be the longest interval centered at $x := c + 2i\eta$ on which $T^r|_H$ is monotone. Then $H \ni z_{-r}$, and $T^{j-l}|_H$ and $T^{j-l-r}|_{H_r}$ are ε-symmetric around x. We will show that H_r satisfies (3.2). Indeed, since $|z_{-r} - c| \leq 9\delta < |z_{N_0} - c|/10$ (so $r > N_0$) by (3.1) and $|x - z_{-r}| \geq \delta/10$, we have $|x_r - c| = s^r|x - z_{-r}| \geq 2^{N_0/2}\delta/10 > \delta$. If $|z_{-r} - \partial H| \geq \delta/10$, then $|c - \partial H_r| > \delta$ for the same reason. If on the other hand there is a point $y \in \partial H$ such that $|y - z_{-r}| < \delta/10$, then y has to be a precritical point. By the choice of r, $y = z_{-r'} \in (c + 8.9\delta, c + 9\delta]$ for some $r' < r$. By the choice of N and Lemma 3.2, $|y_r - c| = |c_{r-r'} - c| \geq \delta$.

This shows that H_r satisfies (3.2), but also $T^{j-l-r}|_{H_r}$ is ε-symmetric around x_r, and this contradicts (IH_{j-l-r}), proving this lemma. $\qquad\square$

Combining the induction hypothesis (IH_n) and Lemma 3.7, we have proved the following stronger property.

Corollary 3.8. If $\tilde{J}$ is centered at c_k for some $1 \leq k \leq 2N$ and $|\tilde{J}| \geq 22\delta$, then $T^j|_{\tilde{J}}$ is not ε-symmetric around c_k for $j \leq n$.

Now we continue the induction on n and assume by contradiction that $T^n|_H$ is ε-symmetric around midpoint x for some H satisfying (3.2) and for ε satisfying (3.3). Let $[a', b'] := H' \subset H$ be centered around x such that $c \in \partial H'$. Assume without loss of generality that $c = a'$ is the left endpoint of H', and let L and R be intervals of length δ at the left and right side adjacent to H'. Since $|H'| \geq \delta$, so $H' \ni z_{-N}$ or $\hat{z}_{-N}$, there is $0 < k \leq N$ minimal such that $c \in H'_k$. Clearly $|H'_k| > |L_k| = |R_k| \geq 100\delta$. We distinguish four cases:

Case I: H'_k satisfies (3.2). Then by (IH_{n-k}), $T^{n-k}|_{H'_k}$ cannot be ε-symmetric around x, and neither can $T^n|_{H'}$ or $T^n|_H$.

Case II: $|x_k - c| < \delta$, see Figure 2 (left). If the length of the interval $T^{n-k}([x_k, c])$ exceeds ε, then since T^{n-k} is also symmetric around c, T^{n-k} must be ε-symmetric on
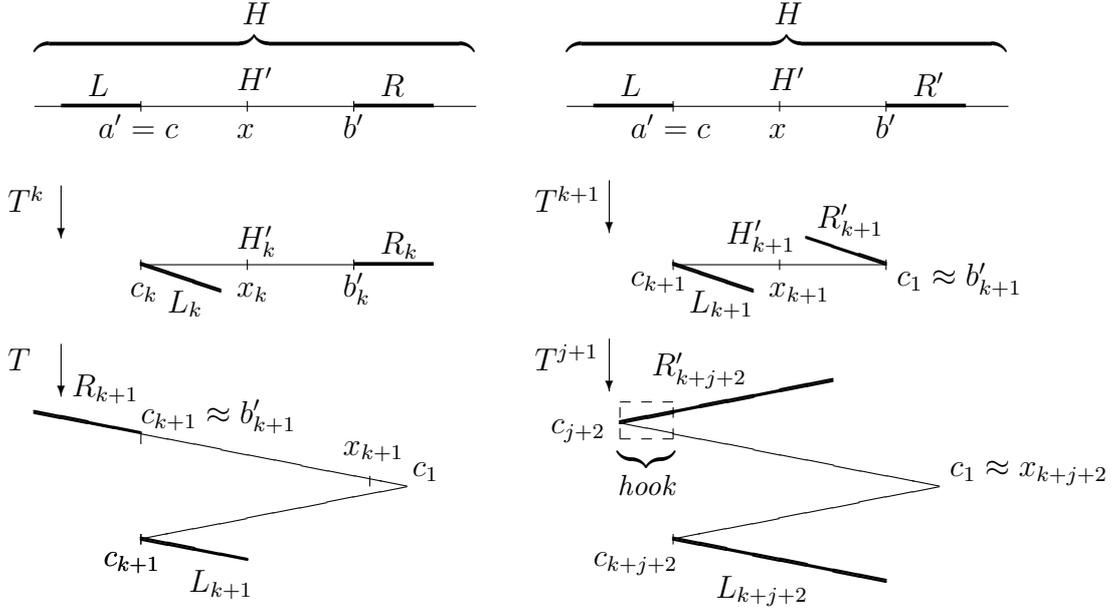
FIGURE 2. An illustration of Cases II (left) and IV (right).

H'_k both with center x_k and with center $\hat{x}_k$, and therefore ε-periodic on H' with period $2\eta := 2|x_k - c|$. We use the same argument as in the proof of Lemma 3.7: T^{n-k} is ε-symmetric on each interval $P_i := [c + (2i-1)\eta, c + (2i+1)\eta]$ for each $i \in \mathbb{Z}$ such that $P_i \subset H'_k$. Since $|H'_k| \geq 100\delta \geq 100\eta$, $P_i \subset H'_k$ for at least $-25 \leq i \leq 25$. Take $r \leq N$ minimal such that $[z_{-r} - \delta/10, z_{-r} + \delta/10] \subset H'_k$, and $i \in \mathbb{Z}$ as in (3.4), and $H'' \subset H'$ such that H''_k is the maximal interval centered at $c + 2i\eta$ on which T^r is monotone. As before, $T^{n-(k+r)}|_{H''_{k+r}}$ is the ε-symmetric but H''_{k+r} satisfies (3.2). This would contradict $(\mathrm{IH}_{n-(k+r)})$, so it cannot occur.

If on the other hand the length of $T^{n-k}([x_k, c])$ is less than ε, then we might as well have chosen x such that $x_k = c$. This means that the intervals L_{k+1} and R_{k+1} are adjacent, see Figure 2 (bottom left). More precisely, they are adjacent except for an error which does not show at ε-scale under the iterate $T^{n-(k+1)}$, so by a negligible adjustment, we can assume that they form an interval of length $\geq 100\delta$ with center c_{k+1}. Since $k + 1 \leq 2N$, Corollary 3.8 implies that $T^{n-(k+1)}|_{L_{k+1} \cup R_{k+1}}$ and hence $T^n|_H$ are not ε-symmetric around x.

Case III: $|a'_k - c| < \delta$. Since $k \leq N$, the choice of δ renders this impossible.

Case IV: $|b'_k - c| < \delta$, see Figure 2 (right). Replace R by the largest interval $R' \subset H \cup R$

with $R' \cap R \neq \emptyset$ such that $c \in \partial R'_k$ and $T^k|_{R'}$ is monotone. If $c \in \partial R'_l$ for some $0 \leq l < k$, then $R'_k = [c, c_{k-l}]$, so $|R'_k| \geq \delta$ by Lemma 3.2. Also rename $H' \setminus R'$ to H'. Hence $T^{k+1}|_{L \cup H' \cup R'}$ has three branches, $s\delta \leq |R'_{k+1}|$ and $100\delta \leq |L_{k+1}| \leq |H'_{k+1}|$.

Let $j > 0$ be minimal such that $T^{k+j+1}(H') \ni c$. If $H'_{k+j+1} = [c_{k+j+1}, c_{j+1}]$, which is centered at x_{k+j+1}, satisfies (3.2), then we can invoke $(\mathrm{IH}_{n-(k+j+1)})$, so assume that this is not the case. Since $|L| \geq \delta$, so $L \ni z_{-N}$ or $\hat{z}_{-N}$, we have $j \leq k + j + 1 \leq N$. Therefore both $|c_{j+1} - c| > \delta$ and $|c_{k+j+1} - c| \geq \delta$.

Thus if (3.2) fails, we must have $|x_{k+j+1} - c| < \delta$. If in the remaining $n - (k+j+1)$ iterates, the arc $[x_{k+j+1}, c]$ grows to length $> \varepsilon$, then, as in Case II, $T^n|_{H'}$ must contain a large ε-periodic arc, to which we apply the same argument as in Case II (*i.e.*, the argument of Lemma 3.7). The remaining possibility is that x_{k+j+1} is so close to c that on an ε-scale, we may as well assume that $x_{n+k+1} = c$.

Both $c_{k+j+2} = a'_{k+j+2}$ and $c_{j+2} \approx b'_{k+j+2}$ are local minima of $T^{k+j+2}|_{L \cup H' \cup R'}$, see Figure 2 (bottom right). Assume without loss of generality that $c_{j+2} < c_{k+j+2}$, so R'_{k+j+2} has a small extra hook before joining up with L_{k+j+2}. As we assumed that $T^n|_H$ is ε-symmetric around x, the effect of this hook needs to be 'ε-repeated' near a' in L. But L_{k+j+2} and R'_{k+j+2} overlap, so in R', the same effect needs to be ε-repeated next to the first hook. Continuing this way, we find that $T^{n-(k+j+2)}$ is ε-periodic over the entire length of R'_{k+j+2}.

Take i minimal such that $R'' := T^i(R'_{k+j+2}) \ni c$. Since $|R'_k| \geq \delta$ we have $j+i+2 < N$, $|R''| \geq 100\delta$ and $|\partial R'' - c| \geq \delta$. Therefore $T^{n-(k+j+i+2)}|_{R''}$ is ε-periodic of period 2η, where the length of the hook after i more iterates is $\eta := |c_{j+i+2} - c_{k+j+i+2}| > \varepsilon$, because $k + j + i + 2 \leq 2N$ and by the choice of ε in (3.3). If $\eta < 10\delta < |R''|/10$, then $T^{n-(k+j+i+2)}|_{R''}$ is ε-periodic with at least 5 adjacent intervals P of length 2η around the center of which $T^{n-(k+j+i+2)}|_{R''}$ is ε-symmetric. So we can find a new interval $H'' \subset R''$ centered around the center of one of these Ps such that H'' satisfies (3.2). But this contradicts $(\mathrm{IH}_{n-(k+j+i+2)})$.

If $\eta \geq 10\delta$, then we let H'' be the arc of length 22δ centered at $c_{k+j+i+2}$. Again, since $k + j + i + 2 \leq 2N$, the iterate $T^{n-(k+j+i+2)}$ cannot be ε-symmetric on H'' around $c_{k+j+i+2}$ by Corollary 3.8. But then the assumed ε-symmetry of $T^n|_H$ does not extend beyond H', and Case IV follows.

This proves the inductive step and the proof of Proposition 3.6 is complete. $\square$

Let $\kappa := \min\{i \geq 3 : c_i \geq c\}$. Then $\kappa < \infty$ provided $1 < s < 2$. Let $\cdots < c_{-3} < c_{-2} < c_{-1} < c_0 = c$ be the successive precritical points on the left of c with $T^j(c_{-j}) = c$. Since $c_{\kappa-1} < c < c_\kappa$, we have $c_{2-\kappa} < c_2 < c_{3-\kappa}$. Let $\delta = |z_{-N} - c|$ as in (3.1) be so small (*i.e.*, N as in Lemma 3.2 so large) that

$$(3.5) \qquad \delta < \frac{1}{30}\min\{|c_{-1} - c_{-2}|, |c_{-1} - \hat{c}_1|, |c_2 - c_{2-\kappa}|\},$$

where $\hat{c}_1 = 1 - c_1 = 1 - s/2$. Assume that $s \in [1, 2]$ is such that c is not (pre)periodic, and take ε is as in (3.3) in the proof of Proposition 3.6.

