+BOJAN PAŽEK

1181211289

Druga zadaća: Parcijalne diferencijalne jednadžbe II

10

1. [2] Odredite $\chi_{[-a,a]} * \cos$.

2. [2] Odredite Fourierovu pretvorbu funkcije

$$f(x) = \frac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1} .$$

3. [2] Izračunajte $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$.

4. [2] Neka je za $a \in \mathbf{C}$ dana funkcija $f_a : \mathbf{R} \to \mathbf{C}$,

$$f_a(x) = \frac{2a}{a^2 + 4\pi^2 x^2} .$$

Za $a, b \in \mathbf{C}$, $\mathsf{Re}\,(a), \mathsf{Re}\,(b) > 0$, izračunajte $f_a * f_b$.

5. [2] Korsteći Fourierovu pretvorbu riješite početnu zadaću na $\mathbf{R}^+ \times \mathbf{R}$:

$$\begin{cases} u_t = 4u_{xx} , \\ u(0,x) = e^{2\pi i x} . \end{cases}$$

Rješenja u pisanom obliku treba predati na vježbama 15. svibnja 2012.

Marko Erceg

+ BOJAN PAŽEK

$\boxed{1.}$ Odredite $\chi_{[-a,a]} * \cos$

Rj. Po definiciji KONVOLUCIJE imamo $(f*g)(x) \overset{DEF}{:=} \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$.

(*) Uočimo da vrijedi sljedeće:

$$(f*g)(x) = (g*f)(x)$$

Naime

$$(f*g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y)\, dy = \left\{ \begin{array}{l} x-y = z \\ y = x-z \\ dy = dz \text{ jer Lebesgueova mjera je TRANSLATORNO INVARIJANTNA} \end{array} \right\}$$

$$= \int_{\mathbb{R}^d} f(z)\, g(x-z)\, dz = \int_{\mathbb{R}^d} g(x-z) f(z)\, dz = (g*f)(x)$$

Naravno, nama ova tvrdnja u ovom zadatku treba za dimenziju $d=1$

Tada se ova tvrdnja trivijalno pokaže:

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(x-y) g(y)\, dy = \left\{ \begin{array}{l} x-y = t \\ y = x-t \\ dy = -dt \\ t(-\infty) = +\infty \\ t(+\infty) = -\infty \end{array} \right\} = -\int_{+\infty}^{-\infty} f(t)\, g(x-t)\, dt = \int_{-\infty}^{+\infty} f(t) g(x-t)\, dt$$

$$= \int_{-\infty}^{+\infty} g(x-t) f(t)\, dt = (g*f)(x) \quad \blacksquare$$

Stoga računamo:

$$(\chi_{[-a,a]} * \cos)(x) = (*) = (\cos * \chi_{[-a,a]})(x) = (\text{definicija konvolucije}) = \int_{-\infty}^{+\infty} \cos(x-y) \chi_{[-a,a]}(y)\, dy$$

$$\int_{-\infty}^{+\infty} \cos(x-y)\chi_{[-a,a]}(y)\, dy = \int_{\mathbb{R}} \cos(x-y)\chi_{[-a,a]}(y)\, dy = \int_{\mathbb{R}\cap[-a,a]} \cos(x-y)\, dy = \int_{-a}^{a} \cos(x-y)\, dy$$

$$= \int_{-a}^{a} \cos(x-y)\, dy = (-1) \sin(x-y) \Big|_{-a}^{a} = (-1)\cdot[\sin(x-a) - \sin(x+a)] =$$

$$= \sin(x+a) - \sin(x-a) \overset{\text{adicijska formula}}{=} \cancel{\sin(x)\cos(a)} + \cos(x)\sin(a) - \cancel{\sin(x)\cos(a)} + \cos(x)\sin(a)$$

$$= 2\cos(x)\sin(a) = 2\sin(a)\cos(x)$$

✓ $\blacksquare$

2. Odrediti Fourierovu pretvorbu funkcije:

$$f(x) = \frac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1}$$

Rj. Definirajmo sa $h(x) := x^3 - x^2 + x - 1$. Uočimo da je $h(1) = 1 - 1 + 1 - 1 = 0$. Dakle polinom $(x-1)$ dijeli polinom $h(x)$.

$$\begin{array}{l} x^3 - x^2 + x - 1 : (x-1) = x^2 + 1 \\ \underline{-x^3 + x^2} \\ \quad\quad x - 1 \\ \quad\quad \underline{-x + 1} \\ \quad\quad\quad 0 \end{array}$$

Dakle vrijedi: $h(x) = (x-1)(x^2+1)$.

Nadalje, uočimo da vrijedi: $\dfrac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1} = \dfrac{1}{x-1} + \dfrac{1}{x-i} + \dfrac{1}{x+i}$

Naime: $\dfrac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1} = \dfrac{A}{x-1} + \dfrac{Bx + C}{x^2 + 1} \Rightarrow 3x^2 - 2x + 1 = Ax^2 + A + (Bx + C)(x - 1)$

$= Ax^2 + Bx^2 - Bx + Cx + A - C$

Iz jednakosti polinoma dobivamo

$$\left.\begin{array}{l} A + B = 3 \\ -B + C = -2 \end{array}\right\} + \Rightarrow A + C = 1$$

$$A - C = 1 \longrightarrow A - C = 1$$

$$+ \Rightarrow 2A = 2 \Rightarrow \boxed{A = 1}$$

Dakle $\boxed{A = 1}$ $\boxed{C = 0}$ $\boxed{B = 2}$

tj: $\dfrac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1} = \dfrac{1}{x-1} + \dfrac{2x}{x^2+1}$

Također vrijedi:

$$\frac{2x}{x^2+1} = \frac{2x}{(x-i)(x+i)} = \frac{A}{x-i} + \frac{B}{x+i} \Rightarrow 2x = A(x+i) + B(x-i)$$

$= Ax + Bx + iA - iB$

Iz jednakosti polinoma dobivamo:

$A + B = 2 \longrightarrow 2A = 2$

$iA - iB = 0 \Rightarrow \boxed{A = B}$

$\boxed{A = 1}$

$\boxed{B = 1}$

tj $\dfrac{2x}{x^2+1} = \dfrac{1}{x-i} + \dfrac{1}{x+i}$

- Definirajmo: $f_1(x) := \dfrac{1}{x-1}$

  $f_2(x) := \dfrac{1}{x-i}$

  $f_3(x) := \dfrac{1}{x+i}$

- Odredimo Fourierovu pretvorbu od f_1, f_2 i f_3.

