1) a) $xT = \delta_0$

Neka je $\theta \in \mathcal{D}(\mathbb{R})$ t.d. $\theta(0) = 1$. Tada $\overbrace{\exists \psi_\varphi \in \mathcal{D}(\mathbb{R})}^{\forall \varphi \in \mathcal{D}(\mathbb{R})}$ t.d.

$$\varphi - \varphi(0)\theta = x\psi_\varphi$$

$$\begin{aligned}
\langle T, \varphi \rangle &= \langle T, \varphi(0)\theta \rangle + \langle T, x\psi_\varphi \rangle \\
&= \langle T, \theta \rangle \overline{\varphi(0)} + \langle xT, \psi_\varphi \rangle \\
&= \langle T, \theta \rangle \langle \delta_0, \varphi \rangle + \overline{\psi_\varphi(0)} \\
&= \langle T, \theta \rangle \langle \delta_0, \varphi \rangle - \overline{\theta'(0)}\,\overline{\varphi(0)} + \overline{\varphi'(0)} \\
&= \underbrace{\left(\langle T, \theta \rangle - \overline{\theta'(0)} \right)}_{C} \langle \delta_0, \varphi \rangle + \langle \delta_0, \varphi' \rangle \\
&= C \langle \delta_0, \varphi \rangle - \langle \delta_0', \varphi \rangle \\
&= \langle C\delta_0 - \delta_0', \varphi \rangle
\end{aligned}$$

$$\begin{aligned}
\psi_\varphi(0) &= (\varphi - \varphi(0)\theta)'(0) \\
&= \varphi'(0) - \varphi(0)\theta'(0)
\end{aligned}$$

$\Rightarrow \quad T = C\delta_0 - \delta_0'$ (T je nužno tog oblika)

Pogledajmo je li to i dovoljan uvjet.

$$\begin{aligned}
\langle x(C\delta_0 - \delta_0'), \varphi \rangle &= C \underbrace{\langle \delta_0, x\varphi \rangle}_{=0} - \langle \delta_0', x\varphi \rangle \\
&= \langle \delta_0, (x\varphi)' \rangle \\
&= \langle \delta_0, x\varphi' + \varphi \rangle \\
&= \langle \delta_0, \varphi \rangle
\end{aligned}$$

$\Rightarrow \quad x(C\delta_0 - \delta_0') = \delta_0$

$\Rightarrow \quad xT = \delta_0 \iff T = C\delta_0 - \delta_0'$, $C =$ konst.

b) $(x-a)T = \delta_b$ KORISTIMO: $\underline{\tau_a \delta_b = \delta_{b+a}}$

$(\tau_a x) T = \delta_b$

$\tau_a (x (\tau_{-a} T)) = \delta_b \quad / \tau_{-a}$

$x(\tau_{-a} T) = \tau_{-a}\delta_b = \delta_{b-a}$

$S := \tau_{-a} T$

$x S = \delta_{b-a}$

1) $\underline{b=a}$

$xS = \delta_0$

a) nam daje

$xS = \delta_0 \Longleftrightarrow S = C\delta_0 - \delta_0'$

$\Rightarrow \quad \tau_{-a} T = C\delta_0 - \delta_0' \; / \tau_a$

$$\boxed{T = C\delta_a - \delta_a'}$$

2) $\underline{b \neq a}$

$xS = \delta_{b-a}$

$\langle S, \varphi \rangle = \langle S, \varphi(0)\theta \rangle + \langle S, x\Psi_\varphi \rangle$

$= \langle S, \theta \rangle \overline{\varphi(0)} + \langle \delta_{b-a}, \Psi_\varphi \rangle$

$= \langle S, \theta \rangle \langle \delta_0, \varphi \rangle + \overline{\Psi_\varphi(b-a)}$

$\sim \cdot \sim$

$\Psi_\varphi(x) = \dfrac{\varphi(x) - \varphi(0)\theta(x)}{x}$

$\Psi_\varphi(b-a) = \dfrac{\varphi(b-a) - \varphi(0)\theta(b-a)}{b-a}$

$= \dfrac{1}{b-a}\varphi(b-a) - \dfrac{\theta(b-a)}{b-a}\varphi(0)$

$\sim \cdot \sim$

$= \langle S, \theta \rangle \langle \delta_0, \varphi \rangle + \dfrac{1}{b-a}\langle \delta_{b-a}, \varphi \rangle -$

