$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2} \Rightarrow \frac{p}{q} \text{ konv. verižnog razlomka od } \alpha$$

Ako je α irac. onda $\exists$ ∞ mnogo $\frac{p}{q}$ t.d. $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$

Napomena 6.1. Tvrdnja Korolara 6.2. ne vrijedi ako je $\alpha \in \mathbb{Q}$.

Neka je $\alpha = \frac{u}{v}$ i $\frac{p}{q} \neq 0$, onda

$$\left|\alpha - \frac{p}{q}\right| = \left|\frac{u}{v} - \frac{p}{q}\right| = \left|\frac{uq - vp}{vq}\right| \geq \frac{1}{vq}$$

$$\leq \frac{1}{q \cdot q}$$

Neka je α proiz. $\in \mathbb{R}$. Stavimo $a_0 = \lfloor \alpha \rfloor$.

Ako je $a_0 \neq \alpha$, onda

$$\alpha = a_0 + \frac{1}{\alpha_1} \quad \text{t.d. } \alpha_1 > 1$$

$$a_1 = \lfloor \alpha_1 \rfloor \;;\; \alpha_1 = a_1 + \frac{1}{\alpha_2} \;;\; \alpha_2 > 1$$

i stavimo $a_2 = \lfloor \alpha_2 \rfloor$ itd.

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k + \cfrac{1}{\alpha_k}}}}$$

$\alpha \rightsquigarrow a_0, a_1, a_2, \dots, a_n, \dots$ $\qquad \alpha_k$ $\quad \dots \alpha = [a_0, a_1, \dots, a_k, \alpha_k]$

$$[a_0, a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots \cfrac{1}{a_n}}}$$

$$\frac{1}{a_{n-1} + \frac{1}{a_n}}$$

$$[a_0, a_1, \dots, a_n] = \frac{p_n}{q_n}$$

Teorem 6.3. Brojevi p_n, q_n zadovoljavaju rekurziju

$$p_n = a_n p_{n-1} + p_{n-2}, \quad p_0 = a_0, \quad p_1 = a_0 a_1 + 1$$

$$q_n = a_n q_{n-1} + q_{n-2}, \quad q_0 = 1, \quad q_1 = a_1$$

Dokaz: Za $n = 2$ tvrdnja vrijedi. Pretp. da je $n > 2$ i da tvrdnja vrijedi za $n-1$. Ideja: Promatraj

(d.z.)

$$\frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}. \quad \text{Znamo } \frac{p_n}{q_n} = [a_0, a_1, \dots, a_n] = \left[a_0, a_1, \dots, a_{n-2}, a_{n-1} + \frac{1}{a_n}\right].$$

$$= \frac{\left(a_{n-1} + \frac{1}{a_n}\right) p_{n-2} + p_{n-3}}{\left(a_{n-1} + \frac{1}{a_n}\right) q_{n-2} + q_{n-3}} = \frac{p_{n-1} + \frac{p_{n-2}}{a_n}}{q_{n-1} + \frac{q_{n-2}}{a_n}} \begin{matrix} / \cdot a_n \\ / \cdot a_n \end{matrix}$$

$p_{-2}=0 \qquad p_{-1}=1$

$q_{-2}=1 \qquad q_{-1}=0$

$$\det\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix} = (-1)^{n-1}$$

Teorem 6.4. Za sve $n\geq -1$ vrijedi $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$.

Dokaz: Indukcijom. Za $n=-1$ imamo $q_{-1}p_{-2} - p_{-1}q_{-2} = 0\cdot 0 - 1\cdot 1 = (-1)^{-1}$ ✓.

Pretp. da tvrdnja vrijedi za $n-1$. Tada je

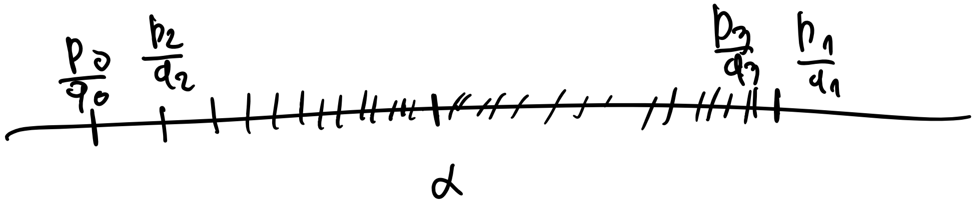
$$q_n p_{n-1} - p_n q_{n-1} = (a_n q_{n-1} + q_{n-2})p_{n-1} - (a_n p_{n-1} + p_{n-2})q_{n-1}$$

$$= -(q_{n-1}p_{n-2} - p_{n-1}q_{n-2}) = -(-1)^{n-1} = (-1)^n \checkmark$$

Korolar 6.5. Brojevi p_n i q_n relativno prosti.