↳ $\hat{f_1}(\xi) = \left(\tau_1 \frac{1}{x}\right)^\wedge(\xi) = e^{-2\pi i \xi} \cdot (-i\pi \operatorname{sign}(\xi))$

$\tau_1 :=$ translacija za 1

kad napišemo $\frac{1}{x}$ mislimo zapravo na $pv(\frac{1}{x})$

$\widehat{\tau_h f}(\xi) = e^{-2\pi i \xi \cdot h} \hat{f}(\xi)$

$(pv(\frac{1}{x}))^\wedge(\xi) = -i\pi \operatorname{sign}(\xi)$

nastavak →

$\boxed{2.}$ + (nastavak):

$\hookrightarrow$ $$\hat{f}_2(\xi) = \left(\frac{1}{-i+x}\right)^{\wedge}(\xi) = \left(\frac{1}{-i(1-\frac{1}{i}x)}\right)^{\wedge}(\xi) = -\frac{1}{i}\left(\frac{1}{1-\frac{1}{i}x}\right)^{\wedge}(\xi) = i\left(\frac{1}{1+ix}\right)^{\wedge}(\xi) =$$

Fourier je linearan operator posebno je homogen tj $\mathcal{F}(\alpha f)(\xi) = \alpha\mathcal{F}(f)(\xi)$ $\forall \alpha \in \mathbb{C}$.

$-\frac{1}{i}\cdot\frac{i}{i} = i$

$$= i\cdot g\left(\frac{x}{2\pi}\right)^{\wedge}(\xi) = i\cdot 2\pi\cdot \widehat{g(2\pi x)}(\xi) = 2\pi i\, e^{2\pi\xi} H(-2\pi\xi)$$

Definiramo $g(x) := \frac{1}{1+2\pi i x}$

$\Rightarrow g\left(\frac{x}{2\pi}\right) = \frac{1}{1+2\pi i\frac{x}{2\pi}} = \frac{1}{1+ix}$

pravilo koje vrijedi:

$\widehat{f(\alpha x)}(\xi) = \frac{1}{|\alpha|^d}\hat{f}\left(\frac{\xi}{\alpha}\right), \forall \alpha\in\mathbb{R}$

$\Rightarrow |\alpha|^d\cdot \widehat{f(\alpha x)}(\xi) = \hat{f}\left(\frac{\xi}{\alpha}\right)$

Prema zadatku s vježbi:
Stavimo $F(x) = e^{-ax}H(x)$, $Re(a)>0$
Tada je $\hat{F}(\xi) = \frac{1}{a+2\pi i\xi}$
Jer $\frac{1}{a+2\pi i\xi}\in L^2(\mathbb{R})$ imamo:
$\hat{\hat{F}}(\xi) = \left(\frac{1}{a+2\pi i\xi}\right)^{\wedge}$
$\|$
$F_\sigma(\xi) = F(-\xi) = e^{a\xi}H(-\xi)$

NAPOMENA: Na vježbama smo izračunali da vrijedi sljedeće:

$$\left(\frac{A}{-ia+x}\right)^{\wedge}(\xi) = \frac{\pi}{a}e^{2\pi a\xi}H(-2\pi\xi) \quad \text{za } a>0$$

gdje je $A := \frac{1}{2ia}$

Stavimo $B = 2ia\cdot A$. Tada linearnost Fouriera povlači:

$$\left(\frac{B}{-ia+x}\right)^{\wedge}(\xi) = 2ia\left(\frac{A}{-ia+x}\right)^{\wedge}(\xi) = 2ia\cdot\frac{\pi}{a}e^{2\pi a\xi}H(-2\pi\xi)$$

Uočimo da je $B=1$, pa za $a=1>0$ imamo:

$$\left(\frac{1}{-i+x}\right)^{\wedge}(\xi) = 2i\pi\, e^{2\pi\xi}H(-2\pi\xi) = 2\pi i\, e^{2\pi\xi}H(-2\pi\xi)$$

Stoga smo $\hat{f}_2(\xi)$ mogli i ovako izračunati. $\blacksquare$

$\hookrightarrow$ $$\hat{f}_3(\xi) = \left(\frac{1}{i+x}\right)^{\wedge}(\xi) = \left(\frac{1}{i(1+\frac{1}{i}x)}\right)^{\wedge}(\xi) = \frac{1}{i}\left(\frac{1}{1+\frac{1}{i}x}\right)^{\wedge}(\xi) = -i\left(\frac{1}{1-ix}\right)^{\wedge}(\xi) = -i\, g\left(\frac{x}{2\pi}\right)^{\wedge}(\xi) =$$

$\frac{1}{i}\frac{i}{i} = -i$

Definiramo $g(x) := \frac{1}{1-2\pi i x}$
$\Rightarrow g\left(\frac{x}{2\pi}\right) = \frac{1}{1-2\pi i\frac{x}{2\pi}} = \frac{1}{1-ix}$

$$= -i\cdot 2\pi\, \widehat{g(2\pi x)}(\xi) = -2\pi i\, e^{-2\pi\xi}H(2\pi\xi).$$

Prema zadatku s vježbi:
Stavimo $F(x) = e^{ax}H(-x)$, $Re(a)>0$
Tada je $\hat{F}(\xi) = \frac{1}{a-2\pi i\xi}$. Jer je $\frac{1}{a-2\pi i\xi}\in L^2(\mathbb{R})$ imamo:
$\hat{\hat{F}}(\xi) = \left(\frac{1}{a-2\pi i\xi}\right)^{\wedge}$
$\|$
$F_\sigma(\xi) = F(-\xi) = e^{-a\xi}H(\xi)$

