Prva zadaća: Parcijalne diferencijalne jednadžbe II

1. [2] Neka je $f : \mathbf{R} \to \mathbf{R}$ zadana s

$$f(x) = \begin{cases} \operatorname{arc\,tg} x & , \quad x \leqslant -1 \\ 2e^x - 1 & , \quad -1 < x \leqslant 0 \\ x + 1 & , \quad 0 < x \leqslant 2 \\ \sin(\pi x) & , \quad x > 2 \, . \end{cases}$$

Dokažite $f \in \mathcal{D}'(\mathbf{R})$ i odredite f' i f'' u smislu distribucija.

2. [4] Distribucija $\mathrm{Pf}\frac{1}{x^2}$, konačni dio $\frac{1}{x^2}$, je definirana preko limesa:

$$\langle \mathrm{Pf}\frac{1}{x^2}, \varphi \rangle := \lim_{\varepsilon \to 0} \int_\varepsilon^{+\infty} \frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^2}\, dx \, .$$

(a) Pokazati da je $\mathrm{Pf}\frac{1}{x^2}$ distribucija reda $\leqslant 2$.

(b) Pokazati da je $\langle \mathrm{Pf}\frac{1}{x^2}, \varphi \rangle = \int_{\mathbf{R}} \frac{\varphi(x)}{x^2}\, dx$ za $\varphi \in \mathcal{D}(\mathbf{R})$, $0 \notin \operatorname{supp}\varphi$, tj. na $\mathbf{R} \setminus \{0\}$ se $\mathrm{Pf}\frac{1}{x^2}$ podudara s $\frac{1}{x^2}$.

(c) Pokazati da je produkt $\mathrm{Pf}\frac{1}{x^2}$ s x^2 jednak 1.

(d) Koristeći (c) riješite diferencijalnu jednadžbu u $\mathcal{D}'(\mathbf{R})$:

$$x^2 T = 1 \, .$$

3. [2] Izračunajte $x\delta_0'$, $x^2\delta_0'$ i $x\delta_0''$.

4. [2] Nađite u koji zadovoljava :

$$\begin{cases} -u'' + u = \delta_0 \, , \\ u(-1) = 0 \, , \\ u(1) = 1 \, . \end{cases}$$

→ neki su probali preko Fouriera, ali onda treba još namještati rubne uvjete t.d. se riješi još jedan PDJ

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Marko Erceg

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PDJ 2 – 1. zadaća
ANA ANUŠIĆ 1191212821

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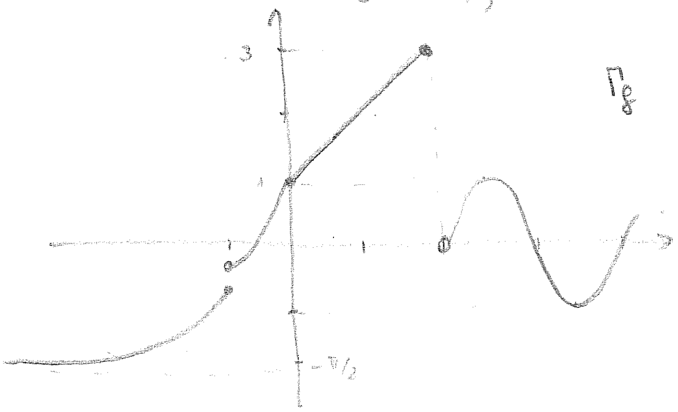
1. $f: \mathbb{R} \to \mathbb{R}$ zadana s:

$$f(x) = \begin{cases} \operatorname{arctg} x, & x \leq -1 \\ 2e^x - 1, & -1 < x \leq 0 \\ x + 1, & 0 < x \leq 2 \\ \sin(\pi x), & x > 2 \end{cases}$$



Očito je $f \in L^1_{loc}(\mathbb{R})$. Naime, f je neprekidna osim u točkama $\{-1, 2\}$. Stoga je jedino potrebno vidjeti da je $\int_K |f| < \infty$ za kompakte K koji sadrže -1 i 2, a to nije teško provjeriti.
Dakle, f definira distribuciju $T_f \in \mathcal{D}'(\mathbb{R})$.