Let $(A_i)_{i \in \mathbb{N}}$ be the sequence of maximal p-link-symmetric arcs with center s_i for every $i \in \mathbb{N}$. Recall that $(s_i)_{i \in \mathbb{N}}$ is the sequence of salient p-points (see Definition 2.7) and that $\text{width}(\mathcal{C}_p) := \max_j |I_p^j|$.

Lemma 3.9. *If* $\text{width}(\mathcal{C}_p) < \varepsilon$, *then* A_i *contains exactly* κ *salient* p-points *for each* $i \geq \kappa - 1$, *namely* $s_{i-\kappa+2}, s_{i-\kappa+3}, \ldots, s_{i+1}$, *and* $s_{i-\kappa+2}$ *is an interior point of* A_i.

Proof. Let H be the interval centered at c_2 such that c is the left endpoint of $H_{\kappa-2} := T^{\kappa-2}(H)$. Then $|H| \geq 22\delta$ by the choice of δ, so by Proposition 3.6 and Lemma 3.7 in particular, $T^{p+i-1}|_H$ cannot be ε-symmetric around c_2.
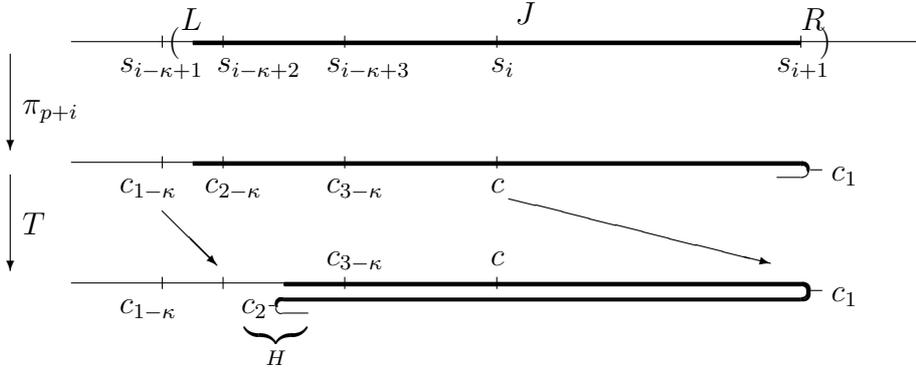

FIGURE 3. The arc J and its image under π_{p+i} and $T \circ \pi_{p+i} = \pi_{p+i-1}$.

Let $J = [x, s_{i+1}]$ be such that $J \ni s_i$ and $\bar{d}(s_i, s_{i+1}) = \bar{d}(x, s_i)$, where $\bar{d}$ is defined in Definition 2.3. Then π_{p+i-1} maps J in a 2-to-1 fashion onto $[c_2, c_1]$, with $\pi_{p+i-1}(s_i) = c_1$ and $\pi_{p+i-1}(s_{i+1}) = c_2$. Therefore J is p-symmetric and also p-link-symmetric around s_i. Since $c_{2-\kappa} < c_2 < c_{3-\kappa}$, we have $\pi_{p+i-1}(J) \not\ni c_{2-\kappa}$. Extend J on either side by equally long arcs L and R such that $\pi_{p+i-1}(L \cup R) = H$, see Figure 3. Since $T^{i-1}|_H$ is

not ε-symmetric around c_2, $A_i \not\supset L \cup J \cup R$ provided width$(\mathcal{C}_p) < \varepsilon$. Hence $A_i \not\supset s_{i-\kappa+1}$ as claimed. $\qquad\square$

Remark 3.10. The bound κ in this lemma is not sharp if T_s has a periodic critical point. For example, for the tent map with $c_2 < c = c_3 < c_1$, the folding pattern is

$$FP(\mathfrak{C}) = * \, 0 \, \underline{1} \, 0 \, \underline{2} \, 0 \, 1 \, \underline{3} \, \overbrace{1 \, 0 \, 2 \, 0 \, \underline{4} \, 0 \, 2 \, 0 \, 1 \, 3 \, 1 \, 0 \, \underline{5} \, 0 \, 1 \, 3 \, 1 \, 0 \, 2 \, 0 \, 4 \, 0 \, 2 \, 0 \, 1}^{\text{maximal } p\text{-symmetric}} \, \underline{6} \, 1 \, 0 \, 2 \, 0 \, 4 \, 0 \, 2 \, 0 \, 1 \ldots$$

with the underbrace labeled "maximal p-link-symmetric"

where p-levels of salient p-points are underlined and $*$ denotes the conventional p-level of α. Since c has period 3, so $c_a = c_{a+3b}$ for all $a, b \in \mathbb{N}$, p-link-symmetric arcs can be longer than p-symmetric arcs. Indeed, the maximal p-symmetric arc centered at salient point s_5 stretches from s_3 to s_6, while maximal p-link-symmetric arc centered at s_5 stretches almost from α to some point with p-level 2. This property holds for all salient points: the maximal p-link-symmetric arc around s_i contains s_j for all $j \leq i+1$.

A pre-periodic example is $s = 2$, *i.e.*, $\varprojlim([0, 1], T_s)$ is the Knaster continuum.

Lemma 3.11. Assume that width$(\mathcal{C}_p) < \varepsilon$ and fix $i \in \mathbb{N}$, $i > \kappa - 1$, and let ℓ^i and ℓ^{i-1} be the links of $\mathcal{C}_p$ containing s_i and s_{i-1} respectively. Let y be such that $s_{i-1} \prec y \prec s_i$ and y is not contained in the same arc-component of ℓ^i as s_i, nor in the same arc-component of ℓ^{i-1} as s_{i-1}. Then the maximal p-link-symmetric arc J with center y contains at most one salient p-point, and $J \subset A_i$.

Proof. Let $\cdots < c_{-2} < c_{-1} < c_0 = c$ be the successive precritical points to the left of c with $T^j(c_{-j}) = c$. Since A_i contains s_{i+1} and its symmetric point around s_i (at least as boundary points), we have $\pi_{p+i}(A_i) \supset [\hat{c}_1, c_1] \supset [c_{-1}, c]$. Let $H := \pi_{p+i}(J)$ with center $x := \pi_{p+i}(y) \in [c_{-1}, c]$. Assume by contradiction that J contains two salient p-points, or that $J \not\subset A_i$. Then $|H| \geq 22\delta$ by the choice of δ in (3.5).

Let $w := (c_{-1} + c)/2$. We distinguish four cases.

   (1) $c - \delta < x < c$. If $|T^i([x, c])| \leq \varepsilon$, then we cannot '$\varepsilon$-distinguish' x from c, violating our assumption that y and s_i do not belong to the same arc-component of the same link. If $|T^i([x, c])| > \varepsilon$, then T^i is ε-symmetric on H with centers x and c, so T^i is ε-periodic on H with period $2|x-c|$. This leads to a contradiction by the argument of the proof of Lemma 3.7.

   (2) $w \leq x \leq c - \delta$. Then H satisfies (3.2), so by Proposition 3.6, $T^i|_H$ cannot be ε-symmetric around x.

(3) $c_{-1} + \delta/s \leq x < w$. Then by assumption H contains one of c, c_{-2} or $\hat{c}_1$ (whence $|H| \geq 22\delta$), and hence $T(H \cap [c_{-2}, c] \cap [\hat{c}_1, c])$ satisfies (3.2), so $T^i|_H$ cannot be ε-symmetric around x by Proposition 3.6.

(4) $c_{-1} < x < c_{-1} + \delta/s$. If $|T^i([c_{-1}, x])| \leq \varepsilon$, then we cannot '$\varepsilon$-distinguish' x from c, violating the assumption that y and s_{i-1} are not contained in the same arc component of ℓ^{i-1}. If $|T^i([c_{-1}, x])| > \varepsilon$ and again, H by assumption contains one of c, c_{-2} or $\hat{c}_1$ (so $|H| \geq 22\delta$), then T^{i-1} is ε-periodic on $T(H)$ which again leads to a contradiction by the argument of the proof of Lemma 3.7.

This proves the lemma. $\qquad\square$

4. Link-Symmetric Arcs and Homeomorphisms

In this section we study the action of homeomorphisms $h : \varprojlim([0,1], T_{s'}) \to \varprojlim([0,1], T_s)$ on salient q-points and q-points in general. Let $q, p, g \in \mathbb{N}_0$ be such that

$$h(\mathcal{C}_q) \preceq \mathcal{C}_p \preceq h(\mathcal{C}_g).$$

Recall that we assumed the slopes s' and s to be such that the critical points c' and c are not (pre)periodic. Clearly h maps the zero-composant $\mathfrak{C}'$ of $\varprojlim([0,1], T_{s'})$ to the zero-composant $\mathfrak{C}$ of $\varprojlim([0,1], T_s)$, and in particular the endpoint α' of $\mathfrak{C}'$ to the endpoint α of $\mathfrak{C}$. Let $\kappa' := \min\{i \geq 3 : c'_i \geq c'\}$, where $c'_i = T^i_{s'}(c')$. Let us denote the salient q-points (*i.e.*, associated with $\mathcal{C}_q$) by s'_i and the salient g-points by s''_i. Therefore, salient q-point s'_i is the same as salient g-point s''_{i+q-g}. Similarly, let A'_i be the maximal q-link-symmetric arc centered at s'_i while as before, A_i denotes the maximal p-link-symmetric arc centered at s_i

Since A'_i is q-link-symmetric, and $h(\mathcal{C}_q) \preceq \mathcal{C}_p$, the image $D_i := h(A'_i) \subset \mathfrak{C}$ is p-link-symmetric and therefore has a well-defined center, we denote it as m_i, and a well-defined central link ℓ_p (see Definition 2.6). In fact, $h(s'_i)$ and m_i belong to the central link ℓ_p and m_i is the p-point with the highest p-level of all p-points of the arc component of ℓ_p which contains $h(s_i)$. Let $M_i := L_p(m_i)$.

Theorem 4.1. $M_{i+1} = M_i + 1$ for all sufficiently large integers $i \in \mathbb{N}$.

Proof. Without loss of generality we can assume that $s' \geq s$, so that $\kappa' \geq \kappa$. We prove first that if $N \geq \kappa$ is so large that m_N lies beyond the κ-th salient p-point of $\mathfrak{C}$, then $L_p(y) < M_N$, for every $y \in (\alpha, m_N)$; *i.e.*, m_N is salient.