NAPOMENA: $\hat{f}_3(\xi)$ mogli smo odmah izračunati koristeći činjenicu da smo računajući $\hat{f}_2(\xi)$ dobili da je $\left(\frac{1}{1+ix}\right)^{\wedge}(\xi) = 2\pi e^{2\pi\xi}H(-2\pi\xi)$. Naime tada odmah imamo $\left(\frac{1}{1-ix}\right)^{\wedge}(\xi) = \left(\left(\frac{1}{1+ix}\right)_\sigma\right)^{\wedge}(\xi) = \left[\left(\frac{1}{1+ix}\right)^{\wedge}(\xi)\right]_\sigma = \left[2\pi e^{2\pi\xi}H(-2\pi\xi)\right]_\sigma = 2\pi e^{-2\pi\xi}H(2\pi\xi)$, a $\hat{f}_3(\xi) = -i\left(\frac{1}{1-ix}\right)^{\wedge}(\xi)$ $\blacksquare$

$\hat{f}_\sigma = (\hat{f})_\sigma$
$f_\sigma(x) := f(-x)$

okreni

- Budući je Fourier LINEARAN OPERATOR (pa je time ADITIVAN) imamo:

$$\mathcal{F}\left(\frac{3x^2-2x+1}{x^3-x^2+x-1}\right)(\xi) = \mathcal{F}\left(\frac{1}{x-1}+\frac{1}{x-i}+\frac{1}{x+i}\right)(\xi) = \mathcal{F}\left(\frac{1}{x-1}\right)(\xi)+\mathcal{F}\left(\frac{1}{x-i}\right)(\xi)+\mathcal{F}\left(\frac{1}{x+i}\right)(\xi)$$

H je Heavisideova funkcija

$$= \hat{f}_1(\xi)+\hat{f}_2(\xi)+\hat{f}_3(\xi)$$

$$= e^{-2\pi i\xi}(-i\pi\,\mathrm{sign}(\xi)) + \underbrace{2\pi i\, e^{2\pi\xi} H(-2\pi\xi) - 2\pi i\, e^{-2\pi\xi} H(2\pi\xi)}_{= -2\pi i\,\mathrm{sign}(\xi)\, e^{-2\pi|\xi|}}$$

$\boxed{4.}$ + Za $a \in \mathbb{C}$ definiramo f-ju $f_a : \mathbb{R} \to \mathbb{C}$ sa:

$$f_a(x) := \frac{2a}{a^2+4\pi^2x^2}.$$

Za $a, b \in \mathbb{C}$, $\mathrm{Re}(a)>0$ i $\mathrm{Re}(b)>0$ izračunajte $f_a * f_b$.

$\boxed{Rj}$ +

- Na vježbama smo izračunali Fourierovu pretvorbu sljedeće funkcije:

$$h(x) := e^{-a|x|}, \quad \mathrm{Re}(a)>0.$$

Vrijedi: $\boxed{\hat{h}(\xi) = \dfrac{2a}{a^2+4\pi^2\xi^2}}$

- Označimo sa $g := f_a * f_b$. Tada imamo:

$$\widehat{f_a * f_b}(\xi) = \hat{g}(\xi) = \hat{f}_a(\xi)\cdot\hat{f}_b(\xi) = (\blacktriangle) = e^{-a|\xi|}\cdot e^{-b|\xi|} = e^{(-a-b)|\xi|}$$

$$= e^{-(a+b)|\xi|}$$

<u>Propozicija</u>: Neka su $f, g \in L^2(\mathbb{R}^d)$. Vrijedi

(i) $\widehat{f*g} = \hat{f}\cdot\hat{g}$

(ii) $\widehat{f\cdot g} = \hat{f}*\hat{g}$. ■

$(\blacktriangle)$ $\hat{h}(\xi) = \dfrac{2a}{a^2+4\pi^2\xi^2} \Big/ \wedge$

$\Rightarrow \hat{\hat{h}}(\xi) = \widehat{\dfrac{2a}{a^2+4\pi^2\xi^2}}$

$\|$

$h(-\xi) = (h$ je parna f-ja$) = h(\xi)$

Dakle $\hat{f}_a = h(\xi)$

- Dakle dobili smo $\hat{g}(\xi) = e^{-(a+b)|\xi|}$. Budući je $\mathrm{Re}(a)>0$ i $\mathrm{Re}(b)>0$

Slijedi da je $\mathrm{Re}(a+b)>0$.

Zato imamo: $\left(e^{-(a+b)|x|}\right)^\wedge(\xi) = \dfrac{2(a+b)}{(a+b)^2+4\pi^2\xi^2}$

Računamo: $\left(e^{-(a+b)|x|}\right)^\wedge(\xi) = \hat{\hat{g}}(\xi) = g(-\xi)$

$\|$

$\dfrac{2(a+b)}{(a+b)^2+4\pi^2\xi^2}$

4. + (nastavak)

• Dakle dobili smo: $g(-\xi) = \dfrac{2(a+b)}{(a+b)^2+4\pi^2\xi^2}$

↳ Uočimo da je funkcija $\dfrac{2(a+b)}{(a+b)^2+4\pi^2\xi^2}$ PARNA stoga je $g(-\xi) = g(-(-\xi)) = g(\xi)$

Dakle: $g(\xi) = \dfrac{2(a+b)}{(a+b)^2+4\pi^2\xi^2}$

• Budući smo sa g označili $f_a * f_b$ dobili smo:

$$\boxed{f_a * f_b(\xi) = \frac{2(a+b)}{(a+b)^2+4\pi^2\xi^2}}$$

5. + Koristeći Fourierovu pretvorbu riješite početnu zadaću na $\mathbb{R}^+ \times \mathbb{R}$:

$$\begin{cases} u_t = 4u_{xx} \\ u(0,x) = e^{2\pi i x} \end{cases}$$

Rj + • Promotrimo prvo $u_t = 4u_{xx} \;/\; \mathcal{F}_x(\;)$ ($\mathcal{F}_x$ ćemo označavati u ovom zadatku sa $\wedge$)

$$\frac{\partial \hat{u}}{\partial t} = -16\pi^2\xi^2\hat{u}$$

derivacija po t i $\mathcal{F}(\,)$ komutiraju ukoliko je fj-a u "dobra" garantirano na predavanju. Također sa predavanja znamo:
$\mathcal{F}_x(4u_{xx}) = 4\mathcal{F}_x(u_{xx}) =$
$= 4\cdot(2\pi i\xi)^2\hat{u} =$
$= 4\cdot 4\pi^2\cdot(-1)\xi^2\hat{u} =$
$= -16\pi^2\xi^2\hat{u}$