Definiramo

$$g(x) := \begin{cases} \frac{1}{1+x^2}, & x < -1 \\ 2e^x, & -1 < x < 0 \\ 1, & 0 < x < 2 \\ \pi\cos(\pi x), & x > 2 \end{cases}$$

$g \in L^1_{loc}(\mathbb{R})$ pa definira distribuciju $T_g \in \mathcal{D}'(\mathbb{R})$.

$$\langle T_f', \varphi \rangle = -\langle T_f, \varphi' \rangle = -\int_{\mathbb{R}} f\varphi' = -\int_{-\infty}^{-1} (\operatorname{arctg} x)\varphi'(x)dx - \int_{-1}^{0} (2e^x - 1)\varphi'(x)dx -$$

$$-\int_0^2 (x+1)\varphi'(x)dx - \int_2^{+\infty} \sin(\pi x)\varphi'(x)dx = (\text{parc.}) -\operatorname{arctg}(x)\varphi(x)\Big|_{-\infty}^{-1} + \int_{-\infty}^{-1} \frac{\varphi(x)}{1+x^2}dx -$$

$$-(2e^x - 1)\varphi(x)\Big|_{-1}^{0} + \int_{-1}^{0} 2e^x\varphi(x)dx - (x+1)\varphi(x)\Big|_0^2 + \int_0^2 \varphi(x)dx - \sin(\pi x)\varphi(x)\Big|_2^{+\infty}$$

$$+\int_2^{+\infty} \pi\cos(\pi x)\varphi(x)dx = (\varphi \text{ s komp. nosačem}) -\operatorname{arctg}(-1)\varphi(-1) + \int_{-\infty}^{-1} \frac{\varphi(x)}{1+x^2}dx -$$

$$-\cancel{\varphi(0)} + (2e^{-1} - 1)\varphi(-1) + \int_{-1}^{0} 2e^x\varphi(x)dx - 3\varphi(2) + \cancel{\varphi(0)} + \int_0^2 \varphi(x)dx + 0 + \int_2^{+\infty} \pi\cos(\pi x)\varphi(x)dx$$

$$= \langle T_g, \varphi \rangle + (2e^{-1} - 1 - \operatorname{arctg}(-1))\langle \delta_{-1}, \varphi \rangle - 3\langle \delta_2, \varphi \rangle$$

$$\Rightarrow \underline{f' = T_f' = T_g + (2e^{-1} - 1 - \operatorname{arctg}(-1))\delta_{-1} - 3\delta_2}.$$

Definiramo

$$h(x) := \begin{cases} -\frac{2x}{(1+x^2)^2}, & x < -1 \\ 2e^x, & -1 < x < 0 \\ 0, & 0 < x < 2 \\ -\pi^2\sin(\pi x), & x > 2 \end{cases}$$

$h \in L^1_{loc}(\mathbb{R})$ pa definira distribuciju $T_h \in \mathcal{D}'(\mathbb{R})$.

$$\langle T_f'', \varphi \rangle = -\langle T_f', \varphi' \rangle = -\langle T_g, \varphi' \rangle - \left(\frac{2}{e} - 1 - \operatorname{arctg}(-1)\right)\langle \delta_{-1}, \varphi' \rangle - 3\langle \delta_2, \varphi' \rangle$$

$$\langle T_g, \varphi' \rangle = -\int_{-\infty}^{-1} \frac{\varphi'(x)}{1+x^2}dx - \int_{-1}^{0} 2e^x\varphi'(x)dx - \int_0^2 \varphi'(x)dx - \int_2^{+\infty} \pi\cos(\pi x)\varphi'(x)dx =$$

$$-\frac{\varphi(x)}{1+x^2}\Big|_{-\infty}^{-1} + \int_{-\infty}^{-1} -\frac{\varphi(x)\cdot 2x}{(1+x^2)^2}dx - 2e^x\varphi(x)\Big|_{-1}^{0} + \int_{-1}^{0} 2e^x\varphi(x)dx - \varphi(x)\Big|_{0}^{2} - \pi\cos(\pi x)\varphi(x)\Big|_{2}^{+\infty} +$$

$$+\int_{2}^{+\infty} -\pi^2\sin(\pi x)\varphi(x)dx = -\frac{\varphi(-1)}{2} - 2\varphi(0) + \frac{2}{e}\varphi(-1) - \varphi(2) + \varphi(0) + \pi\varphi(2) + \langle T_h,\varphi\rangle$$

