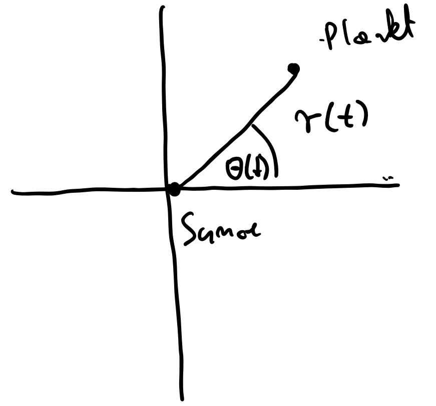
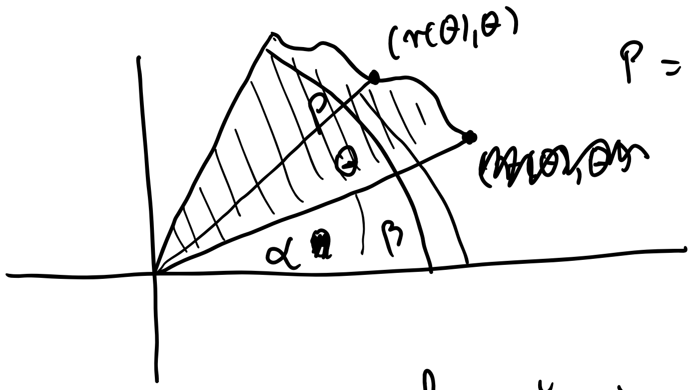


4. Drugi Keplerov zakon

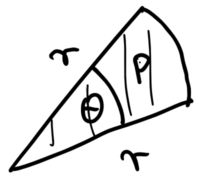
4.1. Površina "prebrisana" radij vektorom



$P = ?$

$$P = \int_{\alpha}^{\beta} \frac{r(\theta)^2}{2}\, d\theta \quad ; \text{ zašto?}$$

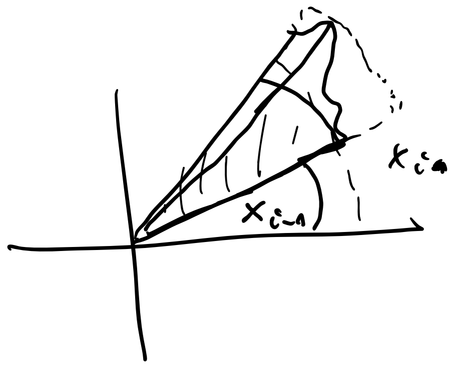
Aproksimacija površine kružnim isječcima:



$$P = r^2\pi \cdot \frac{\theta}{2\pi} = \frac{r^2}{2} \cdot \theta$$

Gledamo subdivizije segmenta $[\alpha, \beta]$ i njima pripadne gornje i donje D. sume:

$$\alpha = x_0 < x_1 < \dots < x_n = \beta$$



$[x_{i-1}, x_i]$

$$m_i = \inf_{\varphi \in [x_{i-1}, x_i]} r(\varphi), \quad M_i = \sup_{[x_{i-1}, x_i]} r(\varphi)$$

$$s = \sum_{i=1}^{n} \frac{m_i^2}{2}(x_i - x_{i-1}), \quad S = \sum_{i=1}^{n} \frac{M_i^2}{2}(x_i - x_{i-1})$$

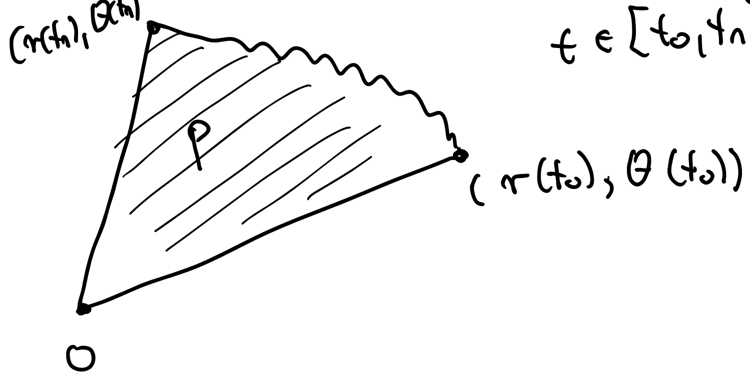
Vrijedi $\frac{m_i^2}{2} = \inf_{[x_{i-1}, x_i]} \frac{r^2(\varphi)}{2} = m_i'$; $\frac{M_i^2}{2} = \sup_{[x_{i-1}, x_i]} \frac{r^2(\varphi)}{2} = M_i'$

$$s = \sum_{i=1}^{n} m_i' (x_i - x_{i-1}) \; ; \; S = \sum_{i=1}^{n} M_i' (x_i - x_{i-1})$$

standardne D. sume pridružene funkciji $\frac{r(\varphi)^2}{2}$

.... $P = \int_{\alpha}^{\beta} \frac{r^2(\varphi)}{2} \, d\varphi.$

4.2. zamjena varijabli u integralu.