• Dakle dobili smo običnu diferencijalnu jednadžbu za $\hat{u}$ po varijabli t koju znamo riješiti:

napomena: $\dot{X} = C(\xi)\cdot X$ (vremenska derivacija)
$\Rightarrow X(t) = D(\xi)e^{C(\xi)t}$

$$\boxed{\hat{u}(t,\xi) = C(\xi)\cdot e^{-16\pi^2\xi^2 t}}$$

• Nadalje imamo $\hat{u}(0,\xi) = C(\xi)\cdot e^{-16\pi^2\xi^2\cdot 0} = C(\xi)$

Dakle: $C(\xi) = (e^{2\pi i x})^\wedge(\xi)$

• Dobili smo:
$$\boxed{\hat{u}(t,\xi) = (e^{2\pi i x})^\wedge(\xi)\cdot e^{-16\pi^2\xi^2 t}}$$

okreni →

- Na vježbama smo izračunali Fourierovu pretvorbu sljedeće funkcije:

$$h(x) := e^{-ax^2}, \quad a > 0.$$

(▲). Vrijedi: $\hat{h}(\xi) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2 \xi^2}{a}}$ ■

- Koristeći formulu (▲) računamo:

$$\left(e^{-16\pi^2 t x^2}\right)^{\wedge}(\xi) \underset{\substack{a = 16\pi^2 t \\ \text{jer } t \in \mathbb{R}^+ \text{ po pretpostavci} \\ \Rightarrow a > 0}}{=} \sqrt{\frac{\pi}{16\pi^2 t}}\, e^{-\frac{\pi^2 \xi^2}{16\pi^2 t}} = \sqrt{\frac{1}{16\pi t}}\, e^{-\frac{\xi^2}{16t}}$$

Uočimo, funkcija $e^{-16\pi^2 t x^2}$ je PARNA pa imamo:

$$e^{-16\pi^2 t \xi^2} = \left(e^{-16\pi^2 t x^2}\right)^{\hat{\wedge}}(\xi) = \left(\sqrt{\frac{1}{16\pi t}}\, e^{-\frac{\xi^2}{16t}}\right)^{\wedge}$$

- Dakle imamo:

$$\hat{u}(t, \xi) = \left(e^{2\pi i x}\right)^{\wedge}(\xi) \cdot e^{-16\pi^2 t \xi^2} = \left(e^{2\pi i x}\right)^{\wedge}(\xi) \cdot \left(\sqrt{\frac{1}{16\pi t}}\, e^{-\frac{x^2}{16t}}\right)^{\wedge}(\xi) =$$

Propozicija: Neka su $f, g \in L^2(\mathbb{R}^d)$. Tada vrijedi
(i) $\widehat{f * g} = \hat{f} \cdot \hat{g}$
(ii) $\widehat{f \cdot g} = \hat{f} * \hat{g}$

$$= \widehat{\left(e^{2\pi i x}\right) * \left(\sqrt{\frac{1}{16\pi t}}\, e^{-\frac{x^2}{16t}}\right)}(\xi) \quad / \quad \bar{\mathcal{F}}$$

$\bar{\mathcal{F}}$ → sada djelujemo "inverznim Fourierom"

$$\Rightarrow \boxed{u(t, x) = \left(e^{2\pi i x}\right) \underset{\text{konvolucija}}{*} \left(\sqrt{\frac{1}{16\pi t}}\, e^{-\frac{x^2}{16t}}\right)}$$

- Za konvoluciju vrijedi da je $(f * g)(x) = (g * f)(x)$, $\forall x \in \mathbb{R}^d$ stoga po definiciji računamo:

$e^{2\pi i y} = \cos(2\pi y) + i \sin(2\pi y)$

$(x-y)^2 = (y-x)^2$

$$u(t, x) = \int_{-\infty}^{+\infty} (16\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{16t}} \cdot e^{2\pi i y}\, dy = (16\pi t)^{-\frac{1}{2}} \left[\int_{-\infty}^{+\infty} e^{-\frac{(y-x)^2}{16t}} \cos(2\pi y)\, dy + i \int_{-\infty}^{+\infty} e^{-\frac{(y-x)^2}{16t}} \sin(2\pi y) \right.$$

$$(1) \quad \int_{-\infty}^{+\infty} e^{-\frac{(y-x)^2}{16t}} \cos(2\pi y)\, dy = \left\{ \begin{array}{l} y - x = s \\ dy = ds \\ y = s + x \\ s(-\infty) = -\infty \\ s(+\infty) = +\infty \end{array} \right\} = \int_{-\infty}^{+\infty} e^{-\frac{s^2}{16t}} \cos(2\pi x)\cos(2\pi s)\, ds - \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{s^2}{16t}} \sin(2\pi x)\sin(2\pi s)\, ds}_{= 0 \text{ (integral neparne fj-e po simetričnoj domeni)}} =$$

$\cos(2\pi(s+x)) = \cos(2\pi s + 2\pi x)$
$= \cos(2\pi s)\cos(2\pi x) - \sin(2\pi s)\sin(2\pi x)$

$$= \cos(2\pi x) \int_{-\infty}^{+\infty} e^{-\frac{1}{16t} s^2} \cos(2\pi s)\, ds = \cos(2\pi x) \sqrt{\frac{\pi}{\frac{1}{16t}}}\, e^{-\frac{(2\pi)^2}{4 \cdot \frac{1}{16t}}} = \sqrt{16\pi t}\, \cos(2\pi x)\, e^{-16\pi^2 t}$$

$$\int_{-\infty}^{+\infty} e^{-ax^2} \cos(bx)\, dx = \sqrt{\frac{\pi}{a}}\, e^{-\frac{b^2}{4a}}$$

