1.) $$f(x) = \begin{cases} x^2+1 & , \ x<0 \\ x & , \ x \geq 0 \end{cases}$$

a) $$\langle f', \varphi \rangle = -\langle f, \varphi' \rangle = -\int_{-\infty}^{+\infty} f(x)\varphi'(x)\,dx = -\int_{-\infty}^{0} f(x)\varphi'(x)\,dx - \int_{0}^{+\infty} f(x)\varphi'(x)\,dx$$

$$= -f(x)\varphi(x)\Big|_{-\infty}^{0} + \int_{-\infty}^{0} f'(x)\varphi(x)\,dx - f(x)\varphi(x)\Big|_{0}^{+\infty} + \int_{0}^{+\infty} f'(x)\varphi(x)\,dx$$

$$= -f(0-)\varphi(0) + \int_{-\infty}^{0} 2x\varphi(x)\,dx + f(0+)\varphi(0) + \int_{0}^{+\infty} \varphi(x)\,dx$$

$$= \int_{-\infty}^{0} 2x\varphi(x)\,dx + \int_{0}^{+\infty} \varphi(x)\,dx - \varphi(0)$$

$$\Rightarrow \quad f' = g - \delta_0 \quad , \quad g(x) = \begin{cases} 2x & , \ x<0 \\ 1 & , \ x>0 \end{cases}$$

b) δ_0 nije funkcija, tj. ne može se reprezentirati funkcijom (dokazano na vježbama) pa onda ni f' nije funkcija.

c) $xT = f'$

rastavljamo na dvije jednadžbe:

① $xT = 2x + \delta_0$

$x(T-2) = \delta_0$

$S := T-2 \in \mathcal{D}'$

$xS = \delta_0 \Rightarrow S = C\delta_0 - \delta_0'$

$$\Rightarrow \quad T = \underbrace{C\delta_0}_{\text{rješenje homogene j.}} + \underbrace{2 - \delta_0'}_{\text{partikularno rj.}}$$

② $xT = 1 + \delta_0$

$x\left(T - pv\left(\frac{1}{x}\right)\right) = \delta_0$

$S := T - pv\left(\frac{1}{x}\right) \in \mathcal{D}'$

$xS = \delta_0 \Rightarrow S = C\delta_0 - \delta_0'$

$\Rightarrow T = C\delta_0 + pv\left(\frac{1}{x}\right) - \delta_0'$

$$\Rightarrow \quad T = C\delta_0 - \delta_0' + \begin{cases} 2 & , \ x<0 \\ pv\left(\frac{1}{x}\right) & , \ x>0 \end{cases}$$

u smislu nosača distribucije

2.) a) $f(x) = \frac{\sin^2 x}{x^2}$

$$g(x) := \frac{\sin x}{x} \Rightarrow f(x) = g(x) \cdot g(x) \quad /\,\hat{}$$

$$\hat{f}(\xi) = \hat{g}(\xi) * \hat{g}(\xi)$$

Znamo iz ~~...~~ tablice:

$$\hat{g}(\xi) = \pi \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}(\xi)$$

$$(\hat{g} * \hat{g})(\xi) = \int_{-\infty}^{+\infty} \hat{g}(\xi - \eta)\hat{g}(\eta)d\eta = \pi^2 \int_{-\frac{1}{2\pi}}^{\frac{1}{2\pi}} \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}(\xi - \eta)\, d\eta$$

$$= \pi^2 \cdot \begin{cases} \frac{1}{\pi} + x, & -\frac{1}{\pi} < x \leq 0 \\ \frac{1}{\pi} - x, & 0 < x < \frac{1}{\pi} \\ 0, & \text{inače} \end{cases}$$

b) $\begin{cases} u_t = 4u_{xx}, \\ u(0,x) = \frac{\sin^2 x}{x^2}; \end{cases}$

$$u_t = 4u_{xx} \quad /\,\hat{}$$

$$\hat{u}_t = -16\pi^2\xi^2\hat{u}$$

$$\Rightarrow \hat{u}(t,\xi) = C(\xi)\, e^{-16\pi^2\xi^2 t}$$

$$\hat{u}(0,\xi) = C(\xi) = \left(\frac{\sin^2 x}{x^2}\right)^{\hat{}}(\xi) = \hat{f}(\xi) \ \dots \text{ a) dio zadatka}$$

$$\Rightarrow u(t,x) = \bar{\mathcal{F}}\left(e^{-16\pi^2\xi^2 t}\hat{f}(\xi)\right)(x) = \mathcal{F}\left(e^{-16\pi^2\xi^2 t}\hat{f}(\xi)\right)(x)$$

3.) $\begin{cases} u_t = i u_{xx} \\ u(0,x) = \sin x \\ u(t,0) = 0 \end{cases}$

Proširimo f-ju na cijeli $\mathbb{R}$ po neparnosti, tj.

$$v : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{C}$$

$$\left.\begin{array}{l} v(t,x) = u(t,x) \quad , x \geq 0 \\ v(t,\cdot) \text{ neparna} \end{array}\right\} , \ t \in \mathbb{R}^+$$

$$\begin{cases} v_t = i v_{xx} \\ v(0,x) = \sin x \end{cases}$$

$$\hat{v}_t = -4\pi^2 \xi^2 i \hat{v}$$

$$\Rightarrow \hat{v}(t,\xi) = C(\xi)\, e^{-4\pi^2\xi^2 i t}$$

$$C(\xi) = \hat{v}(0,\xi) \underset{\substack{\uparrow \\ \text{formule}}}{=} \frac{1}{2i}\left(\delta_{\frac{1}{2\pi}} - \delta_{-\frac{1}{2\pi}}\right)$$