Assume by contradiction that there exists $y \in (\alpha, m_N)$ such that $L_p(y) \geq M_N$. By taking $L_p(y)$ maximal with this property, we can assume that $y = s_{j-1} \prec m_N \prec s_j$ for some $j > \kappa$. More precisely, m_N is not contained in the same arc-component of the link containing s_{j-1} as s_{j-1}, and similarly for s_j. Lemma 3.11 implies that D_N contains at most one salient p-point and that $D_N \subset A_j$. Let us denote by B the p-link-symmetric arc such that s_j is the center of B, $D_N \subset B \subset A_j$ and $\partial D_N \cap \partial B \neq \emptyset$ (see Figure 4). Since $\mathcal{C}_p \preceq h(\mathcal{C}_g)$, the arc $B'' = \sigma^{q-g} \circ h^{-1}(B)$ is g-link-symmetric and contains the arc $\sigma^{q-g} \circ h^{-1}(D_N) = \sigma^{q-g}(A'_N)$. The center z'' of B'' is the center of the arc component of the central link ℓ_g of B'' containing $\sigma^{q-g} \circ h^{-1}(s_j)$. By Lemma 3.9, A'_N contains κ' salient q-points $s'_{N-\kappa'+2}, \ldots, s'_N, s'_{N+1}$.
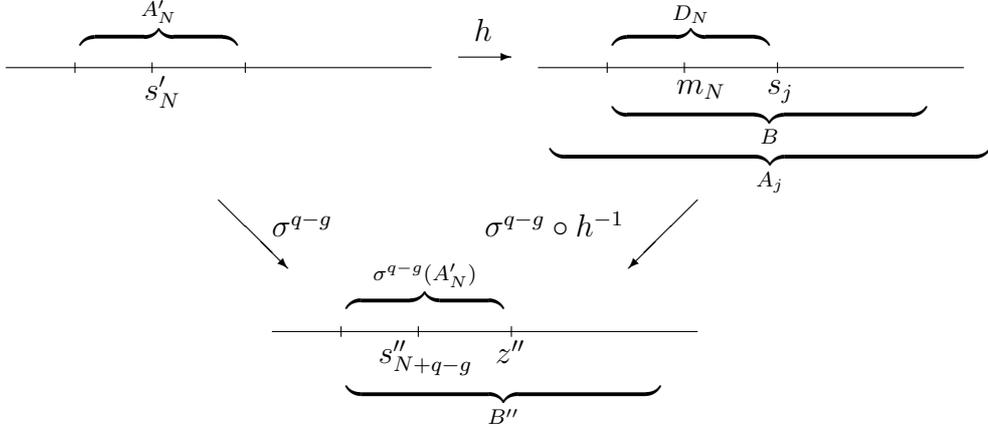

FIGURE 4. The relations between relative salient points and arcs in $\mathcal{C}_q$ (left), $\mathcal{C}_p$ (right), and $\mathcal{C}_g$ (bottom).

The map σ^{q-g} maps the κ' salient q-points $s'_i \in A'_N$ to the κ' salient g-points $s''_{i+q-g} \in \sigma^{q-g}(A'_N)$, and B'' contains at least these κ' salient g-points. If the center z'' of B'' is not salient, then B'' contains at most one salient g-point by Lemma 3.11, so we have a contradiction. Otherwise, if z'' is salient, then even if z'' is the right-most salient g-point of $\sigma^{q-g}(A'_N)$, then still B'' contains $\kappa' - 1$ salient g-points on the left of the center z'', contradicting Lemma 3.9. Therefore, m_N is salient.

Let us consider the arc $D_{N+\kappa'-2} = h(A'_{N+\kappa'-2})$. Since $L_q(s'_{i+1}) - L_q(s'_i) = 1$, the arc $[s'_i, s'_{i+1}]$ contains a q-point of every q-level less than i, so contains q-points of q-levels 1 and 2. Therefore, $\pi_q([s'_i, s'_{i+1}]) = [c_2, c_1]$. Note that two different points from $s'_N, \ldots, s'_{N+\kappa'-1} \in A'_{N+\kappa'-2}$ can be mapped into the same link, say ℓ_p of $\mathcal{C}_p$, but cannot be mapped into the same arc component of ℓ_p. Indeed, if $h([s'_i, s'_{i+1}]) \subset A$, where A is

a arc component of ℓ_p, then $h(\mathcal{C}_q) \subset \ell_p$, a contradiction. Therefore, $s_{M_N}, \ldots, s_{M_{N+\kappa'-1}}$ are all different.

So, the arc $D_{N+\kappa'-2}$ is p-link-symmetric and contains at least κ' salient p-points, $s_{M_N}, \ldots, s_{M_{N+\kappa'-1}}$. By Lemma 3.9, the maximal p-link-symmetric arc $A_{M_{N+\kappa'-2}}$ centered at the salient p-point $s_{M_{N+\kappa'-2}}$ contains κ salient p-points, $s_{M_{N+\kappa'-2}-\kappa+2}, \ldots,$ $s_{M_{N+\kappa'-2}}, s_{M_{N+\kappa'-2}+1}$. Therefore, $D_{N+\kappa'-2} \subseteq A_{M_{N+\kappa'-2}}$, $\kappa' = \kappa$, $s_{M_{N+i}} = s_{M_N+i}$ and $M_{N+i} = M_N + i$ for all $0 \le i \le \kappa - 1$. By induction we get $M_{N+i} = M_N + i$ for all $i \in \mathbb{N}_0$ as well. $\qquad\square$

Every salient p-point $s_i \in \mathfrak{C}$ can be contained in at most two links of $\mathcal{C}_p$, and one of them is always the central link of A_i, which we will denote by $\ell_p^{s_i}$. Let K_{s_i} be the arc component of $\ell_p^{s_i}$ containing s_i. Given a p-point $x \in \mathfrak{C}$ with $L_p(x) = l$, there can be two links of $\mathcal{C}_p$ containing x, but one of them is always $\ell_p^{s_l}$. We denote the arc component of $\ell_p^{s_l}$ containing x by K_x. Let $\ell_q^{s_i'} \in \mathcal{C}_q$ and $K_{s_i'} \subset \ell_q^{s_i'}$ be the similar notation related to $\mathfrak{C}'$ and $\mathcal{C}_q$. Also, for a q-point x' of $\mathfrak{C}'$ with $L_q(x') = k$ let the arc component of $\ell_q^{s_k'}$ containing x' be denoted by $K_{x'}$.

Proposition 4.2. There exists $M \in \mathbb{Z}$ such that the following holds:

(1) Let $l \in \mathbb{N}$ and let x' be a q-point with $L_q(x') = l$. Then $u := h(x') \in \ell_p^{s_{l+M}}$ and the arc component $K_u \subset \ell_p^{s_{l+M}}$ containing u, also contains a p-point x such that $L_p(x) = l + M$.

(2) For $l \in \mathbb{N}_0$ and $i \in \mathbb{N}$, the number of q-points in $[s_i', s_{i+1}']$ with q-level l is the same as the number of p-points in $[s_{M+i}, s_{M+i+1}]$ with p-level $M + l$.

Proof. (1) Recall that the set of q-points in $\mathfrak{C}'$ is denoted by E_q'. By Theorem 4.1, there exists $M \in \mathbb{Z}$ such that $a_i := h(s_i') \in \ell_p^{s_{M+i}}$ for every $i \in \mathbb{N}_0$ and the arc component K_{a_i} of $\ell_p^{s_{M+i}}$ contains s_{M+i}. Therefore, statement (1) is true for all salient q-points.

Also $h([s_1', s_2']) = [a_1, a_2]$, $s_{M+1} \in K_{a_1}$ and $s_{M+2} \in K_{a_2}$. Let q-point $x_1' \in [s_2', s_3']$ be such that the arc $[s_1', x_1']$ is q-symmetric with center s_2'. Then $h([s_1', x_1'])$ is p-link-symmetric with center s_{M+2}. Since there exists a unique p-point b_1 such that the arc $[s_{M+1}, b_1]$ is p-symmetric with center s_{M+2}, we have $h(x_1') \in K_{b_1}$, see Figure 5. Also $L_q(x_1') = 1$ and $L_p(b_1) = M + 1$.

We have $h([s_2', s_3']) = [a_2, a_3]$, $s_{M+2} \in K_{a_2}$ and $s_{M+3} \in K_{a_3}$. Let the q-point $x_2' \in [s_3', s_4']$ be such that the arc $[s_2', x_2']$ is q-symmetric with center s_3'. Therefore $h([s_2', x_2'])$
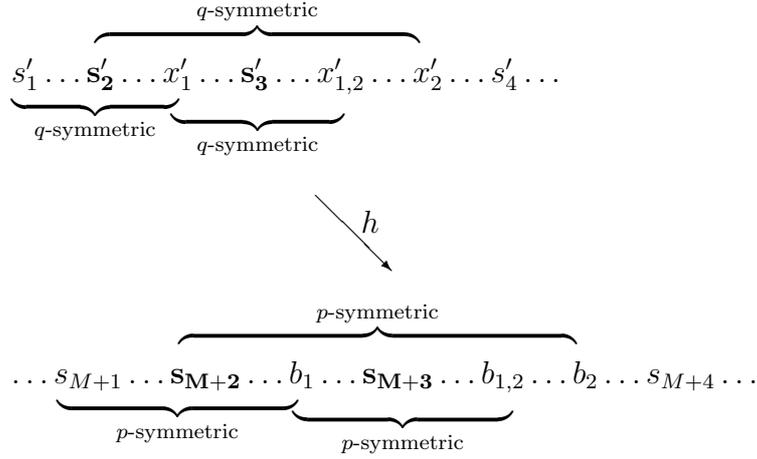
$$\overbrace{s'_1 \ldots \underbrace{\mathbf{s'_2} \ldots x'_1}_{q\text{-symmetric}} \ldots \underbrace{\mathbf{s'_3} \ldots x'_{1,2}}_{q\text{-symmetric}} \ldots x'_2}^{q\text{-symmetric}} \ldots s'_4 \ldots$$

$$\searrow h$$

$$\ldots s_{M+1} \ldots \overbrace{\underbrace{\mathbf{s_{M+2}} \ldots b_1}_{p\text{-symmetric}} \ldots \underbrace{\mathbf{s_{M+3}} \ldots b_{1,2}}_{p\text{-symmetric}} \ldots b_2}^{p\text{-symmetric}} \ldots s_{M+4} \ldots$$

FIGURE 5. The configuration of symmetric arcs.

is p-link-symmetric with center s_{M+3}. There exists a unique p-point b_2 such that the arc $[s_{M+2}, b_2]$ is p-symmetric with center s_{M+3}, so $h(x'_2) \in K_{b_2}$. Also $L_q(x'_2) = 2$ and $L_p(b_2) = M + 2$. Since $[s'_2, x'_2]$ is q-symmetric, there exists a q-point $x'_{1,2} \in [s'_3, x'_2]$ such that the arc $[x'_1, x'_{1,2}]$ is q-symmetric with center s'_3. Then $h([x'_1, x'_{1,2}])$ is p-link-symmetric with center s_{M+3}. Since there exists a unique p-point $b_{1,2}$ such that the arc $[b_1, b_{1,2}]$ is p-symmetric with center s_{M+3}, we have $h(x'_{1,2}) \in K_{b_{1,2}}$, see Figure 5. Also $L_q(x'_{1,2}) = 1$ and $L_p(b_{1,2}) = M + 1$.

The proof of (1) follows by induction. Suppose at step k we have $h([s'_k, s'_{k+1}]) = [a_k, a_{k+1}]$, $s_{M+k} \in K_{a_k}$ and $s_{M+k+1} \in K_{a_{k+1}}$, see Figure 6. Let again q-point $x'_k \in [s'_{k+1}, s'_{k+2}]$ be such that the arc $[s'_k, x'_k]$ is q-symmetric with center s'_{k+1}. Then $h([s'_k, x'_k])$ is p-link-symmetric with center s_{M+k+1}. The unique p-point b_k such that $[s_{M+k}, b_k]$ is p-symmetric with center s_{M+k+1} satisfies $h(x'_k) \in K_{b_k}$. Also $L_q(x'_k) = k$ and $L_p(b_k) = M + k$.