5. + (nastavak)

$$(2)\quad \int_{-\infty}^{+\infty} e^{-\frac{(y-x)^2}{16t}} \sin(2\pi y)\,dy = \left\{ \begin{array}{l} y-x=s \\ dy=ds \\ y=s+x \\ s(-\infty)=-\infty \\ s(+\infty)=+\infty \end{array} \right\} = \int_{-\infty}^{+\infty} e^{-\frac{s^2}{16t}} \cos(2\pi x)\sin(2\pi s)\,ds + \int_{-\infty}^{+\infty} e^{-\frac{s^2}{16t}} \sin(2\pi x)\cos(2\pi s)\,ds =$$

$\int_{-\infty}^{+\infty} e^{-\frac{s^2}{16t}} \cos(2\pi x)\sin(2\pi s)\,ds = 0$ (neparna fj.·a)

$\sin(2\pi(s+x)) = \sin(2\pi s + 2\pi x) =$
$= \sin(2\pi s)\cos(2\pi x) + \cos(2\pi s)\sin(2\pi x)$

$$= \sin(2\pi x)\int_{-\infty}^{+\infty} e^{-\frac{1}{16t}s^2}\cos(2\pi s)\,ds = \sin(2\pi x)\sqrt{\frac{\pi}{\frac{1}{16t}}}\, e^{-\frac{(2\pi)^2}{4\cdot\frac{1}{16t}}} = \sqrt{16\pi t}\,\sin(2\pi x)\, e^{-16\pi^2 t}$$

Dakle imamo:

$$u(t,x) = (16\pi t)^{-1/2}\cdot\sqrt{16\pi t}\,\cos(2\pi x)\,e^{-16\pi^2 t} + (16\pi t)^{-1/2}\cdot i\cdot\sqrt{16\pi t}\,\sin(2\pi x)\,e^{-16\pi^2 t} =$$

(1) i (2)

$$= \left[\cos(2\pi x) + i\sin(2\pi x)\right] e^{-16\pi^2 t} = e^{2\pi i x}\, e^{-16\pi^2 t}$$

• Stoga smo dobili konačno rješenje:

$$\boxed{u(t,x) = e^{-16\pi^2 t}\cdot e^{2\pi i x}}$$

↳ provjera: $u(0,x) = e^{-16\pi^2\cdot 0}\cdot e^{2\pi i x} = e^{2\pi i x}$ ✓

$$u_t = -16\pi^2 e^{-16\pi^2 t}\cdot e^{2\pi i x}$$
$$u_x = e^{-16\pi^2 t}(2\pi i)\, e^{2\pi i x}$$
$$u_{xx} = e^{-16\pi^2 t}(2\pi i)^2 e^{2\pi i x} = -4\pi^2 e^{-16\pi^2 t}\cdot e^{2\pi i x}$$

$$\Rightarrow\quad u_t - 4u_{xx} = -16\pi^2 e^{-16\pi^2 t}\cdot e^{2\pi i x} - 4\cdot\left(-4\pi^2 e^{-16\pi^2 t}\cdot e^{2\pi i x}\right)$$
$$= \cancel{-16\pi^2 e^{-16\pi^2 t} e^{2\pi i x}} + \cancel{16\pi^2 e^{-16\pi^2 t}\cdot e^{2\pi i x}}$$
$$= 0\ ✓$$

3. + Izračunati:

$$\int_{-\infty}^{+\infty} \frac{\sin^2(x)}{x^2}\,dx = ?$$

Rj. +

$$I = \int_{-\infty}^{+\infty} \frac{\sin^2(x)}{x^2}\,dx = \int_{-\infty}^{+\infty} \frac{\sin(x)}{x^2}\cdot\sin(x)\,dx$$

okrenuti →

- Definiramo $f(z) := \frac{\sin(z)}{z^2}$ tj. proširimo funkciju $\frac{\sin(x)}{x^2}$ na čitavu kompleksnu ravninu do funkcije $f: \mathbb{C} \to \mathbb{C}$ tako da stavimo $f(z) := \frac{\sin(z)}{z^2}$, $\forall z \in \mathbb{C}$.
- Stavimo $F(z) := f(z) \cdot e^{iz} = \frac{\sin(z)}{z^2} \cdot e^{iz}$

↳ TVRDNJA 1: $z_0 = 0$ je pol prvog reda za funkciju f (napomena: 0 je nultočka nazivnika kratnosti 2 i brojnika kratnosti 1 pa je "razlika kratnosti" $= 2-1=1$)

dz: $f(z) = \frac{1}{z^2}\sin(z) = \frac{1}{z^2}\sum_{n=0}^{+\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} = \frac{1}{z^2}\left(z - \frac{z^3}{6} + \frac{z^5}{120} - \dots\right) = \frac{1}{z} - \frac{z}{6} + \frac{z^3}{120} - \dots$

Budući je red pola, red najveće potencije od $\frac{1}{z-z_0}$ koje se javljaju u Laurentovu razvoju u red fj-e f s koeficijentima različitim od 0 zaključujemo da je $z_0 = 0$ POL PRVOG reda za funkciju f ■

↳ Budući je funkcija e^{iz} HOLOMORFNA na čitavom $\mathbb{C}$ zaključujemo:
$z_0 = 0$ je POL PRVOG REDA za funkciju F! ■

- NAPOMENA (IZ KOMPLEKSNE ANALIZE: zaobilaženje singulariteta (polova 1. reda) na realnoj osi (kojih ima konačno mnogo)
↳ Ako funkcija ima konačno mnogo polova 1. reda na realnoj osi onda ih zaobilazimo. Uzmemo R dovoljno velik da obuhvatimo singularitete na gornjoj poluravnini i na realnoj osi

↳ radimo na kompleksnoj analizi:

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{\mathrm{Im}(z_k)>0} \mathrm{res}(f, z_k) + \pi i \sum_{\mathrm{Im}(z_k)=0} \mathrm{res}(f, z_k)$$

i $\int_{-\infty}^{+\infty} f(t)\sin(\alpha t)\,dt = \mathrm{Im}(\quad)$ ■

- JORDANOVA LEMA:

Neka je $\max_{z \in C_R} |f(z)| \le M(R)$.