$$= \langle T_h,\varphi\rangle + \left(\frac{2}{e}-\frac{1}{2}\right)\langle\delta_{-1},\varphi\rangle - \langle\delta_0,\varphi\rangle + (\pi-1)\langle\delta_2,\varphi\rangle$$

$$\Rightarrow T_g' = T_h + \left(\frac{2}{e}-\frac{1}{2}\right)\delta_{-1} - \delta_0 + (\pi-1)\delta_2$$

$$\Rightarrow \underline{T_g'' = T_h + \left(\frac{2}{e}-\frac{1}{2}\right)\delta_{-1} - \delta_0 + (\pi-1)\delta_2 + \left(\frac{2}{e} - 1 - \operatorname{arctg}(-1)\right)\delta_{-1}' + 3\delta_2'}\,.$$

✓

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3.

(i) $\langle x\delta_0', \varphi\rangle = \langle\delta_0', x\varphi\rangle = -\langle\delta_0, (x\varphi)'\rangle = -\langle\delta_0, \varphi(x)+x\varphi'(x)\rangle =$

$= -\varphi(0) - 0\cdot\varphi'(0) = -\varphi(0) = -\langle\delta_0,\varphi\rangle \quad , \forall\varphi\in\mathcal{D}(\mathbb{R})$

$\Rightarrow \underline{x\delta_0' = -\delta_0}$.

(ii) $\langle x^2\delta_0', \varphi\rangle = \langle\delta_0', x^2\varphi\rangle = -\langle\delta_0, (x^2\varphi)'\rangle = -\langle\delta_0, 2x\varphi(x)+x^2\varphi'(x)\rangle$

$= -2\cdot 0\cdot\varphi(0) - 0\cdot\varphi'(0) = 0 \quad , \forall\varphi\in\mathcal{D}(\mathbb{R})$

$\Rightarrow \underline{x^2\delta_0' = 0}$.

✓

(iii) $\langle x\delta_0'', \varphi\rangle = \langle\delta_0'', x\varphi\rangle = \langle\delta_0, (x\varphi)''\rangle = \langle\delta_0, 2\varphi'(x)+x\varphi''(x)\rangle =$

$= 2\varphi'(0) + 0\cdot\varphi''(0) = 2\varphi'(0) = \langle -2\delta_0', \varphi\rangle , \forall\varphi\in\mathcal{D}(\mathbb{R})$

$\Rightarrow \underline{x\delta_0'' = -2\delta_0'}$. ✓

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2. $\langle \mathrm{Pf}\frac{1}{x^2}, \varphi\rangle := \lim_{\varepsilon\to 0}\int_{\varepsilon}^{+\infty}\frac{\varphi(x)+\varphi(-x)-2\varphi(0)}{x^2}dx$

(a) Neka su $\alpha\in\mathbb{R}$, $\varphi,\psi\in\mathcal{D}(\mathbb{R})$. Tada

$\langle \mathrm{Pf}\frac{1}{x^2}, \alpha\varphi+\psi\rangle = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{+\infty}\frac{\alpha\varphi(x)+\psi(x)+\alpha\varphi(-x)+\psi(-x)-2\alpha\varphi(0)-2\psi(0)}{x^2}dx =$

$= \alpha\lim_{\varepsilon\to 0}\int_{\varepsilon}^{+\infty}\frac{\varphi(x)+\varphi(-x)-2\varphi(0)}{x^2}dx + \lim_{\varepsilon\to 0}\int_{\varepsilon}^{+\infty}\frac{\psi(x)+\psi(-x)-2\psi(0)}{x^2}dx = \alpha\langle\mathrm{Pf}\frac{1}{x^2},\varphi\rangle + \langle\mathrm{Pf}\frac{1}{x^2},\psi\rangle$

dakle, $\mathrm{Pf}\frac{1}{x^2}$ je linearan funkcional.

Primijetimo da je svaki kompaktan skup u $\mathbb{R}$ sadržan u skupu oblika $[-a,a]$. Stoga, neka je $K\in\mathcal{K}(\mathbb{R})$, $K\subseteq[-a,a]$, te neka je $\varphi\in\mathcal{D}(\mathbb{R})$ t.d. $\mathrm{supp}\,\varphi\subseteq[-a,a]$.