Let us suppose by induction that for every q-point $x' \in E_q$, $L_q(x') > 0$, $x' \prec x'_k$, we have $u = h(x') \in \ell_p^{s_{r+M}}$, where $r = L_q(x')$, and the arc component $K_u \subset \ell_p^{s_{r+M}}$ contains a p-point x such that $L_p(x) = r + M$. Since $L_q(x'_k) = k$, $L_q(s'_{k+1}) = k + 1$ and $L_q(s'_{k+2}) = k + 2$, for every q-point $x' \in (s'_{k+1}, s'_{k+2})$, $x' \neq x'_k$, we have $L_q(x') < L_q(x'_k)$. Hence for every q-point $y' \in (x'_k, s'_{k+2})$ there exists a q-point $x' \in (s'_{k+1}, x'_k)$ such that the arc $[x', y']$ is q-symmetric with center x'_k. So the arc $h([x', y'])$ is p-link-symmetric
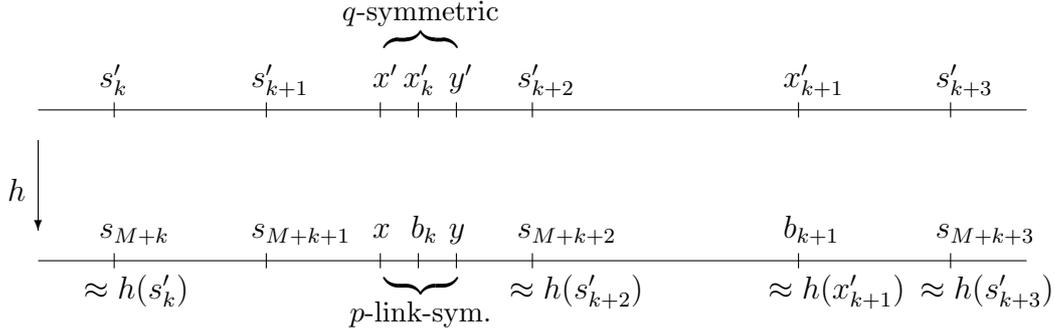
FIGURE 6. The relative point in the induction step. Here $\approx$ stands for "belongs to the same arc component in the same link".

with center b_k. The induction hypothesis implies that for $u = h(x')$, the arc component $K_u \in \ell_p^{s_r+M}$ contains a p-point x such that $L_p(x) = r + M$, where $r = L_q(x')$.

Since $L_p(b_k) = M + k$, $L_p(s_{M+k+1}) = M + k + 1$ and $L_p(s_{M+k+2}) = M + k + 2$, we have $L_p(v) < L_p(b_k)$ for every p-point $v \in (s_{M+k+1}, s_{M+k+2})$, $v \neq b_k$. Hence for every p-point $v \in (b_k, s_{M+k+2})$ there exists a p-point $w \in (s_{M+k+1}, b_k)$ such that the arc $[w, v]$ is p-symmetric with center b_k. Therefore, and since $h([x', y'])$ is p-link-symmetric with center b_k, there exists a unique p-point y such that the arc $[x, y]$ is p-symmetric with center b_k. Also, $h(y') \in K_y$ and $L_p(y) = L_p(x)$, so $L_p(y) = L_q(y') + M$. This proves that for every q-point $x' \in E_q$, $L_q(x') > 0$, $x' \prec s'_{k+2}$, we have $u = h(x') \in \ell_p^{s_r+M}$, where $r = L_q(x')$, and the arc component $K_u \subset \ell_p^{s_r+M}$ contains a p-point x such that $L_p(x) = r + M$.

Next $h([s'_{k+1}, s'_{k+2}]) = [a_{k+1}, a_{k+2}]$, $s_{M+k+1} \in K_{a_{k+1}}$ and $s_{M+k+2} \in K_{a_{k+2}}$. Let the q-point $x'_{k+1} \in [s'_{k+2}, s'_{k+3}]$ be such that the arc $[s'_{k+1}, x'_{k+1}]$ is q-symmetric with center s'_{k+2}. Then $h([s'_{k+1}, x'_{k+1}])$ is p-link-symmetric with center s_{M+k+2}. Since there exists a unique p-point b_{k+1} such that the arc $[s_{M+k+1}, b_{k+1}]$ is p-symmetric with center s_{M+k+2}, it follows that $h(x'_{k+1}) \in K_{b_{k+1}}$. Also, $L_q(x'_{k+1}) = k + 1$ and $L_p(b_{k+1}) = M + k + 1$. Since $[s'_{k+1}, x'_{k+1}]$ is q-symmetric with center s'_{k+2} and $[s_{M+k+1}, b_{k+1}]$ is p-symmetric with center s_{M+k+2}, the same argument as above shows that for every q-point $x' \in E_q$, $L_q(x') > 0$, $x' \prec x'_{k+1}$, we have $u = h(x') \in \ell_p^{s_r+M}$, where $r = L_q(x')$, and the arc component $K_u \subset \ell_p^{s_r+M}$ contains a p-point x such that $L_p(x) = r + M$. This proves the induction step.

(2) Let x be a p-point such that $L_p(x) > 0$ and $v = h^{-1}(x)$ lies beyond the κ-th salient g-point. Since h^{-1} is also a homeomorphism and $h^{-1}(\mathcal{C}_p) \prec \mathcal{C}_g$, (1) implies

that there exists M' such that $v \in \ell_g^{s''_{r+M'}}$, where $r = L_p(x)$. Also the arc component $K_v \subset \ell_g^{s''_{r+M'}}$ contains a g-point x'' such that $L_g(x'') = r + M'$.

Let x' be a q-point such that $L_q(x') > 0$, x' lies beyond the κ-th salient g-point and $u = h(x')$ lies beyond the κ-th salient p-point. Then $u \in \ell_p^{s_{r'+M}}$, where $r' = L_q(x')$, and the arc component $K_u \subset \ell_p^{s_{r'+M}}$ contains a p-point x such that $L_p(x) = r' + M$. Also $v = h^{-1}(x) \in \ell_g^{s''_{r'+M+M'}}$ and the arc component $K_v \subset \ell_g^{s''_{r'+M+M'}}$ contains a g-point x'' such that $L_g(x'') = L_q(x') + M + M'$. Since $h^{-1} \circ h = id$, we have $x'' = x'$. Also $L_g(x'') = L_q(x') + q - g$ implies that $M + M' = q - g$. Since the number of q-points in $[s'_i, s'_{i+1}]$ with q-level l, $l \in \mathbb{N}_0$, is the same as the number of g-points in $[s''_{q-g+i}, s''_{q-g+i+1}]$ with g-level $q - g + l$, it follows that this number is the same as the number of p-points in $[s_{M+i}, s_{M+i+1}]$ with p-level $M + l$.   □

Definition 4.3. A point $x \in \varprojlim([0,1], T_s)$ is called a *folding point* if it has no closed neighborhood that is homeomorphic to a Cantor set of arcs.

Proof of Theorem 1.1. Folding points $x = (\dots, x_{-2}, x_{-1}, x_0)$ are characterized by the fact that each entry x_{-k} belongs to the omega-limit set $\omega(c)$ of the turning point $c = \frac{1}{2}$, see [29]. If the critical point c of T_s has (eventual) period n, then $\varprojlim([c_2, c_1], T_s)$ contains exactly n folding points. If, on the other hand, $\omega(c)$ is infinite, then there are infinitely many folding points.

Since folding points map to folding points under a homeomorphism, $\varprojlim([0,1], T_s)$ and $\varprojlim([0,1], T_{s'})$ must have the same cardinality of folding points. If this cardinality is finite, then the Ingram Conjecture was proved in [32]. So from now on, we can assume that the critical points of T_s and $T_{s'}$ have infinite orbits. Therefore the above proposition shows that

$$FP_q([s'_k, s'_{k+1}]) = FP_{p+M}([s_{M+k}, s_{M+k+1}]) = FP_p([s_k, s_{k+1}]),$$

for every positive integer k, and therefore $FP(\mathfrak{C}') = FP(\mathfrak{C})$.

Since orbits of c and c' are infinite, we have $c_i, c'_i \neq c$ for all $i \in \mathbb{N}$. Note that $c = c' = 1/2$, $c_1, c'_1 > c$, $c_2, c'_2 < c$ and $FP_p([\alpha, s_2]) = * \; 0 \; 1 \; 0 \; 2 = FP_q([\alpha', s'_2])$. It is well-known (see *e.g.* [12]) that $s = s'$ if and only if c_i and c'_i are on the same side of c for all $i \in \mathbb{N}$ (*i.e.,* if the kneading sequences of T_s and $T_{s'}$ are the same). Therefore, we only need to prove that $FP(\mathfrak{C}) = FP(\mathfrak{C}')$ if and only if c_i and c'_i are on the same side of c for all $i \in \mathbb{N}$. We prove this by induction.

Let us suppose that c_i and c'_i are on the same side of c for all $i \leq k$, and that $FP_p([\alpha, s_k]) = FP_q([\alpha', s'_k])$. Let us denote by x the first p-point on the left of s_k (i.e., $x \in [\alpha, s_k]$ and between x and s_k there are no other p-points), and let x' be the first q-point on the left of s'_k. Let $l = L_p(x) = L_q(x')$.

If $l = 0$, then $L_p(\sigma(x)) = 1 = L_q(\sigma(x'))$. If $c_{k+1}, c'_{k+1} < c$, then $FP_p([\sigma(x), s_{k+1}]) = FP_p(\sigma([x, s_k])) = 1\, 0\, k+1 = FP_q([\sigma(x'), s'_{k+1}])$. If $c_{k+1}, c'_{k+1} > c$, then $FP_p([\sigma(x), s_{k+1}]) = 1\, k+1 = FP_q([\sigma(x'), s'_{k+1}])$. Similarly $c_{k+1} < c$ and $c'_{k+1} > c$, or vice versa, implies $FP_p([\sigma(x), s_{k+1}]) \neq FP_q([\sigma(x'), s'_{k+1}])$.

If $l \neq 0$, then since $x \in (s_{k-1}, s_k)$, we have $l < k - 1$, i.e., $l + 1 < k$, and by the induction hypothesis c_{l+1} and c'_{l+1} are on the same side of c. Therefore, if c_{k+1} and c'_{k+1} are on the same side of c as c_{l+1}, then $FP_p([\sigma(x), s_{k+1}]) = l+1\, k+1 = FP_q([\sigma(x'), s'_{k+1}])$. If c_{k+1} and c'_{k+1} are on the same side of c, but on the opposite side of c_{l+1}, then $FP_p([\sigma(x), s_{k+1}]) = l+1\, 0\, k+1 = FP_q([\sigma(x'), s'_{k+1}])$. Similarly, $c_{k+1} < c$ and $c'_{k+1} > c$, or vice versa, implies $FP_p([\sigma(x), s_{k+1}]) \neq FP_q([\sigma(x'), s'_{k+1}])$.

In every case we conclude that c_i and c'_i are on the same side of c for all $i \leq k + 1$ if and only if $FP_p([\alpha, s_{k+1}]) = FP_q([\alpha', s'_{k+1}])$, and by induction that c_i and c'_i are on the same side of c for all $i \in \mathbb{N}$ if and only if $FP(\mathfrak{C}) = FP(\mathfrak{C}')$.