(1) Ako je $\lim_{R\to\infty}[R \cdot M(R)] = 0$ onda vrijedi $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$.

(2) Ako je $\lim_{R\to\infty} M(R) = 0$, onda $\forall \alpha > 0$ i $\alpha \in \mathbb{R}$ vrijedi $\lim_{R\to\infty}\int_{C_R} f(z) e^{i\alpha z}\,dz = 0$ ■

- Uočimo:

$$F(z) = \frac{\sin(z)}{z^2} \cdot e^{iz} = \frac{e^{iz} - e^{-iz}}{2iz^2} \cdot e^{iz} = \frac{e^{2iz}}{2iz^2} - \frac{1}{2iz^2}$$

(i) $\max_{|z|=R}\left|\frac{1}{2iz^2}\right| = \max_{|z|=R}\frac{1}{2|z|^2} \le \frac{1}{R^2}$; vrijedi $\lim_{R\to\infty} R \cdot \frac{1}{R^2} = 0$, zato je i

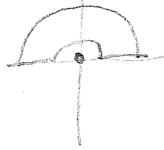
$2R^2 \ge R^2$
$\Rightarrow \frac{1}{2R^2} \le \frac{1}{R^2}$

$\lim_{R\to\infty}\int_{C_R}\frac{1}{2iz^2}\,dz = 0$ (Jordanova lema (1))

(ii) $\max_{|z|=R}\left|\frac{1}{2iz^2}\right| \le \frac{1}{R^2} \xrightarrow{R\to\infty} 0$. Tada $\forall \alpha > 0, \alpha \in \mathbb{R}$ pa i za $\alpha = 1$ vrijedi:

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{2iz^2} = 0 \quad \text{(Jordanova lema (2))}$$

$\boxed{3.}$ + (nastavak)



• Sada direktno iz (i) + (ii) čitamo:

$$\boxed{\lim_{R\to\infty}\int_{C_R} F(z)\,dz = 0}$$

■

napomena: Nismo mogli odmah primijeniti Jordanovu lemu da ustanovimo $\int_{C_R} F(z)dz \xrightarrow{R\to\infty} 0$ jer $\lim_{R\to\infty}\max_{|z|=R}\left|\frac{\sin(z)}{z^2}\right| \neq 0$

SADA IMAMO:

$$I = \int_{-\infty}^{+\infty}\frac{\sin^2(x)}{x^2}\,dx = \int_{-\infty}^{+\infty}\frac{\sin(x)}{x^2}\cdot\sin(x)\,dx = \operatorname{im}\left(\pi i\cdot \operatorname{res}(F,0)\right)$$

nemamo singularitete u gornjoj poluravnini već samo na realnoj osi

im → znači imaginarni dio

$$\operatorname{res}(F,0) = ?$$

razvijmo fj-u e^{iz} u red potencija. Vrijedi:

$$e^{iz} = \sum_{n=0}^{+\infty}\frac{(iz)^n}{n!} = 1 + iz + \frac{(iz)^2}{2}+\dots$$

razvili smo fj-u $\frac{\sin(z)}{z^2}$ u Laurentov red: $\frac{\sin(z)}{z^2} = \frac{1}{z}-\frac{z}{6}+\frac{z^3}{120}-\dots$

Iz ovoga zaključujemo:

$$\boxed{\operatorname{res}(F,0)=1}$$

koeficient uz a_{-1} u razv u Laurentov red funkcije $\frac{\sin(z)}{z^2}\cdot e^{iz}$ (to je bitno za ovaj zaključak)

$$\frac{\sin(z)}{z^2}\cdot e^{iz} = \left(\frac1z-\frac z6+\frac{z^3}{120}-\dots\right)\cdot\left(1+iz+\frac{(iz)^2}{2}+\dots\right) = \left(\frac1z-\frac z6+\frac{z^3}{120}-\dots\right)+\left(i\cdot \dots \frac{iz^2}{6}+\dots\right)+\dots$$

$\boxed{a_{-1}=1}$

Dakle:

$$I = \operatorname{im}(\pi i\cdot 1) = \operatorname{im}(\pi i) = \pi$$

$$\boxed{\text{Rješenje: } \int_{-\infty}^{+\infty}\frac{\sin^2(x)}{x^2}\,dx = \pi}$$

■

• RJEŠENJE NA DRUGI NAČIN (puno elegantnije):

TM: Fourierova pretvorba se na jedinstven način proširuje sa $S(\mathbb{R}^d)$ do unitarnog operatora na $L^2(\mathbb{R}^d)$ i vrijedi Parsevalova jednakost:

(PLANCHEREL)

$$\int_{\mathbb{R}^d}\hat f(\xi)\hat g(\xi)\,d\xi = \int_{\mathbb{R}^d} f(x)g(x)\,dx$$

■

okreni →

- Na vježbama smo pokazali da vrijedi sljedeće:

$$f(x) := \chi_{[a,b]}(x) = \begin{cases} 1, & x \in [a,b] \\ 0, & \text{inače} \end{cases} \implies \hat{f}(\xi) = \begin{cases} b-a, & \xi = 0 \\ \frac{\sin(\pi(b-a)\xi)}{\pi\xi} e^{-i\pi(a+b)\xi}, & \xi \neq 0 \end{cases}$$

$$\forall a,b \cap a \leq b.$$

Dakle vrijedi: $\hat{f}(\xi) = \frac{\sin(\pi(b-a)\xi)}{\pi\xi} e^{-i\pi(a+b)\xi}$ (g.sv) → Za integrale dovoljno

