Taylorovi redovi

Neka je $f: \langle c-r, c+r \rangle \to \mathbb{R}$ zadana s $\underline{f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n}$

gdje je $r>0$ radijus konvergencije reda. Tada je $f \in C^\infty(\langle c-r, c+r \rangle)$ i vrijedi

$$f(c) = a_0$$

$$f'(c) = a_1$$

$$f''(c) = 2a_2 \qquad \Rightarrow \quad a_n = \frac{f^{(n)}(c)}{n!} \quad \text{za } n = 0,1,2,\dots$$

$$\vdots$$

$$f^{(n)}(c) = n!\, a_n$$

Def. 6.6 Neka je $f \in C^\infty(\langle c-r, c+r \rangle)$. Red

$$(*) \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

zovemo <u>Taylorov red.</u> fje. f oko točke c.

Taylorovi polinomi $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k$ su parcijalne sume Taylorovog reda.

$$f \rightsquigarrow \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \rightsquigarrow r' \rightsquigarrow \text{fj. } \tilde{f} : \langle c-r', c+r' \rangle \to \mathbb{R}$$

Pitanje je: je li $f = \tilde{f}$?

Teorem 6.17. Neka je $I \subseteq \mathbb{R}$ otvoren interval, $c \in I$ i $f : I \to \mathbb{R}$ klase $C^\infty(I)$. Ako postoji $n_0 \in \mathbb{N}$ i $\delta > 0$, $M > 0$ i $C > 0$ t.d.

$$\forall n \geq n_0 \quad |f^{(n)}(x)| \leq C \cdot M^n \cdot n! \quad \forall x \in I' = \langle c-\delta, c+\delta \rangle \cap I,$$

onda red $(*)$ konvergira k $f(x)$ $\forall x \in I' \cap \langle c - \frac{1}{M}, c + \frac{1}{M} \rangle$.

Dokaz: $\forall x \in I$ po Taylorovom teoremu srednje vrijednosti postoji c_x između c i x t.d.

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-c)^{n+1}$$

Odatle je,

$$|f(x) - T_n(x)| = \frac{|f^{(n+1)}(c_x)|}{(n+1)!}|x-c|^{n+1} \leq \frac{C \cdot M^{n+1}(n+1)!}{(n+1)!}|x-c|^{n+1}$$

$$= C(M|x-c|)^{n+1}.$$

Za $x \in I' \cap \langle c-\frac{1}{M}, c+\frac{1}{M} \rangle$ je $M|x-c| < 1$

pa je $\lim\limits_{n\to+\infty} |f(x) - T_n(x)| = 0$, tj.

$$\lim_{n\to+\infty} T_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(x).$$

$\blacksquare$.

Teorem 6.14. Neka je $I \subseteq \mathbb{R}$ otvoreni interval, $c \in I$ i $f: I \to \mathbb{R}$ klase $C^\infty(I)$. Ako postoji $\delta > 0$, $C > 0$ i $M > 0$ t.d.

$$\forall n \in \mathbb{N} \quad |f^{(n)}(x)| \leq C \cdot M^n, \quad \forall x \in I' = \langle c-\delta, c+\delta \rangle \cap I$$

onda red $(*)$ konvergira k $f(x)$ $\forall x \in I'$.

Primjer 6.17. Za fju. $f(x) = e^x$, $\forall x \in \mathbb{R}$ i $c = 0$ je $f^{(n)}(0) = 1$ pa je Taylorov red oko nule $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Iz prethodnog teorema slijedi da red konvergira $\forall x \in \mathbb{R}$ funkciji $f(x)$. Zašto? e^x je ograničena na $\langle c-\delta, c+\delta \rangle$ $\forall \delta > 0$.

$(M = 1)$.

Primjer 6.18. Za fj. $f(x)=\sin x$ i $f(x)=\cos x$ je

$|f^{(n)}(x)| \leq 1$ $\forall x\in\mathbb{R}$ pa vrijedi

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{i} \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \forall x\in\mathbb{R}.$$

Primjer 6.19. (Binomni red) Za $\alpha\in\mathbb{R}$ i $k\in\mathbb{N}$ definiramo

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} = \cancel{\alpha\cdot\frac{\alpha-1}{1}\cdot\frac{\alpha-2}{2}\cdots\frac{\alpha-k+1}{k} = \alpha\cdot\frac{\alpha-1}{1}\cdot\left(\frac{\alpha}{2}-1\right.}$$

$$= \frac{\alpha}{1}\cdot\frac{\alpha-1}{2}\cdot\frac{\alpha-2}{3}\cdots\cdots\frac{\alpha-k+1}{k}$$ s time da je

$\binom{\alpha}{0}=1$. Promotrimo red potencija

$\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$. Po D'Alemb. kriteriju

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\binom{\alpha}{n+1}|x|^{n+1}}{\binom{\alpha}{n}|x|} = \frac{|\alpha-n|}{n+1}|x| \xrightarrow{n\to\infty} |x|$$

Dakle, za $|x|<1$ red apsolutno konvergira, a za $|x|>1$ divergira. Neka je $f:\langle -1,1\rangle \to \mathbb{R}$ f-ja def s

$f(x) := (1+x)^{\alpha}$. Vrijedi $f^{(n)}(x) = \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n}$

$$= \binom{\alpha}{n} n!\,(1+x)^{\alpha-n}.$$

pa je $\frac{f^{(n)}(0)}{n!} = \binom{\alpha}{n}$ $\forall n \in \mathbb{N}$. (tj. $\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ je Taylorov red fje. f.).

Pokažimo da je $f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ $\forall x \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$.

Neka je $n_0 \in \mathbb{N}$, $n_0 > \alpha+1$. Tada za $n > n_0$ imamo $n-\alpha>1$.

pa $\left|\binom{\alpha}{n}\right| = |\alpha| \cdot \left|1-\frac{\alpha+1}{2}\right| \cdot \ldots \cdot \left|1-\frac{\alpha+1}{n_0-1}\right| \cdot \left(1-\frac{\alpha+1}{n_0}\right) \cdots \left(1-\frac{\alpha+1}{n}\right)$

$$\leq |\alpha| \cdot \left|1-\frac{\alpha+1}{2}\right| \cdots \left|1-\frac{\alpha+1}{n_0-1}\right| = \left|\binom{\alpha}{n_0-1}\right| \text{ (d.z.)}$$

Odatle je

$$\left|\frac{f^{(n)}(x)}{n!}\right| \doteq \left|\binom{\alpha}{n}\right| (1+x)^{\alpha-n} \le \left|\binom{\alpha}{n_0-1}\right| (1+x)^{\alpha} \cdot (1+x)^{-n}$$

U slučaju $|x| \le \frac{1}{2}$ je $1+x \ge \frac{1}{2}$ pa vrijedi

$$(1+x)^{-n} \le 2^n \quad \forall n \in \mathbb{N}$$

pa na intervalu $\langle -\frac{1}{2}, \frac{1}{2} \rangle$ vrijedi uvjet iz Teorema 6.13.

za $C = \left|\binom{\alpha}{n_0-1}\right| \max\left\{\left(\frac{3}{2}\right)^{\alpha}, \left(\frac{1}{2}\right)^{\alpha}\right\}$ i $M = 2$.

Dakle, pokazali smo da se fje. $\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ i $(1+x)^{\alpha}$ obje definirane

na $\langle -1, 1 \rangle$ podudaraju na $\langle -\frac{1}{2}, \frac{1}{2} \rangle$. Može se pokazati

da se podudaraju na cijelom intervalu $\langle -1, 1 \rangle$. ■