This proves the Ingram Conjecture. $\square$

5. PSEUDO-ISOTOPY

Throughout this section, $h : \varprojlim([0, 1], T_s) \rightarrow \varprojlim([0, 1], T_s)$ will be an arbitrary self-homeomorphism. We will extend Proposition 4.2 in order to prove the result on pseudo-isotopy. Note that (1) and (2) of Proposition 4.2 together show that h induces an order preserving injection $h_{q,p}$ from E_q to E_p such that $h_{q,p}(E_{q,i}) = E_{p,M+i} = E_{p+M,i}$ for every $i \in \mathbb{N}_0$, where $E_{r,l}$ denotes the set of all r-points with r-level l (see Definition 2.5). In fact $h_{q,p}$ is an order preserving bijection from E_q to E_{p+M} and is defined as follows:

Definition 5.1. Let $x \in E_q$. If $x = s'_i$ for some $i \in \mathbb{N}$, we define $h_{q,p}(s'_i) = s_{M+i} \in E_p$. For all other $x \in E_q$, there exists $i \in \mathbb{N}$ such that $x \in (s'_i, s'_{i+1})$. By Proposition 4.2, the number of q-points of (s'_i, s'_{i+1}) is the same as the number of $(p + M)$-points of (s_{M+i}, s'_{M+i+1}). Let $(s'_i, s'_{i+1}) \cap E_q = \{x^0, \ldots, x^n\}$ and $(s_{M+i}, s'_{M+i+1}) \cap E_{p+M} = \{y^0, \ldots, y^n\}$. We define $h_{q,p}(x^i) = y^i$, $i = 0, \ldots, n$.

The next lemma shows that $h_{q,p}$ is essentially independent of q and p.

Lemma 5.2. If $q_1, p_1 \in \mathbb{N}$ are such that $h(\mathcal{C}_{q_1}) \prec \mathcal{C}_{p_1} \prec h(\mathcal{C}_q) \prec \mathcal{C}_p$, then $h_{q_1,p_1}|_{E_{q_1}} = h_{q,p}|_{E_{q_1}}$.

Proof. By Proposition 4.2, $h(\mathcal{C}_q) \prec \mathcal{C}_p$ implies that there exists $M \in \mathbb{Z}$ such that $h_{q,p}(E_{q,i}) = E_{p,M+i}$ for every $i \in \mathbb{N}_0$. Also, $h(\mathcal{C}_{q_1}) \prec \mathcal{C}_{p_1}$ implies that there exists $M_1 \in \mathbb{Z}$ such that $h_{q_1,p_1}(E_{q_1,i}) = E_{p_1,M_1+i}$ for every $i \in \mathbb{N}_0$. Let $r, l \in \mathbb{N}$ be such that $q_1 = q + r$ and $p_1 = p + l$. Since $E_{q+r,i} = E_{q,r+i}$, we have

$$h_{q,p}(E_{q+r,i}) = h_{q,p}(E_{q,r+i}) = E_{p,M+r+i},$$

and also

$$h_{q+r,p+l}(E_{q+r,i}) = E_{p+l,M_1+i} = E_{p,M_1+l+i}.$$

We want to prove that $M + r = M_1 + l$. To see this it suffices to pick a convenient point x in $E_{q+r,j}$ for some $j \in \mathbb{N}$, and to prove that $h_{q,p}(x) = y = h_{q+r,p+l}(x)$. Then the fact that $y \in E_{p,M+r+j}$ and $y \in E_{p,M_1+l+j}$ implies that $M + r + j = L_p(y) = M_1 + l + j$. For us, the convenient choice of $x \in E_{q+r} \subset E_q$ is a salient $(q + r)$-point.

Let us denote the salient $(q + r)$-points by $\hat{s}'_i$ and the salient $(p + l)$-points by $\hat{s}_i$, while as before s'_i denotes the salient q-points and s_i denotes the salient p-points. Note that the salient $(q + r)$-point $\hat{s}'_i$ is the same as the salient q-point s'_{i+r}, and the salient $(p+l)$-point $\hat{s}_i$ is the same as the salient p-point s_{i+l}. Let us denote the maximal $(q+r)$-link-symmetric arc with the center $\hat{s}'_i$ by $\hat{A}'_i$, and the maximal $(p+l)$-link-symmetric arc with the center $\hat{s}_i$ by $\hat{A}_i$, while as before A'_i denotes the maximal q-link-symmetric arc with the center s'_i, and A_i denotes the maximal p-link-symmetric arc with the center s_i. Note that $h(\hat{A}'_i) \subseteq \hat{A}_{M_1+i}$, $h(A'_{i+r}) \subseteq A_{M+i+r}$ and $\hat{s}'_i = s'_{i+r}$. Also, the center of $\hat{A}_{M_1+i}$ is $\hat{s}_{M_1+i} = s_{M_1+i+l}$ and the center of A_{M+i+r} is s_{M+i+r}. Therefore, $s_{M+i+r} = s_{M_1+i+l}$ and $M + r = M_1 + l$. $\square$

Corollary 5.3. $R = M + p - q$ does not depend on M, p, q.

Proof. By Lemma 5.2, $M_1 + l = M + r$. Therefore $R_1 = M_1 + p_1 - q_1 = M_1 + (p + l) - (q + r) = M + r + p - q - r = R$. $\square$

Definition 5.4. We call an arc $B \subset \mathfrak{C}$ a *p-bridge* if the boundary points of B are p-points with p-level 0, and if $L_p(x) \neq 0$ for every p-point $x \in \text{Int } B$.

Corollary 5.5. Let $B' \subset \mathfrak{C}$ be a $(q + 1)$-bridge and $\partial B' = \{a', b'\}$. There exists a $(p + M + 1)$-bridge B such that for $\partial B = \{a, b\}$ we have $h(B') \subset K_a \cup B \cup K_b$ and

$h(a') \in K_a$, $h(b') \in K_b$, where K_a and K_b are the arc-components of the link $\ell_p^{s_M+1}$ of $\mathcal{C}_p$ containing a and b respectively.

Proof. Proposition 4.2 dealt with points in $E_{q,j}$ for $j \geq 1$, but bridges involve points of level zero. Since $E_{q,1} = E_{q+1,0}$, in this corollary we can work with $(q+1)$-bridges.

For each $j \geq 1$, $E_{q,j}$ is contained in a single link $\ell_q^{s'_j} \in \mathcal{C}_q$ and by Proposition 4.2, for $\ell_p^{s_M+j} \supseteq h(\ell_q^{s'_j})$, every point of $h(E_{q,j})$ is contained in an arc component of $\ell_p^{s_M+j}$ which contains a p-point of $E_{p,M+j} = E_{p+M,j}$. Since $E_{q+1,0} = E_{q,1}$ and $E_{p+M+1,0} = E_{p+M,1}$, every point of $h(E_{q+1,0}) = h(E_{q,1})$ is contained in an arc component of $\ell_p^{s_M+1}$ which contains a point of $E_{p+M,1} = E_{p+M+1,0}$.

Every two adjacent points of $E_{q+1,0}$ are the boundary points of a $(q+1)$-bridge, and every two adjacent points of $E_{p+M+1,0}$ are the boundary points of a $(p+M+1)$-bridge. We also have $h_{q,p+M}(E_{q+1,0}) = h_{q,p+M}(E_{q,1}) = E_{p,M+1} = E_{p+M+1,0}$. Therefore, for every $(q+1)$-bridge B' there exists a $(p+M+1)$-bridge B such that $h_{q,p+M}(B') = B$. More precisely, for every $(q+1)$-bridge B' and $\partial B' = \{a', b'\}$, there exists a $(p+M+1)$-bridge B such that for $\partial B = \{a, b\}$ we have $h(B') \subset K_a \cup B \cup K_b$ with $h(a') \in K_a$ and $h(b') \in K_b$. Note that if B' is a $(q+1)$-bridge with center z' and $\partial B' = \{a', b'\}$ and B' is contained in a single link $\ell_{q+1}^{s'_1}$, then $h(B')$ is contained in the arc component $K_a = K_b$ which contains also a $(p+M+1)$-point z such that $L_{p+M+1}(z) = L_{q+1}(z')$. So the arc component K_a contains a $(p+M+1)$-bridge B with center z and we have again $h(B') \subset K_a \cup B \cup K_b$. □

Example 5.6. A $\sin\frac{1}{x}$-*continuum* is a homeomorphic copy of

$$\left(\{0\} \times [-1,1]\right) \cup \left\{(x, \sin\frac{1}{x}) : x \in (0,1]\right\}$$

and the arc $\{0\} \times [-1,1]$ is called the *bar* of the $\sin\frac{1}{x}$-continuum. Assume that $s > \sqrt{2}$ is such that the inverse limit $\varprojlim([0,1], T_s)$ contains a $\sin\frac{1}{x}$-continuum H. (Such s exist in abundance, *cf.* [2] and [14].) Then $\{\sigma^{-n}(H)\}_{n=0}^{\infty}$ is a sequence of pairwise disjoint $\sin\frac{1}{x}$-continua with $\mathrm{diam}(\sigma^{-n}(H)) \to 0$ as $n \to \infty$. There is then a sequence of disjoint neighborhoods U_n of $\sigma^{-n}(H)$ with $diam(U_n) \to 0$. For each n, $U_n \cap \mathfrak{C}$ contains arbitrarily long arcs. Pick a sequence of arcs $A_n \subset U_n \cap \mathfrak{C}$ of arc-length $\geq n+1$, and construct a bijection $h : \varprojlim([0,1], T_s) \circlearrowleft$ such that h is the identity on $\varprojlim([0,1], T_s) \setminus \cup_n A_n$ and on each A_n, h fixes ∂A_n, but moves some points in A_n homeomorphically such that there is $x_n \in A_n$ with $\bar{d}(x_n, h(x_n)) = n$. Since $\mathrm{diam}(U_n) \to 0$, we find that h

is continuous and bijective. Finally the compactness of $\varprojlim([0,1], T_s)$ implies that h is a homeomorphism. Even though h is isotopic to the identity, $\sup_{x \in \mathfrak{C}} \bar{d}(x, h(x)) = \infty$.

Therefore we cannot assume that a general self-homeomorphism of $\varprojlim([0,1], T_s)$ has an $R \in \mathbb{Z}$ such that $\sup_x \bar{d}(h(x), \sigma^R(x)) < \infty$. Block et al. [8, Theorem 4.2] used this property to conclude that h and σ^R are pseudo-isotopic, *i.e.*, they permute the composants of $\varprojlim([c_2, c_1], T_s)$ in the same way. However, since $\sigma^{-R} \circ h$ preserves $(q+1)$-bridges for some $R \in \mathbb{Z}$ and q sufficiently large, we can still follow the argument from [8].

Proof of Theorem 1.3. Let $P = s/(1+s) > 1/2$ be the orientation reversing fixed point of T_s and Q the center between c_2 and c_1. Let $\varepsilon = \text{mesh}(\mathcal{C}_p)$ in Definition 2.4. Without loss of generality, we can take $\varepsilon/2 < \min\{|c - P|, |c - Q|\}$. Let $x \in \varprojlim([0,1], T_s) \setminus \mathfrak{C} = \varprojlim([c_2, c_1], T_s)$ be arbitrary. Recall that the composant of x in $\varprojlim([c_2, c_1], T_s)$ is the union of all proper subcontinua of $\varprojlim([c_2, c_1], T_s)$ containing x. Without loss of generality we can fix $q \in \mathbb{N}$ such that $\pi_{q+1}(x) \geq P$. Fix $p \in \mathbb{N}$ and $M \in \mathbb{Z}$ as in Proposition 4.2 such that $h(\mathcal{C}_q) \preceq \mathcal{C}_p$ and h sends $(q+1)$-bridges to $(p+M+1)$-bridges in terms of Corollary 5.5. Let $R = M + p - q$, so $p + M + 1 = q + R + 1$. Since by Corollary 5.3, R does not depend on q and p, we can take q and p larger than $|R|$ without loss of generality.