Primjenom teorema srednje vrijednosti dobivamo da $\exists c_1\in[0,x]$, $\exists c_2\in[-x,0]$ t.d. $\varphi(x)-\varphi(0)-(\varphi(0)-\varphi(-x)) = x\varphi'(c_1) - x\varphi'(c_2) = x(\varphi'(c_1)-\varphi'(c_2))$. Ponovo primjenom teorema srednje vrijednosti

dobivamo da postoji $c_3 \in [c_2, c_1]$ takav da je $\varphi'(c_1) - \varphi'(c_2) = (c_1 - c_2)\varphi''(c_3)$.
Stoga, jer je $c_1 - c_2 \le 2x$, imamo

$$\varphi(x) - \varphi(0) + \varphi(-x) - \varphi(0) \le 2x^2 \varphi''(c_3) \Rightarrow \left| \frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^2} \right| \le 2 \|\varphi''\|_{L^\infty([-x,x])}$$

Prema tome, jer je $\operatorname{supp}\varphi \subset [-a,a]$, imamo

$$|\langle \mathrm{Pf}\tfrac{1}{x^2}, \varphi \rangle| \le 2\|\varphi''\|_{L^\infty(K)} \lim_{\varepsilon \to 0} \int_\varepsilon^a dx = 2a \|\varphi''\|_{L^\infty(K)}.$$

Iz toga zaključujemo da je $\mathrm{Pf}\frac{1}{x^2}$ distribucija reda ≤ 2.

(b) Neka je $\varphi \in \mathcal{D}(\mathbb{R})$ t.d. $0 \notin \operatorname{supp}\varphi \Rightarrow \varphi(0) = 0$. Tada je

$$\langle \mathrm{Pf}\tfrac{1}{x^2}, \varphi \rangle = \lim_{\varepsilon \to 0} \int_\varepsilon^{+\infty} \frac{\varphi(x) + \varphi(-x)}{x^2} dx = \lim_{\varepsilon \to 0} \int_{|x| \ge \varepsilon} \frac{\varphi(x)}{x^2} dx.$$

Kako je $\varphi(0) = 0$, onda integral $\int_{\mathbb{R}} \frac{\varphi(x)}{x^2} dx$ postoji i vrijedi

$$0 \le \left| \int_{\mathbb{R}} \frac{\varphi(x)}{x^2} dx - \int_{|x| \ge \varepsilon} \frac{\varphi(x)}{x^2} dx \right| = \left| \int_{|x| < \varepsilon} \frac{\varphi(x)}{x} \right| \le \int_{|x| < \varepsilon} \left| \frac{\varphi(x)}{x} \right| dx \xrightarrow{\varepsilon \to 0} 0$$

$$\Rightarrow \lim_{\varepsilon \to 0} \int_{|x| \ge \varepsilon} \frac{\varphi(x)}{x^2} dx = \int_{\mathbb{R}} \frac{\varphi(x)}{x^2} dx.$$

(c) Za $\varphi \in \mathcal{D}(\mathbb{R})$ imamo:

$$\langle x^2 \mathrm{Pf}\tfrac{1}{x^2}, \varphi \rangle = \langle \mathrm{Pf}\tfrac{1}{x^2}, x^2\varphi \rangle = \lim_{\varepsilon \to 0} \int_\varepsilon^{+\infty} \frac{x^2\varphi(x) + x^2\varphi(-x)}{x^2} dx = \lim_{\varepsilon \to 0} \int_{|x| \ge \varepsilon} \varphi(x) dx =$$

$$= (\varphi \in C^\infty(\mathbb{R})) \int_{\mathbb{R}} \varphi(x) dx = \langle 1, \varphi \rangle \Rightarrow x^2 \mathrm{Pf}\tfrac{1}{x^2} = 1$$

(d) $x^2 T = 1. \Rightarrow x^2 T = x^2 \mathrm{Pf}\frac{1}{x^2} \Rightarrow x^2(T - \mathrm{Pf}\frac{1}{x^2}) = 0$

Označimo sa $S := T - \mathrm{Pf}\frac{1}{x^2} \in \mathcal{D}'(\mathbb{R})$. Tada je $x^2 S = 0$.