Teorem 6.6.

1) $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \dots$



2) $\frac{p_1}{q_1} > \frac{p_3}{q_3} > \dots$

3) Ako je n paran, a m neparan, onda je $\frac{p_n}{q_n} < \frac{p_m}{q_m}$.

Dokaz: Promotrimo

$$\frac{p_{n-2}}{q_{n-2}} - \frac{p_n}{q_n} = \frac{p_{n-2}(a_n q_{n-1} + q_{n-2}) - (a_n p_{n-1} + \cancel{p_{n-2}}) \cdot q_{n-2}}{q_n q_{n-2}}$$

$$= \frac{a_n (p_{n-2} q_{n-1} - p_{n-1} q_{n-2})}{q_n q_{n-2}} = \frac{(-1)^{n-1}}{q_n q_{n-2}}$$

Slijedi 1) i 2).

3) d.z.

Teorem 6.7. $\lim_{n\to\infty} \frac{p_n}{q_n} = \alpha$

Dokaz: Budući da je $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \dots < \frac{p_1}{q_1}$ pa postoji $\lim_n \frac{p_{2n}}{q_{2n}}$.

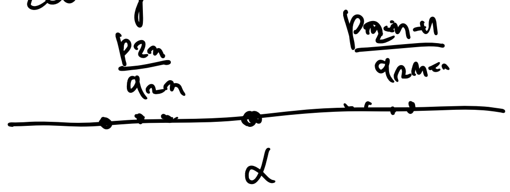
Slično postoji i $\lim_n \frac{p_{2n+1}}{q_{2n+1}}$. Ova dva limesa su jednaka jer

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{p_{n-1} q_n - q_{n-1} p_n}{q_{n-1} q_n} = \frac{(-1)^n}{q_{n-1} q_n} \Rightarrow \underline{\lim_n \left(\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n}\right) = 0}$$

$$\Longrightarrow \lim_n \frac{p_{2n}}{q_{2n}} = \lim_n \frac{p_{2n+1}}{q_{2n+1}} \quad \text{(d.z.)}.$$

Sada možemo zaključiti da $\alpha \in \mathbb{Q}$ ima konačan verižni razlomak.

Zašto?:



$$\left|\alpha - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_{n+1} \cdot q_n} < \frac{1}{q_n^2}$$

Ako je $\alpha \in \mathbb{Q}$, onda zbog Napomene 6.1. slijedi da postoji konačno mnogo ~~razlomaka~~ ~~pa~~ konvergenti (jer za svaki konvergent vrijedi $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$).

Definicija 6.1. Ako je a_0 cijeli broj, $a_1, \dots, a_n$ prirodni brojevi, te ako je $\alpha = [a_0, \dots, a_n]$ onda taj izraz zovemo razvoj broja α u konačni jednostavni verižni razlomak; $\frac{p_i}{q_i} = [a_0, \dots, a_i]$ je i-ta konvergenta od α, a_i je i-ti parcijalni kvocijent od α ($\alpha_i = [a_i, a_{i+1}, \dots, a_n]$ je i-ti potpuni kvocijent od α.

Ako je α irac. broj, onda uvodimo oznaku $\lim_{n\to\infty} [a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n, \ldots]$

Ako $\alpha = [a_0, a_1, \ldots]$ onda ovaj izraz zovemo razvoj od α u jednostavni verižni razlomak : $\frac{p_i}{q_i} = [a_0, \ldots, a_i]$ je i-ta konvergenta, a_i, α_i

$$|\alpha - \frac{p}{q}| < \frac{1}{2q^2} \rightsquigarrow \left(|\alpha - \frac{p}{q}| < \frac{1}{2q^2}, \right.$$