Recall that the links ℓ_p^k of $\mathcal{C}_p$ are of the form $\ell_p^k = \pi_p^{-1}(I_p^k)$ of width $\leq \varepsilon s^{-p}/2$. The map σ^{-R} maps the chain $\mathcal{C}_p$ to a chain $\tilde{\mathcal{C}}_{p-R}$ whose links are of the form $\pi_{p-R}^{-1}(I_p^k)$ and hence also with width $\leq \varepsilon s^{-p}/2$; this chain is coarser than $\mathcal{C}_{p-R}$ if $R < 0$. Furthermore, the σ^{-R}-image of a $(q+R+1)$-bridge is a $(q+1)$-bridge.

Take $\tilde{h} = \sigma^{-R} \circ h$. Since $h(\mathcal{C}_q) \preceq \mathcal{C}_p$, we have $\tilde{h}(\mathcal{C}_q) \preceq \tilde{\mathcal{C}}_{p-R}$ and $\tilde{h}$ sends $(q+1)$-bridges to $(q+1)$-bridges, but the 'error' allowed in Corollary 5.5, *i.e.*, the arc-components of links from $\mathcal{C}_p$, must now be replaced by arc-components of links of $\tilde{\mathcal{C}}_{p-R}$. Recall that $\text{width}(\mathcal{C}_p) = \max_j |I_p^j|$, and $|\pi_{p-i}(\ell_p^j)| = |\pi_p(\ell_p^j)| s^i = |I_p^j| s^i$, for every $0 \leq i \leq p$. Therefore, $\pi_{p-R}(\tilde{\ell}_{p-R}^j) = \pi_{q-M}(\tilde{\ell}_{q-M}^j) \leq \varepsilon s^{-p}/2$, and $\pi_{q+1}(\tilde{\ell}_{p-R}^j) = \pi_{q+1}(\tilde{\ell}_{q-M}^j) = \pi_{q-M}(\tilde{\ell}_{q-M}^j) s^{-M-1} \leq \varepsilon s^{-p-M-1}/2$. Thus, the $(q+1)$-th projection of links of $\tilde{\mathcal{C}}_{p-R}$ are intervals of length $\leq \varepsilon s^{-(p+M+1)}/2 = \varepsilon s^{-(q+R+1)}/2$, see Figure 7.

The $(q+1)$-bridges that are small enough to belong to one or two links of $\mathcal{C}_q$ will map to arcs contained in the link $\tilde{\ell}_{p-R}$. Since $\pi_{q+1}(x) \geq P$ and $\varepsilon s^{-(q+1)}/2 < |c - P|$, no such short bridge can be close to x. On the longer $(q+1)$-bridges of $\mathcal{C}_q$ that map
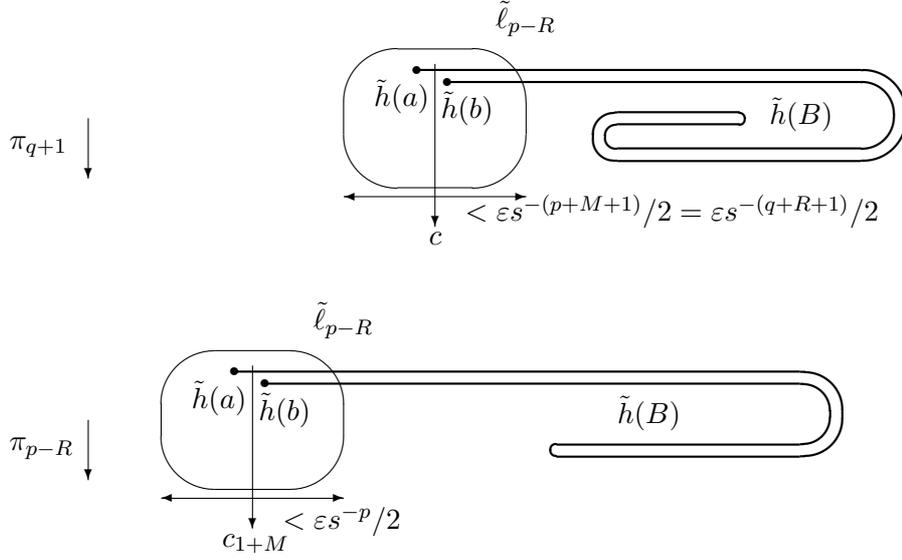
FIGURE 7. The $(p-R)$-th and $(q+1)$-th projection of 'the bridge' $\tilde{h}(B)$ with relevant link $\tilde{\ell}_{p-R}$. The picture is suggestive of $M+1 \leq 0$; if instead $M+1 > 0$, then $\tilde{h}(B)$ contains fewer $(q+1)$-points than $(p-R)$-points.

outside of $\tilde{\ell}_{p-R}$, $\tilde{h}$ acts as a trivial one-to-one correspondence, sending the first such bridge to the first, the second to the second, etc.

Find a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathfrak{C}$ such that $x_n \to x$. Then for large n, x_n belongs to a long $(q+1)$-bridge, and by the above argument, $\tilde{h}(x_n)$ and x_n belong to the same $(q+1)$-bridge up to an 'error' of at most $\varepsilon s^{-(q+R+1)}/2$. Take $H_n = [\tilde{h}(x_n), x_n]$ and a subsequence such that $H_{n_j} \to H$ in Hausdorff topology. Clearly H is a continuum and $x, \tilde{h}(x) \in H$. Since $\pi_{q+1}(x) \geq P$, the arcs H_{n_j} belong to arcs whose $(q+1)$-projections belong to $[c - \varepsilon s^{-(q+R+1)}/2, c_1]$ for all sufficiently large j. Since $q + R + 1 \geq 1$ and $\varepsilon/2 < c - Q$, we have $Q < c - \varepsilon/2 < c - \varepsilon s^{-(q+R+1)}/2$ implying $[c - \varepsilon s^{-(q+R+1)}/2, c_1] \subset [Q, c_1]$.

Therefore $\pi_{q+1}(H_{n_j}), \pi_{q+1}(H) \subset [Q, c_1]$, and since $[Q, c_1]$ is a proper subset of $[c_2, c_1]$ and the inclusion holds for arbitrarily large q, H is a proper subcontinuum of $\varprojlim([c_2, c_1], T_s)$. It follows that $\tilde{h}(x)$ and x belong to the same composant of $\varprojlim([c_2, c_1], T_s)$. Apply σ^R to find that $h(x)$ and $\sigma^R(x)$ belong to the same composant as well. $\qquad \square$

Pseudo-isotopy of h implies that the number of composants being mapped to themselves is the same for h^n and σ^{nR}. This number grows like s^{nR}, which in [9] provides a proof of the Ingram conjecture for tent maps with periodic critical point. In this situation, [9] in fact also shows that h is isotopic to a power of the shift. Due to the existence of composants that are not arc-connected, this is not so clear in the general case.

Remark 5.7. Not every pseudo-isotopy is an isotopy. For instance, a homeomorphism flipping the bar of a $\sin \frac{1}{x}$-continuum cannot be isotopic to the identity. If the bonding map is a quadratic map within the first period doubling cascade, then the inverse limit space is a finite collection of $\sin \frac{1}{x}$-continua, see [7], and we can indeed construct homeomorphism that are pseudo-isotopic but not isotopic to the identity. Among those tent maps T_s, $s \in [\sqrt{2}, 2]$, whose inverse limit space is known to contain $\sin \frac{1}{x}$-continua, both in [2] and [14], the topology is much more complicated, as more than a single ray can be expected to accumulate on their bars. Thus the following question is very relevant:

> Is every self-homeomorphism of $\varprojlim([0,1], T_s)$ isotopic to a power of the shift?

We know this to be true if c is periodic or non-recurrent [9, 10], but this case is simpler, because the only proper subcontinua of $\varprojlim([c_2, c_1], T_s)$ are arcs or points.

6. The Logistic Family

In this section we prove of Theorem 1.2. Logistic maps are by far the best studied unimodal maps, and we give here a list of its properties that simplify our task, although they are not essential: Theorem 1.2 works for arbitrary C^2 families just as well.

The map $f_a(x) = ax(1-x)$ has negative Schwarzian derivative, *i.e.*, $Sf_a := \frac{f_a'''}{f_a'} - \frac{3}{2}(\frac{f_a''}{f_a'})^2 < 0$ wherever defined. Due to Singer [31], this implies that every p-periodic point x with multiplier $|(f_a^p)'(x)| \leq 1$ is attracting from at least one side, and has the critical point in its immediate basin of its orbit, *i.e.*, there is a point $y \in \mathrm{orb}(x)$ such that $\cap_k f_a^{pk}([y,c]) = \{y\}$. Another consequence of the Schwarzian derivative being negative [19] is that every interval K on which $f^n|_K$ is a homeomorphism for all $n \geq 0$ must be attracted by a periodic attractor, so there are no *wandering intervals*. The

absence of wandering intervals was proved for general C^2 families, see [25, Theorem 6.2. page 156].

Much more complicated results have been proved for the logistic family as well. The entropy $h_{top}(f_a)$ is a non-decreasing function of its parameter, see Douady & Hubbard [17] and Milnor & Thurston [26]. Furthermore, if f_a and f_b are topological conjugate and have no periodic attractor, then $a = b$. This is known as "denseness of hyperbolicity" and was proved in [21, 24].

The logistic family $(f_a)_{a\in[0,4]}$ is richer than the tent family $(T_s)_{s\in[0,2]}$ in the sense that it allows renormalization of all types.

Definition 6.1. A unimodal map $f : [0, 1] \to [0, 1]$ with critical point c is *renormalizable* if there is a neighborhood $J \ni c$ and period $p \geq 2$ such that

(6.1) $f^p(J) \subset J, f^p(\partial J) \subset \partial J$ and $J, f(J), \ldots, f^{p-1}(J)$ have disjoint interiors.

In this case, $f^p : J \to J$ is a new unimodal map; it is called a *renormalization* of f.

If f_a is renormalizable, then $f_a^p|_J$ is conjugate with another member, $f_{a'}$, of the logistic family. This is because the logistic family is *full* in the sense that it witnesses every possible combinatorial type of unimodal map without wandering intervals, [25, Section II.4]. The renormalization of a unimodal map can itself be renormalizable, etc. This gives rise to infinitely renormalizable maps, for which there is a nested sequence $(J^k)_{k\geq 1}$ of p_k-periodic neighborhoods of c, such that $C := \cap_k \cup_{j=0}^{p_k-1} f^j(J^k)$ is an invariant Cantor set, called a *solenoidal attractor*. It coincides with $\omega(c)$, is Lyapunov stable and the orbit of Lebesgue-a.e. point converges to C; yet arbitrarily close to C there are periodic orbits that are not contained in C.

Definition 6.2. We call a point x *non-wandering* if for every neighborhood $U \ni x$, there is $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. The collection $\Omega(f)$ of non-wandering points is called the *non-wandering set*. The *reduced non-wandering set* of a logistic map f_a, $\Omega(f_a)/\sim_{sn}$, is the non-wandering set of f_a with each saddle-node pair identified to a point. This set inherits an order from $[0, 1]$, and we say that two logistic maps f_a and $f_b : [0, 1] \to [0, 1]$ are *order-preserving conjugate on their reduced non-wandering sets* if there is an order preserving homeomorphism $h : \Omega(f_a)/\sim_{sn} \to \Omega(f_b)/\sim_{sn}$ such that $f_b \circ h([x]) = h \circ f_a([x])$ for all $[x] \in \Omega(f_a)/\sim_{sn}$.

Clearly, periodic points are non-wandering, as are points in the closure of the set of periodic points. For logistic maps, the only wandering points are those that belong to the basin of a periodic attractor (but are not periodic themselves), or that belong to the basin of a solenoidal attractor.