Označimo sa $U := xS \in \mathcal{D}'(\mathbb{R})$. Iz zad. 8. s vježbi slijedi da je $U = C\delta_0$, gdje je $C \in \mathbb{C}$ konstanta.

Rješavamo $xS = C\delta_0$.

Neka je $\theta \in \mathcal{D}(\mathbb{R})$ t.d. $\theta(0) = 1$. Tada $(\forall \varphi \in \mathcal{D}(\mathbb{R}))(\exists \psi_\varphi \in \mathcal{D}(\mathbb{R}))$ t.d. $\varphi = \varphi(0)\theta + x\psi_\varphi$ (ovo smo pokazali na vježbama.)

Neka je $\varphi \in \mathcal{D}(\mathbb{R})$. Tada je $\psi_\varphi(0) = \chi'(0)$, pri čemu je $\chi = \varphi - \varphi(0)\theta$.

$\Rightarrow \psi_\varphi(0) = \varphi'(0) - \varphi(0)\theta'(0)$.

Imamo: $\langle S, \varphi \rangle = \langle S, \varphi(0)\theta + x\psi_\varphi \rangle = \overline{\varphi(0)} \langle S, \theta \rangle + \langle \overbrace{xS}^{C\delta_0}, \psi_\varphi \rangle =$

$$= \overline{\varphi(0)} \langle S, \theta \rangle + C\overline{\psi_\varphi(0)} = \overline{\varphi(0)} \langle S, \theta \rangle + C\overline{\varphi'(0)} - C\overline{\varphi(0)}\,\overline{\theta'(0)}$$

$$= \overline{\varphi(0)} \underbrace{(\langle S, \theta \rangle - C\overline{\theta'(0)})}_{=: D \in \mathbb{C}} + C\overline{\varphi'(0)}$$

$$= D\langle \delta_0, \varphi \rangle - C\langle \delta_0', \varphi \rangle \Rightarrow S = D\delta_0 - C\delta_0' \Rightarrow$$

$\Rightarrow \underline{T = \mathrm{Pf}\frac{1}{x^2} - D\delta_0 + C\delta_0'}$, $C, D \in \mathbb{C}$.

Provjerimo je li $\forall C, D$ ovo rješenje:

$$\langle x^2 T, \varphi \rangle = \langle x^2 \mathrm{Pf}\tfrac{1}{x^2}, \varphi \rangle - D\langle x^2\delta_0, \varphi \rangle + C\langle x^2\delta_0', \varphi \rangle \overset{(c)}{=} \langle 1, \varphi \rangle - D\langle \delta_0, \overline{x^2\varphi} \rangle +$$

$$+ C\langle \delta_0', x^2\varphi \rangle = \langle 1, \varphi \rangle - D \cdot 0\overline{\varphi(0)} - C\langle \delta_0, 2x\varphi + x^2\varphi' \rangle = \langle 1, \varphi \rangle - C \cdot \overline{2 \cdot 0\varphi(0) + 0 \cdot \varphi'(0)}$$

$= \langle 1, \varphi \rangle, \forall \varphi \in \mathcal{D}(\mathbb{R})$. ☒

4. $$\begin{cases} -u'' + u = \delta_0 \\ u(-1) = 0 \\ u(1) = 1 \end{cases}$$

Rješenje.

Primijetimo prvo da mora vrijediti

$$(-u''+u)(x) = \begin{cases} 0, & x<0 \\ 0, & x>0 \end{cases}$$

Dakle, $u(x) = \begin{cases} c_1 e^{x} + c_2 e^{-x}, & x<0 \\ c_3 e^{x} + c_4 e^{-x}, & x>0 \end{cases}$, gdje su $c_1, c_2, c_3, c_4 \in \mathbb{C}$ konstante.