- Odaberimo $a = \frac{-1}{2}$ i $b = \frac{1}{2}$. Promatrajmo $f(x) := \chi_{[-1/2,\,1/2]}(x)$ → možemo odabrati bilo koji simetrični interval oko 0 oblika $[-\frac{1}{n}, \frac{1}{n}]$ mogli smo to odabrati jer $-\frac{1}{n} + \frac{1}{n} = 0$

Vrijedi: $$\hat{f}(\xi) = \frac{\sin(\pi \cdot (1/2 + 1/2)\xi)}{\pi\xi} e^{-i\pi(-1/2+1/2)\xi} = \frac{\sin(\pi\xi)}{\pi\xi} e^{0} = \frac{\sin(\pi\xi)}{\pi\xi} \quad (\text{g.sv})$$

Stoga je $$\hat{f}\left(\frac{\xi}{\pi}\right) = \frac{\sin(\xi)}{\xi} \Big|^{\wedge} \quad (\text{g.sv})$$

$$\implies \hat{\hat{f}}\left(\frac{\xi}{\pi}\right) = \left(\frac{\sin(x)}{x}\right)^{\wedge}(\xi) \quad (\text{g.sv})$$

Nadalje uočimo: $$\hat{\hat{f}}\left(\frac{\xi}{\pi}\right) = f\left(-\frac{\xi}{\pi}\right) = \chi_{[-1/2,1/2]}\left(-\frac{\xi}{\pi}\right) = \begin{cases} 1, & -1/2 \leq -\frac{\xi}{\pi} \leq 1/2 \\ 0, & \text{inače} \end{cases} = \begin{cases} 1, & -\frac{\pi}{2} \leq -\xi \leq \frac{\pi}{2} \\ 0, & \text{inače} \end{cases}$$

$$= \begin{cases} 1, & \frac{\pi}{2} \geq \xi \geq -\frac{\pi}{2} \\ 0, & \text{inače} \end{cases} = \begin{cases} 1, & -\frac{\pi}{2} \leq \xi \leq \frac{\pi}{2} \\ 0, & \text{inače} \end{cases} = \chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(\xi)$$

Vrijedi: $$\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi) \cdot \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi) = \chi_{[-\frac{\pi}{2},\frac{\pi}{2}] \cap [-\frac{\pi}{2},\frac{\pi}{2}]}(\xi) = \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi)$$

- Sada lagano dobivamo rješenje:

$$\int_{-\infty}^{+\infty} \frac{\sin^2(x)}{x^2}\,dx = \int_{-\infty}^{+\infty} \frac{\sin(x)}{x} \cdot \frac{\sin(x)}{x}\,dx \overset{\text{TEOREM (PLANCHEREL)}}{=} \int_{-\infty}^{+\infty} \left(\frac{\sin(x)}{x}\right)^{\wedge}(\xi) \cdot \left(\frac{\sin(x)}{x}\right)^{\wedge}(\xi)\,d\xi \overset{\text{Ako je } f=g \text{ g.s. tada je } \int f = \int g}{=}$$

$$= \int_{-\infty}^{+\infty} \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi) \cdot \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi)\,d\xi = \int_{-\infty}^{+\infty} \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi)\,d\xi = \int_{-\pi/2}^{\pi/2} d\xi = \xi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Dakle: $$\boxed{\int_{-\infty}^{+\infty} \frac{\sin^2(x)}{x}\,dx = \pi}$$

Lenka Vukšić

PDJ 2, DRUGA ZADAĆA

10

1. $$(\chi_{[-a,a]} * \cos)(x) = \int_{-\infty}^{+\infty} \cos(x-y)\, \chi_{[-a,a]}(y)\, dy = \int_{-a}^{a} \cos(x-y)\, dy =$$

$$= -\sin(x-y)\Big|_{y=-a}^{a} = -\sin(x-a) + \sin(x+a) = \boxed{2 \sin a \cos x}$$

2. $$f(x) = \frac{3x^2-2x+1}{x^3-x^2+x-1}$$

$$x^3-x^2+x-1 = x^2(x-1) + x - 1 = (x^2+1)(x-1) = (x-i)(x+i)(x-1)$$

$$\frac{3x^2-2x+1}{x^3-x^2+x-1} = \underbrace{\frac{A}{x-i}}_{f_1(x)} + \underbrace{\frac{B}{x+i}}_{f_2(x)} + \underbrace{\frac{C}{x-1}}_{f_3(x)}$$
$$= \left[A\,(x^2+(i-1)x - i) + B\,(x^2-(i+1)x+i) + C\,(x^2+1)\right] / (x^3-x^2+x-1)$$

$$\Rightarrow \left.\begin{aligned} &A+B+C = 3 \\ &(i-1)A - (i+1)B = -2 \\ &-iA + iB + C = 1 \end{aligned}\right\} \quad A = B = C = 1 \text{ je rješenje}$$

$$\hat{f} = \hat{f}_1 + \hat{f}_2 + \hat{f}_3$$

$$\left(\frac{1}{x-1}\right)^{\wedge}(\xi) = e^{-2\pi i \xi}\,(-i\pi)\, \mathrm{sign}(\xi)$$

$$\left(\frac{1}{2\pi i x + a}\right)^{\wedge}(\xi) = -e^{-a\xi}\, H(\xi), \quad a>0 \qquad a=1 \Rightarrow \left(\frac{1}{2\pi i x - 1}\right)^{\wedge}(\xi) = -e^{-\xi}\, H(\xi)$$

$$\left(\frac{1}{2\pi i x + a}\right)^{\wedge}(\xi) = e^{a\xi}\, H(-\xi), \quad a>0 \qquad \left(\frac{1}{2\pi i x + 1}\right)^{\wedge}(\xi) = e^{\xi}\, H(-\xi)$$

$$\frac{1}{i}\,\frac{1}{x-i} = \frac{1}{ix+1} \Rightarrow \left(\frac{1}{i} f_1(2\pi x)\right)^{\wedge}(\xi) = \left(\frac{1}{2\pi i x + 1}\right)^{\wedge}(\xi) = e^{\xi}\, H(-\xi)$$