Let us describe in some detail the first periodic doubling cascade, occurring at parameters $1 = a_0 < a_1 < a_2 < \dots$ on the parameter interval $[0, a_\infty]$, see Figure 8 where $a_\infty = \lim_k a_k$ is the so-called Feigenbaum-Coullet-Tresser parameter [16, 18], at which f_a has the simplest kind of solenoidal attractor. The map f_{a_∞} is also called the 2^∞-map, because it has periodic points of period 2^k for all k, and none of higher period in the Sharkovskiĭ order. These dynamics are not present in the tent family. It is also well-known that $h_{top}(f_a) = 0$ if and only if $a \le a_\infty$, see [28]. The inverse limits spaces $\varprojlim([0, 1], f_a)$ for $a \in [0, a_\infty]$ were described in [7].
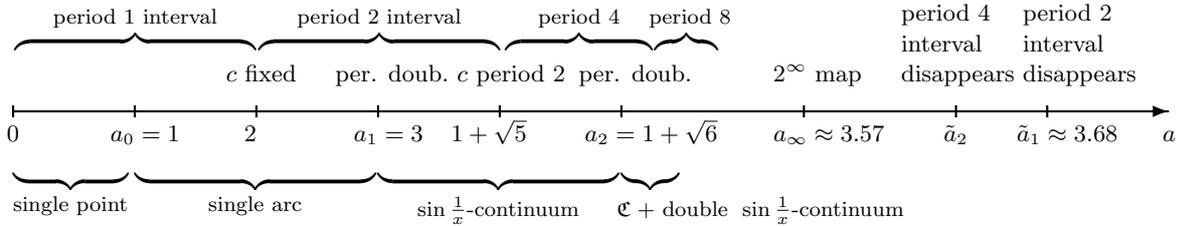

FIGURE 8. Sketch (not to scale) of the bifurcations in the first period doubling cascade. The upper braces indicate parameter intervals in which J^k is the *smallest* periodic interval as in (6.1). The lower braces indicate parameter intervals where the inverse limit space is as indicated.

- For $a \in [0, a_0]$, f_a has a single fixed point 0, attracting every $x \in [0, 1]$. The inverse limit $\varprojlim([0, 1], f_a)$ is a single point $(\dots, 0, 0, 0)$.
- At $a = a_0$, the fixed point 0 becomes unstable and splits off another stable fixed point $p_a = \frac{a-1}{a}$ with multiplier $f'_a(p_a) = 2 - a$, attracting every point $x \in (0, 1)$. For all $a \in (a_0, a_1 = 3]$, p_a remains stable and $\varprojlim([0, 1], f_a)$ is a single arc.
- At $a = a_1$, p_a undergoes a period doubling bifurcation. It becomes unstable itself, and splits off a stable period 2 orbit $q_a^\pm = \frac{a+1 \pm \sqrt{(a-1)^2 - 4}}{2a}$ with multiplier $(f_a^2)'(q_a^\pm) = 5 - (a-1)^2$. For all $a \in (a_1, a_2 = 1 + \sqrt{6}]$, $q_a^\pm$ remain stable and $\varprojlim([0, 1], f_a)$ is a $\sin \frac{1}{x}$-continuum, consisting of two arc components: the zero-composant and the bar (an arc containing $(\dots, p_a, p_a, p_a)$).

- At $a = a_2$, $q_a^\pm$ undergo a period doubling bifurcation. They become unstable themselves, splitting off a stable period 4 orbit. For all $a \in (a_2, a_3]$, this period 4 orbit remains stable and $\varprojlim([0,1], f_a)$ has four arc-components: two arcs containing $(\ldots, q_a^-, q_a^+, q_a^-, q_a^+)$ and $(\ldots, q_a^+, q_a^-, q_a^+, q_a^-)$ respectively, one ray containing $(\ldots, p_a, p_a, p_a)$ and forming a double $\sin \frac{1}{x}$-continuum with the two arcs, and the zero-composant compactifying on all of the above.

- In general, for $a \in (a_k, a_{k+1}]$, $k \geq 1$, the inverse limit space $\varprojlim([0,1], f_a)$ has 2^k arc-components arranged in a hierarchical manner. There are 2^{k-1} arcs, each containing a 2^{k-1}-periodic point of the shift homeomorphism. At the next level, there are 2^{k-2} rays, each containing a 2^{k-2}-periodic point and forming a double $\sin \frac{1}{x}$-continuum with two arcs of the first level. The next level consists of 2^{k-2} rays, each containing a 2^{k-2}-periodic point, and compactifying on two double $\sin \frac{1}{x}$-continua (including four arcs) of the previous levels. This structure continues, until, at the highest level, the zero-composant compactifies on all of the above.

- At $a = a_\infty$, this structure contains infinitely many levels, and compactifies on an additional Cantor set of points x such that $x_{-n} \in \omega(c)$ for all $n \geq 0$.

We see that the inverse limit $\varprojlim([0,1], f_a)$ is different at every stage of the period doubling cascade.

Each saddle node bifurcation occurring at $a_* > a_\infty$, is followed by its own period doubling cascade. At $a = a_*$, a fresh neutrally attracting p-periodic orbit emerges, which immediately splits into an unstable and a stable p-periodic orbit, but this does not affect the inverse limit space. For this reason we introduced the reduced non-wandering set which does not discriminate between the twin p-periodic orbits emerging from a saddle node bifurcation. The non-wandering sets $\Omega(f_{a_*})$ and $\Omega(f_{a_*+\varepsilon})$ are different, but their reduced non-wandering sets are the same, and their inverse limit spaces homeomorphic.

For $a > a_\infty$, f_a has positive topological entropy, and Milnor & Thurston [26] showed that there is an entropy-preserving semi-conjugacy

$$\psi_a : [0,1] \to [0,1] \text{ such that } \psi_a \circ f_a = T_s \circ \psi_a$$

for $\log s = h_{top}(f_a)$. The map ψ_a collapses to a point every interval K on which the local entropy $h_{top}(f_a|_K) = \lim_n \frac{1}{n} \log l(f_a^n|_K) < h_{top}(f_a)$. Here $l(f_a^n|_K)$ is the *lap-number* (*i.e.*, the number of maximal intervals of monotonicity) of $f_a^n|_K$. This applies to wandering

intervals (but logistic maps have none) and (pre)periodic intervals, with the exception of those created in the first period doubling cascade. These 2^k-periodic intervals (let us denote them by J^k), emerge after the critical point becomes 2^{k-1}-periodic. For $k = 1$ and 2 this happens at $a = 2$ and $1 + \sqrt{5}$ respectively, see Figure 8.

For $a \in (a_\infty, \tilde{a}_k]$, the semi-conjugacy ψ_a does not collapse J^k to a point, but maps it to a 2^k-periodic interval $L^k = [1 - v_k, v_k]$. One can compute that $v_k = \frac{1}{2} + \frac{1}{2} \prod_{j=0}^{k-1} \frac{s^{2^j} - 1}{s^{2^j} + 1}$, and v_k or $1 - v_k$ is the 2^{k-1}-periodic boundary point, according to whether k is odd or even. So L^k plays the role of J^k; it exists as long as $1 < s \leq 2^{1/2^k}$.

The interval J^k continues to exist until the parameter $\tilde{a}_k > a_\infty$, where it disappears in a homoclinic bifurcation. At this parameter, $f_{\tilde{a}_k}^{2^k}(c)$ and $f_{\tilde{a}_k}^{2^{k+1}}(c)$ are the two boundary points of J^k, and $f_{\tilde{a}_k}^{2^{k+1}}(c)$ is 2^k-periodic. The parameters $\tilde{a}_k$ form a decreasing sequence, and $\lim_k \tilde{a}_k = a_\infty$, due to the denseness of hyperbolicity.

Proof of Theorem 1.2. We start with the "if"-direction. Recall that a logistic map can have at most one periodic attractor. Suppose that f_a and f_b are conjugate on their reduced non-wandering sets. If f_a and hence f_b have no periodic attractor (but possibly a solenoidal attractor), then denseness of hyperbolicity implies that $a = b$. In fact, in this case, the reduced non-wandering set coincides with the non-wandering set, and (without invoking denseness of hyperbolicity) the conjugacy $h : \Omega(f_a) \to \Omega(f_b)$ extends to a conjugacy $h : [0, 1] \to [0, 1]$ such that $h \circ f_b = f_a \circ h$. Hence $\varprojlim([0, 1], f_a)$ and $\varprojlim([0, 1], f_b)$ are homeomorphic.

If f_a and hence f_b have a periodic attractor, then they have the same period p and we can choose attracting p-periodic points x_a and x_b such that spatial order $(f_a^i(x_a))_{i=0}^{p-1}$ and $(f_b^i(x_b))_{i=0}^{p-1}$ are the same. This means that f_a and f_b are in the same stage of the same period doubling cascade, and the above description implies that $\varprojlim([0, 1], f_a)$ and $\varprojlim([0, 1], f_b)$ are homeomorphic.

Now, for the "only if"-direction, assume that $\varprojlim([0, 1], f_a)$ and $\varprojlim([0, 1], f_b)$ are homeomorphic. We gave a complete description of the inverse limit spaces if $a < a_\infty$, so it suffices to consider the case $a_\infty < a < b$. Given a p-periodic interval J as in (6.1), define the subcontinuum

(6.2) $$D_J := \{x \in \varprojlim([c_2, c_1], f_a) : x_{np} \in J \text{ for all } n \leq 0\}.$$

Clearly, D_J is p-periodic under the shift homeomorphism, and if a' is the parameter such that $f_{a'}$ is conjugate to $f_a^p : J \to J$, then D_J is homeomorphic with

$\varprojlim([0,1], f_{a'})$. Moreover, all folding points of $\varprojlim([c_2, c_1], f_a)$ belong to $\cup_{i=0}^{p-1} \sigma^i(D_J)$, so points $x \in \varprojlim([c_2, c_1], f_a)$ outside $\cup_{i=0}^{p-1} \sigma^i(D_J)$ all have Cantor set of arcs neighborhoods in $\varprojlim([c_2, c_1], f_a)$.

If J is a maximal interval satisfying (6.1), then D_J and its images $\sigma^i(D_J)$, $0 < i < p$, are *maximal* subcontinua in the sense that there is no indecomposable proper subcontinuum of the core $\varprojlim([c_2, c_1], f_a)$ that properly contains D_J (and similarly for $\sigma^i(D_J)$, $0 < i < p$). This shows that if $\varprojlim([0,1], f_a)$ and $\varprojlim([0,1], f_b)$ are homeomorphic, then their maximal periodic intervals J_a and J_b must have the same period, and their respective subcontinua D_{J_a} and D_{J_b} must be homeomorphic.

Next we will show that J_a and J_b are also of the same type (*i.e.*, the spatial order of $(f_a^i(J_a))_{i=0}^{p-1}$ and $(f_b^i(J_b))_{i=0}^{p-1}$ is the same). For the factor map

$$\Psi_a \; : \varprojlim([0,1], f_a) \to \Psi_a(\varprojlim([0,1], f_a)),$$
$$(\dots x_{-2}, x_{-1}, x_0) \mapsto (\dots \psi_a(x_{-2}), \psi_a(x_{-1}), \psi_a(x_0)).$$

$\Psi_a(\varprojlim([0,1], f_a))$ is homeomorphic with $\varprojlim([0,1], T_s)$ for $\log s = h_{top}(f_a)$. Since ψ_a collapses a periodic interval J only if it does not emerge from the first period doubling cascade (*i.e.*, $J \neq J^k$), we will from now on assume that J is the maximal periodic interval other than (so possibly contained in some) J^k. We will show (Claims 1 and 2, below) that Ψ_a consists of a sequence of quotient maps that, other than squeezing a single orbit of subcontinua $(\sigma^i(D_J))_{i=0}^{p-1}$ to p endpoints, has no effect on the topology. If J_a and J_b are not of the same type, then $\Psi_a(\varprojlim([0,1], f_a))$ and $\Psi_b(\varprojlim([0,1], f_b))$ are homeomorphic with the inverse limit spaces of different tent maps, and, due to Theorem 1.1, not homeomorphic with each other. Since Ψ_a and Ψ_b only squeeze finitely many well-defined subcontinua to points, $\varprojlim([0,1], f_a)$ and $\varprojlim([0,1], f_b)$ could not have been homeomorphic.