$- \dfrac{\overline{\theta(b-a)}}{b-a}\langle \delta_0, \varphi \rangle$

$= \langle C\delta_0 + \dfrac{1}{b-a}\delta_{b-a}, \varphi \rangle$

$\Rightarrow$ S je nužno oblika

$$S = C\delta_0 + \frac{1}{b-a}\delta_{b-a}$$

$$\langle x(C\delta_0 + \frac{1}{b-a}\delta_{b-a}), \varphi\rangle = C\langle\delta_0, x\varphi\rangle + \frac{1}{b-a}\langle\delta_{b-a}, x\varphi\rangle$$
$$= 0 + \frac{1}{b-a}(b-a)\overline{\varphi(b-a)}$$
$$= \overline{\varphi(b-a)}$$
$$= \langle\delta_{b-a}, \varphi\rangle$$

$$\Rightarrow \quad xS = \delta_{b-a} \iff S = C\delta_0 + \frac{1}{b-a}\delta_{b-a}, \quad C = \text{konst.}$$

$$\Rightarrow \quad \tau_{-a}T = C\delta_0 + \frac{1}{b-a}\delta_{b-a} \quad / \tau_a$$
$$\boxed{T = C\delta_a + \frac{1}{b-a}\delta_b}$$

c) $\underline{\underline{b = a}}$

$S := T'$

$S = C\delta_a - \delta_a'$

$T' = C\delta_a - \delta_a'$

$$\Rightarrow \quad T_x = \begin{cases} A, & x < a \\ B, & x > a \end{cases} - \delta_a$$

Skok je C što je proizvoljna konstanta pa su i A i B proizvoljni.

$\underline{\underline{b \neq a}}$

$S := T'$

$S = C\delta_a + \frac{1}{b-a}\delta_b$

$T' = C\delta_a + \frac{1}{b-a}\delta_b$

$$T_x = \begin{cases} A, & x < a \\ B, & a < x < b \\ C, & x > b \end{cases}$$

(BSO) $a < b$

A i B su proizvoljni, a za C imamo:

$C - B = \frac{1}{b-a}$

$$\Rightarrow \boxed{C = \frac{1}{b-a} + B}$$

2.)

a) $$\overline{\mathcal{F}f} = \overline{\int_{-\infty}^{+\infty} e^{-2\pi i x\xi} f(x)\,dx} = \int_{-\infty}^{+\infty} \overline{e^{-2\pi i x \xi} f(x)}\,dx$$

$$= \int_{-\infty}^{+\infty} \overline{e^{-2\pi i x\xi}}\;\overline{f(x)}\,dx \underset{\substack{\uparrow \\ f \text{ realna}}}{=} \int_{-\infty}^{+\infty} e^{2\pi i x \xi} f(x)\,dx$$

$$= \left\{ \begin{matrix} y = -x \\ dy = -dx \end{matrix} \right\} = \int_{-\infty}^{+\infty} e^{-2\pi i y \xi} f(-y)\,dy$$

$$= \begin{cases} \mathcal{F}f, & f \text{ parna} \\ -\mathcal{F}f, & f \text{ neparna} \end{cases}$$

$$\Rightarrow \quad \operatorname{Im} \mathcal{F}f = 0, \quad f \text{ parna}$$
$$\operatorname{Re} \mathcal{F}f = 0, \quad f \text{ neparna}$$

b) $$f(x) = \frac{1}{1+x^2}$$

$$\hat{f}(\xi) = \pi e^{-2\pi|\xi|}$$

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} = \|f\|_{L^2(\mathbb{R})} \underset{\substack{\uparrow \\ \text{Plancharel}}}{=} \|\hat{f}\|_{L^2(\mathbb{R})} = \pi^2 \int_{-\infty}^{+\infty} e^{-4\pi|\xi|}\,d\xi$$

$$= 2\pi^2 \int_0^{+\infty} e^{-4\pi\xi}\,d\xi = 2\pi^2 \left. \frac{e^{-4\pi\xi}}{-4\pi} \right|_0^{+\infty} = \frac{2\pi^2}{4\pi} = \frac{\pi}{2}$$

3)

a) $$\begin{cases} u_{tt} + u_{xxxx} = 0 & \text{u } \mathbb{R}^+ \times \mathbb{R} \\ u(0,\cdot) = u_0 \\ u_t(0,\cdot) = u_1 \end{cases}$$