$t \in [t_0, t_n]$

<u>Teorem</u>: Neka su I i J otvoreni intervali, φ dif. fja na J i F primitivna fja. od f na I, gdje $f : I \to \mathbb{R}$ neprekidna fja.

Neka je $\varphi(J) \subseteq I$, t.j. $f \circ \varphi$ je def. na J.

Tada je $G = F \circ \varphi$ primitivna fja. od $(f \circ \varphi) \cdot \varphi'$ na I.

Također, $\forall \alpha, \beta \in J$ vrijedi $\int_{\alpha}^{\beta} f(\varphi(t)) \varphi'(t) dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x) \, dx.$

Dokaz: Zbog $\varphi(J) \subseteq I$ je fja. $G = F \circ \varphi$ definirana i dif. na J. Vrijedi $G'(x) = F'(\varphi(x)) \cdot \varphi'(x) = f(\varphi(x)) \cdot \varphi'(x)$ što daje

$$\int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\,dt = G(\beta) - G(\alpha) = F(\varphi(\beta)) - F(\varphi(\alpha))$$

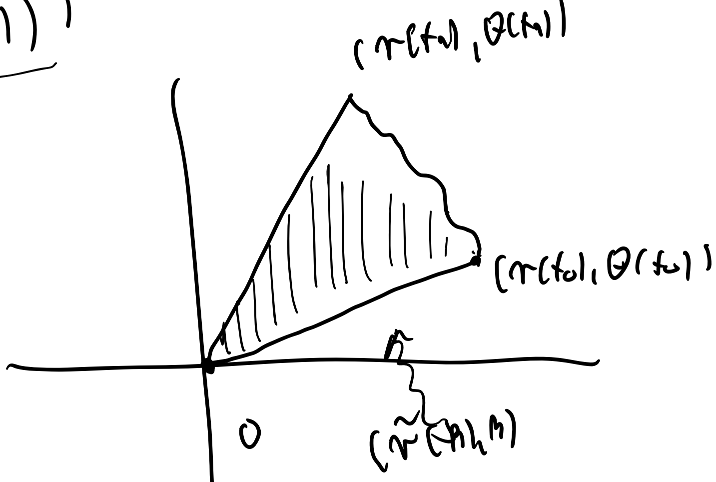
$$= \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx \quad \checkmark$$

$\square$

d.z.

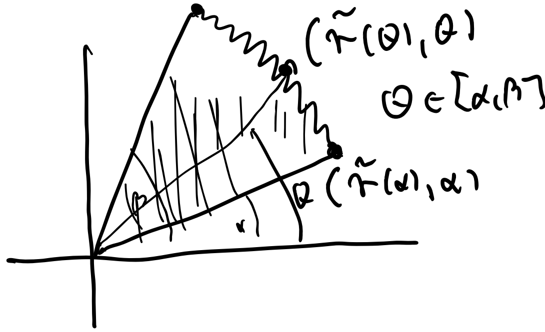
Definirajmo fju. $\tilde{r} : [\alpha, \beta] \to \mathbb{R}$ formulom

$\tilde{r}(\theta(t)) := r(t)$ (tj. $\tilde{r}(\alpha) :\overset{?\checkmark}{=} r(\theta^{-1}(\alpha))$)



Tada $P = \int_\alpha^\beta \frac{\tilde{r}^2(\theta)\,d\theta}{2}$ gdje je $\alpha = \theta(t_0)$ i $\beta = \theta(t_1)$.

Imamo $$\int_{t_0}^{t_1} \frac{\tilde{r}(\theta(t))^2}{2} \cdot \dot{\theta}(t)\,dt = \int_\alpha^\beta \frac{\tilde{r}^2(\theta)}{2}\,d\theta$$



$$\Rightarrow \boxed{P = \int_{t_0}^{t_1} \frac{r(t)^2}{2} \cdot \dot{\theta}(t)\,dt}$$

$$P = \int_{t_0}^{t_1} \frac{r(t)^2}{2} \cdot \dot{\theta}(t)\, dt$$

Označimo sa $K = \frac{1}{2} r^2 \cdot \dot{\theta}$.

Računamo $$\dot{K} = \frac{1}{2} \cdot 2r \cdot \dot{r} \cdot \dot{\theta} + \frac{1}{2} r^2 \cdot \ddot{\theta}$$

$$= \frac{r}{2} \Big(\underbrace{r \ddot{\theta} + 2 \dot{r} \dot{\theta}}_{\overset{!!}{0}} \Big) \rightsquigarrow \text{jednadžba gibanja}$$

$$= 0$$

$\Rightarrow$ K je konstanta.

$$\Rightarrow P = K \cdot \int_{t_0}^{t_1} 1\, dt \Rightarrow \text{2. Keplerov zakon.}$$