Therefore, J_a and J_b are of the same type, and since the subcontinua D_{J_a} and D_{J_b} are homeomorphic, $f_a|_{J_a}$ and $f_b|_{J_b}$ must be in the same stage of their period doubling cascade, and hence their reduced non-wandering sets are conjugate. This finishes the proof, except for the announced claims:

Claim 1: Let Φ_a be the quotient map under the equivalence relation $x \sim y$ if $x = y$ or x, y both belong to $\sigma^i(D_{J_a})$ for the same $0 \le i < p$. Then $\Phi_a(\varprojlim([0,1], f_a))$ has p endpoints in the core (that are p-periodic under the shift) and no other folding points.

Proof. Since $J_a \neq J^k$ for any of the 2^k-periodic intervals emerging from the first period doubling cascade, ∂J_a contains a p-periodic orientation preserving point z_a. Let $\bar{z}_a = (\ldots f_a^{p-1}(z_a), z_a, f_a(z_a), \ldots, f_a^{p-1}(z_a))$; it is a point in D_{J_a} and one "half" of its arc-component serves as the zero-composant of D_{J_a}, while the other "half" coils densely in $\varprojlim([c_2, c_1], f_a)$ (or densely in D_{J^k} if $J_a \subset J^k$ for some maximal k), cf. [4]. The image $\Phi_a(D_{J_a}) = \Phi_a(\bar{z}_a)$ is now an end-point of the other "half" of the arc-component of $\bar{z}_a$. A similar statement holds for the points $\sigma^i(\bar{z}_a)$, $0 < i < p$. Since $\cup_{i=0}^{p-1} D_{J_a}$ contains all folding points of $\varprojlim([c_2, c_1], f_a)$, $\Phi_a(\varprojlim([c_2, c_1], f_a))$ contains no other folding points than the p-periodic endpoints $\Phi_a(\sigma^i(\bar{z}_a))$, $0 \leq i < p$. $\qquad\square$

Claim 2: $\Psi_a(\varprojlim([0, 1], f_a))$ is homeomorphic with $\Phi_a(\varprojlim([0, 1], f_a))$.

Proof. If K is a maximal interval such that $f_a^m(K) = J_a$ for some $m \geq 0$, then there is an open neighborhood U of K on which $f_a^m : U \to f_a^m(U)$ is a diffeomorphism. Take $n \geq 0$ and set

$$\bar{K}_n = \{x \in \varprojlim([0, 1], f_a) : x_{-n} \in K\} \setminus \cup_{i=0}^{p-1} \sigma^i(D_{J_a}).$$

Since all folding points of $\varprojlim([c_2, c_1], f_a)$ are contained in $\cup_{i=0}^{p-1} \sigma^i(D_{J_a})$, $\bar{K}_n$ is a zero-dimensional set of arcs (more precisely, this zero-dimensional set is a Cantor set together with a countable set coming from the zero-composant). Similarly

$$\bar{U}_n = \{x \in \varprojlim([0, 1], f_a) : x_{-n} \in U\} \setminus \cup_{i=0}^{p-1} \sigma^i(D_{J_a}).$$

is a zero-dimensional set of open arcs, and $\bar{U}_n$ compactly contains $\bar{K}_n$. Set $x \sim_{\bar{K}_n} y$ if $x = y$ or if x and y both belong to $\bar{K}_n$. It follows that $\varprojlim([0, 1], f_a)$ is homeomorphic with $\varprojlim([0, 1], f_a)/\sim_{\bar{K}_n}$ for each component K of $\cup_{m \geq 0} f_a^{-m}(J_a)$ and $n \geq 0$.

The collection $\mathcal{K} := \{\bar{K}_n : K \text{ is component of } \cup_{m \geq 0} f_a^{-m}(J_a), n \geq 0\}$ is a null-sequence, *i.e.,* for fixed $\delta > 0$, there are only finitely many elements in $\mathcal{K}$ with $\text{diam}(\bar{K}_n) > \delta$. Therefore, if we denote by $\sim_{\mathcal{K}}$ the intersection of all equivalence relations $\sim_{\bar{K}_n}$, $\bar{K}_n \in \mathcal{K}$, $\Phi_a(\varprojlim([0, 1], f_a))$ is homeomorphic with $\Phi_a(\varprojlim([0, 1], f_a))/\sim_{\mathcal{K}}$ which in turn is homeomorphic with $\Psi_a(\varprojlim([0, 1], f_a))$. $\qquad\square$

This proves the claims and hence Theorem 1.2 $\qquad\square$

References

[1] J. Anderson, I. Putnam, *Topological invariants for substitution tilings and their associated C^*-algebras,* Ergod. Th. & Dynam. Sys. **18** (1998), 509–537.

[2] M. Barge, K. Brucks, B. Diamond, *Self-similarity in inverse limit spaces of the tent family,* Proc. Amer. Math. Soc. **124** (1996), 3563–3570.

[3] M. Barge, B. Diamond, *Homeomorphisms of inverse limit spaces of one-dimensional maps,* Fund. Math. **146** (1995), 171-187.

[4] M. Barge, B. Diamond, *Inverse limit spaces of infinitely renormalizable maps,* Topology Appl. **83** (1998), 103–108.

[5] M. Barge, B. Diamond, *Subcontinua of the closure of the unstable manifold at homoclinic tangency,* Ergod. Th. & Dynam. Sys. **19** (1999), 289-307.

[6] M. Barge, S. Holte, *Nearly one-dimensional Hénon attractors and inverse limits,* Nonlinearity **8** (1995), 29-42.

[7] M. Barge, W. Ingram, *Inverse limits on $[0,1]$ using logistic bonding maps,* Topology Appl. **72** (1996), 159-172.

[8] L. Block, S. Jakimovik, J. Keesling, *On Ingram's conjecture,* Spring Topology and Dynamical Systems Conference. Topology Proc. **30** (2006), 95–114.

[9] L. Block, S. Jakimovik, J. Keesling, L. Kailhofer, *On the classification of inverse limits of tent maps,* Fund. Math. **187** (2005), no. 2, 171–192.

[10] L. Block, J. Keesling, B. Raines, S. Štimac, *Homeomorphisms of unimodal inverse limit spaces with a non-recurrent critical point,* Topology Appl. **156** (2009), 2417 - 2425.

[11] K. Brucks, H. Bruin, *Subcontinua of inverse limit spaces of unimodal maps,* Fund. Math. **160** (1999), 219–246.

[12] K. Brucks, B. Diamond, *Monotonicity of auto-expansions*, Physica D **51** (1991), 39–42.

[13] H. Bruin, *Planar embeddings of inverse limit spaces of unimodal maps,* Topology Appl. **96** (1999), 191–208.

[14] H. Bruin, *Subcontinua of Fibonacci-like unimodal inverse limit spaces,* Topology Proc. **31** (2007), 37-50.

[15] H. Bruin, *Inverse limit spaces of post-critically finite tent maps,* Fund. Math. **165** (2000), 125–138.

[16] P. Coullet, C. Tresser, *Itération d'endomorphismes et groupe de renormalisation,* C. R. Acad. Sci. Paris Sér. A-B **287** (1978), no. 7, A577-A580.

[17] A. Douady, *Topological entropy of unimodal maps: Monotonicity for quadratic polynomials.* In: Real and Complex Dynamical Systems, B. Branner and P. Hjorth, Eds. Dordrecht: Kluwer (1995) 65–87.

[18] M. J. Feigenbaum, *Quantitative universality for a class of nonlinear transformations,* J. Stat. Phys. **19** (1978), no. 1, 25-52.

[19] J. Guckenheimer, *Sensitive dependence to initial conditions for one-dimensional maps,* Comm. Math. Phys. **70** (1979), 133-160.

[20] F. Hofbauer, P. Raith, K. Simon, *Hausdorff dimension for some hyperbolic attractors with overlaps and without finite Markov partition,* Ergod. Th. & Dynam. Sys. **27** (2007), 1143–1165.

[21] J. Graczyk, G. Świątek, *Generic hyperbolicity in the logistic family.* Ann. Math. **146** (1997), 1–52.

[22] W. T. Ingram, *Inverse limits on* [0, 1] *using tent maps and certain other piecewise linear bounding maps*, Continua with the Houston problem book (Cincinnati, OH, 1994), in: H. Cook et al. (Eds.), Lecture Notes in Pure and Appl. Math. **170**, Dekker (1995), 253–258.

[23] L. Kailhofer, *A classification of inverse limit spaces of tent maps with periodic critical points*, Fund. Math. **177** (2003), 95–120.

[24] M. Lyubich, *Dynamics of quadratic polynomials, I and II.* Acta Math. **178** (1997), 185–247, 247–297.

[25] W. de Melo, S. van Strien, *One-dimensional dynamics,* Springer (1993).

[26] J. Milnor, W. Thurston, *On iterated maps of the interval: I, II*, Preprint 1977. Published in Lect. Notes in Math. **1342**, Springer, Berlin New York (1988) 465-563.

[27] M. Misiurewicz, *Embedding inverse limits of interval maps as attractors,* Fund. Math. **125** (1985), 23–40.

[28] M. Misiurewicz, W. Szlenk, *Entropy of piecewise monotone mappings,* Studia Math. **67** (1980), 45–63.

[29] B. Raines, *Inhomogeneities in non-hyperbolic one-dimensional invariant sets,* Fund. Math. **182** (2004), 241–268.

[30] B. Raines, S. Štimac, *A classification of inverse limit spacers of tent maps with non-recurrent critical point,* Algebraic and Geometric Topology **9** (2009), 1049–1088.

[31] D. Singer, *Stable orbits and bifurcation of maps of the interval,* SIAM J. Appl. Math. **35** (1978), no. 2, 260-267.

[32] S. Štimac, *A classification of inverse limit spaces of tent maps with finite critical orbit,* Topology Appl. **154** (2007), 2265–2281.

[33] R. Swanson, H. Volkmer, *Invariants of weak equivalence in primitive matrices,* Ergod. Th. & Dynam. Sys. **20** (2000), 611–626.

[34] W. Szczechla, *Inverse limits of certain interval mappings as attractor in two dimensions,* Fund.Math. **133** (1989), 1–23.

[35] R. F. Williams, *One-dimensional non-wandering sets*, Topology **6** (1967), 473–487.

[36] R. F. Williams, *Classification of one dimensional attractors*, 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) pp. 341–361 Amer. Math. Soc., Providence, R.I.

Marcy Barge, Department of Mathematical Sciences, Montana State University, Bozeman, MT 59717, USA

umsfmbar@math.montana.edu

http://www.math.montana.edu/ umsfmbar/

Henk Bruin, Faculty of Mathematics, University of Vienna Nordbergstrasse 15, 1090 Vienna, Austria

henk.bruin@univie.ac.at

http://homepage.univie.ac.at/henk.bruin/

Sonja Štimac, Department of Mathematics, University of Zagreb, Bijenička 30, 10 000 Zagreb, Croatia

sonja@math.hr

http://www.math.hr/ sonja