Mora vrijediti $\langle T_u'', \varphi \rangle + \langle \delta_0, \varphi \rangle = \langle T_u, \varphi \rangle$, $\forall \varphi \in \mathcal{D}(\mathbb{R})$

$$\Rightarrow \langle T_u, \varphi'' \rangle + \varphi(0) = \langle T_u, \varphi \rangle$$

$$\langle T_u, \varphi'' \rangle + \varphi(0) = c_1 \int_{-\infty}^{0} e^{x} \varphi''(x)\,dx + c_2 \int_{-\infty}^{0} e^{-x} \varphi''(x)\,dx + c_3 \int_{0}^{+\infty} e^{x} \varphi''(x)\,dx + c_4 \int_{0}^{+\infty} e^{-x} \varphi''(x)\,dx$$

$$+ \varphi(0) = c_1 \left(e^{x}\varphi'(x)\Big|_{-\infty}^{0} - e^{x}\varphi(x)\Big|_{-\infty}^{0} + \int_{-\infty}^{0} e^{x}\varphi(x)\,dx \right) + c_2 \left(e^{-x}\varphi'(x)\Big|_{-\infty}^{0} + e^{-x}\varphi(x)\Big|_{-\infty}^{0} + \right.$$

$$\left. + \int_{-\infty}^{0} e^{-x}\varphi(x)\,dx \right) + c_3 \left(e^{x}\varphi'(x)\Big|_{0}^{+\infty} - e^{x}\varphi(x)\Big|_{0}^{+\infty} + \int_{0}^{+\infty} e^{x}\varphi(x)\,dx \right) +$$

$$c_4 \left(e^{-x}\varphi'(x)\Big|_{0}^{+\infty} + e^{-x}\varphi(x)\Big|_{0}^{+\infty} + \int_{0}^{+\infty} e^{-x}\varphi(x)\,dx \right) + \varphi(0)$$

$= (\varphi$ s kompaktnim nosačem$)$ $c_1(\varphi'(0) - \varphi(0)) + c_2(\varphi'(0) + \varphi(0)) +$

$$+ c_3(-\varphi'(0) + \varphi(0)) + c_4(-\varphi'(0) - \varphi(0)) + \varphi(0) + \langle T_u, \varphi \rangle$$

Dakle, tražimo konstante takve da

$$(c_1 + c_2 - c_3 - c_4)\varphi'(0) + (-c_1 + c_2 + c_3 - c_4 + 1)\varphi(0) = 0 \qquad \forall \varphi \in \mathcal{D}(\mathbb{R})$$

$$\Rightarrow \quad c_1 + c_2 - c_3 - c_4 = 0 \qquad (1)$$

$$-c_1 + c_2 + c_3 - c_4 + 1 = 0 \qquad (2)$$

Još imamo uvjete: $u(-1) = c_1 e^{-1} + c_2 e = 0$ (3)

$u(1) = c_3 e + c_4 e^{-1} = 1$ (4)

$(1)+(2) \Rightarrow 2c_2 = 2c_4 - 1 \Rightarrow c_2 = c_4 - \frac{1}{2} \Rightarrow c_4 = c_2 + \frac{1}{2}$

$(1)-(2) \Rightarrow 2c_1 = 2c_3 + 1 \Rightarrow c_1 = c_3 + \frac{1}{2} \Rightarrow c_3 = c_1 - \frac{1}{2} = -c_2 e^2 - \frac{1}{2}$

$(3) \Rightarrow c_1 = -c_2 e^2$

$\Rightarrow (4)$ $-c_2 e^3 - \frac{1}{2} e + c_2 e^{-1} + \frac{1}{2} e^{-1} = 1 \Rightarrow c_2(e^{-1} - e^3) = 1 + \frac{1}{2} e - \frac{1}{2} e^{-1} \Rightarrow$

$$c_2 = \frac{1 + \frac{1}{2} e - \frac{1}{2} e^{-1}}{e^{-1} - e^3} = \frac{e + \frac{1}{2} e^2 - \frac{1}{2}}{1 - e^4}$$

$$\Rightarrow c_1 = -e^2 \frac{e + \frac{1}{2} e^2 - \frac{1}{2}}{1 - e^4}, \quad c_3 = -e^2 \frac{e + \frac{1}{2} e^2 - \frac{1}{2}}{1 - e^4} - \frac{1}{2}, \quad c_4 = \frac{e + \frac{1}{2} e^2 - \frac{1}{2}}{1 - e^4} + \frac{1}{2}$$

Iz gornjeg računa zaključujemo da u uz ovakve konstante zadovoljava početnu jednadžbu.

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