$$\Rightarrow \hat{v}(t,\xi) = \frac{1}{2i} e^{-4\pi^2\xi^2 i t} \left(\delta_{\frac{1}{2\pi}} - \delta_{-\frac{1}{2\pi}}\right)$$

$$\langle f\delta_a, \varphi \rangle = \langle \delta_a, f\varphi \rangle = \langle \delta_a, f(a)\varphi \rangle = \langle f(a)\delta_a, \varphi \rangle$$

$$\Rightarrow \hat{v}(t,\xi) = \underbrace{\frac{1}{2i} e^{-it}}_{\substack{\text{ne ovisi} \\ \text{o } \xi}} \underbrace{\delta_{\frac{1}{2\pi}}(\xi)}_{\text{ovisi o } \xi} - \underbrace{\frac{1}{2i} e^{-it}}_{\substack{\text{ne} \\ \text{ovisi o } \xi}} \underbrace{\delta_{-\frac{1}{2\pi}}(\xi)}_{\text{ovisi o } \xi}$$

$$v(t,x) = \frac{1}{2i} e^{-it} e^{-2\pi i \frac{1}{2\pi} x} - \frac{1}{2i} e^{-it} e^{-2\pi i \frac{-1}{2\pi} x}$$

$$= \frac{1}{2i} e^{-it}\left(e^{-ix} - e^{ix}\right)$$

$$= \frac{1}{2i} e^{-it}\left(-2i \sin x\right)$$

$$= \underline{-e^{-it} \sin x}$$

$$\boxed{u(t,x) = -e^{-it} \sin x \ , \ \begin{array}{l} t \geq 0 \\ x \geq 0 \end{array}}$$

4.) $\langle \mathrm{Pf}\ \frac{H(x)}{x}, \varphi \rangle = \int_0^a \frac{\varphi(x)-\varphi(0)}{x}\,dx + \varphi(0)\ln a$, $\varphi \in \mathcal{D}(\mathbb{R})$, supp $\varphi \subseteq [-a,a]$.

a) supp $\varphi \subseteq [-b,b]$, $b<a$

$$\langle \mathrm{Pf}\ \frac{H(x)}{x}, \varphi \rangle = \int_0^a \frac{\varphi(x)-\varphi(0)}{x}\,dx + \varphi(0)\ln a$$

$$= \int_0^b \frac{\varphi(x)-\varphi(0)}{x}\,dx + \int_b^a \frac{\overset{=0}{\varphi(x)}-\varphi(0)}{x}\,dx + \varphi(0)\ln a$$

$$= \int_0^b \frac{\varphi(x)-\varphi(0)}{x}\,dx + \int_b^a \frac{-\varphi(0)}{x}\,dx + \varphi(0)\ln a$$

$$= \int_0^b \frac{\varphi(x)-\varphi(0)}{x}\,dx - \cancel{\varphi(0)\ln a} + \varphi(0)\ln b + \cancel{\varphi(0)\ln a}$$

$$= \int_0^b \frac{\varphi(x)-\varphi(0)}{x}\,dx + \varphi(0)\ln b$$

b) $\varphi(x)-\varphi(0) = x\,\varphi'(c_x)$, $c_x \in \langle 0, x\rangle$

$$\Rightarrow \left|\langle \mathrm{Pf}\ \frac{H(x)}{x}, \varphi \rangle\right| \leq \int_0^a |\varphi'(c_x)|\,dx + |\ln a|\,|\varphi(0)|$$

supp $\varphi \subseteq [-a,a]$

$$\leq a\,\|\varphi'\|_{L^\infty} + |\ln a|\,\|\varphi\|_{L^\infty}$$

$$\leq C\left(\|\varphi\|_{L^\infty} + \|\varphi'\|_{L^\infty}\right)$$

C ovisi samo o a, tj. o kompaktu koji sadrži nosač od φ, a ne i o φ

$\Rightarrow$ $\mathrm{Pf}\ \frac{H(x)}{x}$ reda ≤ 1

c) $\varphi \in \mathcal{D}(\mathbb{R})$, $\operatorname{supp}\varphi \subseteq \langle -\infty, 0 \rangle$

$(\exists a > 0)$ $\operatorname{supp}\varphi \subseteq [-a, 0] \subseteq [-a, a]$

$$\langle \operatorname{Pf}\frac{H(x)}{x}, \varphi \rangle = \int_0^a \frac{\overset{=0}{\varphi(x)} - \overset{=0}{\varphi(0)}}{x}\,dx + \underbrace{\varphi(0)}_{=0}\ln a = 0$$

$\Rightarrow$ $\operatorname{supp}\operatorname{Pf}\frac{H(x)}{x} \subseteq [0, +\infty\rangle$

d) $xT' + T = \delta_0$

$(xT)' = \delta_0$

$S := xT$

$$S' = \delta_0 \Rightarrow S = \begin{cases} A, & x < 0 \\ B, & x > 0 \end{cases}$$

imamo skok za 1 pa je zapravo:

$S = H(x) + C$ ⟶ konstanta

$\Rightarrow$ $xT = H(x) + C$

$\Rightarrow$ $x\left(T - \operatorname{Pf}\frac{H(x)}{x} - C\operatorname{Pf}\frac{1}{x}\right) = 0$

$\Rightarrow$ $T - \operatorname{Pf}\frac{H(x)}{x} - C\operatorname{Pf}\frac{1}{x} = D\delta_0$

$\Rightarrow$ $\boxed{T = D\delta_0 + \operatorname{Pf}\frac{H(x)}{x} + C\operatorname{Pf}\frac{1}{x}}$

NAP. Koristili smo $x\operatorname{Pf}\frac{H(x)}{x} = H(x)$ pa i to stavili da se pokaže