$$\frac{1}{i}\,\frac{1}{x+i} = \frac{1}{ix-1} \Rightarrow \left(\frac{1}{i} f_2(2\pi x)\right)^{\wedge}(\xi) = \left(\frac{1}{2\pi i x - 1}\right)^{\wedge}(\xi) = -e^{-\xi}\, H(\xi)$$

$$\Rightarrow (f_1(2\pi x))^{\wedge}(\xi) = i e^{\xi} H(-\xi), \quad (f_2(2\pi x))^{\wedge}(\xi) = -i e^{-\xi} H(\xi)$$

$$\Rightarrow \hat{f}_1(\xi) = 2\pi i\, e^{2\pi\xi}\, H(-2\pi\xi) = 2\pi i\, e^{2\pi\xi}\, H(-\xi)$$

$$\hat{f}_2(\xi) = -2\pi i\, e^{-2\pi\xi}\, H(2\pi\xi) = -2\pi i\, e^{-2\pi\xi}\, H(\xi)$$

$$(\hat{f}_1 + \hat{f}_2)(\xi) = -2\pi i\, e^{-2\pi|\xi|}\, \mathrm{sign}(\xi)$$

$$\Rightarrow \hat{f}(\xi) = -i\pi\, \mathrm{sign}(\xi)\left[e^{-2\pi i\xi} + 2e^{-2\pi|\xi|}\right]$$

3. $$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = ?$$

$$(\chi_{[-\frac{1}{2},\frac{1}{2}]})^{\wedge}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi} \quad \Leftarrow \quad (\chi_{[a,b]})^{\wedge}(\xi) = \begin{cases} b-a, & \xi = 0 \\ \frac{\sin(\pi(b-a)\xi)}{\pi\xi}\, e^{-i\pi(a+b)}, & \xi \neq 0 \end{cases}$$

Parsevalova jednakost na L^2: $\langle \hat{u} | \hat{v} \rangle = \langle u | v \rangle \Rightarrow \|u\|_{L^2}^2 = \|\hat{u}\|_2^2$

$$u(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x) \Rightarrow \underbrace{\int_{-\infty}^{\infty} \left[\chi_{[-\frac{1}{2},\frac{1}{2}]}(x)\right]^2 dx}_{1} = \int_{-\infty}^{+\infty} \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2 d\xi$$

$$\Rightarrow 1 = \int_{-\infty}^{+\infty} \frac{\sin^2(\pi\xi)}{\pi^2\xi^2}\, d\xi = \left[\begin{aligned} \pi\xi &= x \\ d\xi &= \tfrac{1}{\pi} dx \end{aligned}\right] = \int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, \frac{1}{\pi}\, dx$$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx = \pi}$$

4. $a \in \mathbb{C}$, $f_a : \mathbb{R} \to \mathbb{C}$, $f_a(x) = \frac{2a}{a^2 + 4\pi^2 x^2}$

$a, b \in \mathbb{C}$, $\mathrm{Re}\, a, \mathrm{Re}\, b > 0$, $f_a * f_b = ?$

$$(f_a * f_b)^{\wedge} = \hat{f}_a \cdot \hat{f}_b$$

$$\hat{f}_a(\xi) = e^{-a|\xi|} \quad \Leftarrow \quad (e^{-a|x|})^{\wedge}(\xi) = \frac{2a}{a^2 + 4\pi^2\xi^2}, \quad \mathrm{Re}\, a > 0$$

$$(f_a * f_b)^{\wedge}(\xi) = e^{-(a+b)|\xi|}, \quad \mathrm{Re}(a+b) > 0 \qquad \left(\frac{2a}{a^2 + 4\pi^2 x^2}\right)^{\wedge}(\xi) = e^{-a|\xi|}$$

$$\Rightarrow \boxed{(f_a * f_b)(x) = \frac{2(a+b)}{(a+b)^2 + 4\pi^2 x^2}}$$

5) $$\begin{cases} u_t = 4u_{xx} \\ u(0,x) = e^{2\pi i x} \end{cases} \quad \Big| \mathcal{F} \quad \Big| \mathcal{F}_1 \text{ variable } x$$

$$\hat{\delta}_a(\xi) = e^{-2\pi i a\xi} \qquad \hat{\delta}_a(\xi) = e^{-2\pi i a \xi}$$

$$a=-1 \qquad \hat{\delta}_{-1}(\xi) = e^{2\pi i \xi}$$

$$\hat{\hat{\delta}}_{-1}(x) = (e^{2\pi i \xi})^{\wedge}(x)$$

$$\hat{\hat{\delta}}_{-1}(x) = \delta_{-1}(-x) = \delta_1(x)$$

$$\Rightarrow (e^{2\pi i \xi})^{\wedge}(x) = \delta_1(x)$$

$$\hat{u}_t(t,\xi) = 4\,\widehat{u_{xx}}(t,\xi) = 4\,(2\pi i \xi)^2\,\hat{u}(t,\xi)$$

$$\begin{cases} \hat{u}_t = -16\xi^2 \hat{u} \\ \hat{u}(0,\xi) = \delta_1(\xi) \end{cases} \Rightarrow \hat{u}(t,\xi) = A e^{-16\pi^2\xi^2 t} \qquad \hat{u}(0,\xi) = A$$

$$\Rightarrow \hat{u}(t,\xi) = \delta_1(\xi)\, e^{-16\pi^2\xi^2 t}$$

$$u(t,x) = \overline{\mathcal{F}}\left(\delta_1(\xi)\, e^{-16\pi^2\xi^2 t}\right)(x) = \int_{-\infty}^{\infty} e^{+2\pi i \xi x}\, e^{-16\pi^2\xi^2 t}\, \delta_1(\xi)\, d\xi =$$

$$= \langle \delta_1, e^{-2\pi i \xi x}\, e^{-16\pi^2\xi^2 t} \rangle = e^{+2\pi i \xi x}\, e^{-16\pi^2\xi^2 t}\Big|_{\xi=1} =$$

$$= \boxed{e^{+2\pi i x - 16\pi^2 t}}$$