Djelujemo s $\mathcal{F}_x$ na jednadžbu :

$$\begin{cases} \hat{u}_{tt} + 16\pi^4 \xi^4 \hat{u} = 0 \\ \hat{u}(0,\cdot) = \hat{u}_0 \\ \hat{u}_t(0,\cdot) = \hat{u}_1 \end{cases}$$

$$\hat{u}(t,\xi) = A(\xi) \sin(4\pi^2 \xi^2 t) + B(\xi) \cos(4\pi^2 \xi^2 t)$$

$$\hat{u}_0(\xi) = \hat{u}(0,\xi) = B(\xi)$$

$$\hat{u}_1(\xi) = \hat{u}_t(0,\xi) = 4\pi^2 \xi^2 A(\xi) \Rightarrow A(\xi) = \frac{\hat{u}_1(\xi)}{4\pi^2\xi^2}$$

$$\Rightarrow \hat{u}(t,\xi) = \underbrace{\hat{u}_1(\xi)}_{\in\, \mathcal{S}(\mathbb{R})} \underbrace{\frac{\sin(4\pi^2\xi^2 t)}{4\pi^2\xi^2}}_{\in\, C^\infty(\mathbb{R}^+, \mathcal{O}(\mathbb{R}))} + \underbrace{\hat{u}_0(\xi)}_{\in\, \mathcal{S}(\mathbb{R})} \underbrace{\cos(4\pi^2\xi^2 t)}_{\in\, C^\infty(\mathbb{R}^+; \mathcal{O}(\mathbb{R}))} \in C^\infty(\mathbb{R}^+; \mathcal{S}(\mathbb{R}))$$

funkcije koje ne rastu brže od polinoma

$$\Rightarrow \boxed{u(t,x) = \bar{\mathcal{F}}\left(\hat{u}_1(\xi) \frac{\sin(4\pi^2\xi^2 t)}{4\pi^2\xi^2} + \hat{u}_0(\xi) \cos(4\pi^2\xi^2 t)\right) \in C^\infty(\mathbb{R}^+; \mathcal{S}(\mathbb{R}))}$$

b) $$\hat{u}_1(\xi) = 0, \quad \hat{u}_0(\xi) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2\xi^2}{a}}$$

$$u(t,x) = \sqrt{\frac{\pi}{a}}\, \bar{\mathcal{F}}\Big(\underbrace{e^{-\frac{\pi^2\xi^2}{a}} \cos(4\pi^2\xi^2 t)}_{\text{parna}}\Big) = \sqrt{\frac{\pi}{a}}\, \mathcal{F}\left(e^{-\frac{\pi^2\xi^2}{a}} \cos(4\pi^2\xi^2 t)\right)$$

$$\alpha := \frac{\pi^2}{a}, \quad \beta := 4\pi^2 t > 0, \quad \lambda := \alpha + i\beta$$

$$\Rightarrow u(t,x) = \sqrt{\frac{\pi}{a}} \frac{1}{\sqrt{\frac{\pi^4}{a^2} + 16\pi^4 t^2}} \sqrt{\frac{\pi}{2}}\, e^{-\frac{\pi^4\xi^2}{\frac{\pi^4}{a} + 16\pi^4 a t^2}} \left(\sqrt{|\lambda| + \alpha} \cos\left(\frac{\pi^2\beta}{|\lambda|^2}\xi^2\right) + \sqrt{|\lambda| - \alpha} \sin\left(\frac{\pi^2\beta}{|\lambda|^2}\xi^2\right)\right).$$

4) $\inf J > -\infty$

$\exists (h_n)_n \subseteq H$, $J(h_n) \longrightarrow \inf J$

$(h_n)_n$ je omeđen

Kad ne bi bio, postojao bi podniz za koji vrijedi

$\|h_{n_k}\| \longrightarrow \infty$

Ali tada po c) imamo $J(h_{n_k}) \longrightarrow \infty$ što je u kontradikciji s izborom niza $(h_n)_n$.

Budući da je $(h_n)_n$ omeđen, postoji slabo konvergentan podniz $h_{n_k} \rightharpoonup h \in H$.

b) $\Rightarrow$ $$J(h) \leq \liminf_k J(h_{n_k}) = \lim_n J(h_n) = \inf J$$

podniz ima isti limes

S druge strane mora biti $\inf J \leq J(h)$ pa imamo $J(h) = \inf J$.

$(\forall g \in H) \quad J(h) = \inf J \leq J(g)$ ✓