1) Metodom karakteristika riješite Cauchyjevu zadaću

$$\begin{cases} u x u_x - y u u_y = y^2 - x^2 & \text{u } \mathbb{R}^2 \\ u(x,x) = x^2+1 \end{cases}$$

Rj: Uvjet je zadan na krivulji $S = \{(x,x) : x \in \mathbb{R}\}$, pa je normala na S u točki (x,x) dana s

$$\nu(x,x) = \nu = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Ispitajmo postoje li karakteristične točke:

$$\begin{bmatrix} u(x,x)\,x \\ -u(x,x)\,x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = xu + xu = 2xu(x,x) = 2x(x^2+1) = 0 \iff x = 0$$

$\Rightarrow$ ishodište $(0,0)$ je jedina karakt. točka   2

Odredimo karakteristike:

$$\boxed{\begin{aligned} x' &= zx \\ y' &= -zy \\ z' &= y^2 - x^2 \end{aligned}}$$

$\rightsquigarrow$ ovo je komplicirano za računati pa ćemo cijelu jednadžbu podijeliti s u

$$u x u_x - y u u_y = y^2 - x^2 \quad /: u \neq 0$$

$$\boxed{x u_x - y u_y = \frac{y^2 - x^2}{u}}$$

Ove jednadžbe nisu ekvivalentne samo ako je u u nekoj točki 0. Neka je $u(x_0,y_0)=0$ za neku $(x_0,y_0) \in \mathbb{R}^2$. Tada iz početne jed. slijedi

$$0 \cdot x_0 u_x(x_0,y_0) - y_0 \cdot 0 \cdot u_y(x_0,y_0) = y_0^2 - x_0^2$$

$$\Rightarrow \quad 0 = y_0^2 - x_0^2$$

$$\Rightarrow \quad 0 = (y_0 - x_0)(y_0 + x_0)$$

Dakle, u može biti 0 samo na pravcima $y=x$ i $y=-x$. Na pravcu $y=x$ se to ne može dogoditi jer je $u(x,x) = x^2+1 > 0$, a pravac $y=-x$ siječe S samo u $(0,0)$ koja je karakteristična pa nam nije bitno da promijenimo jednadžbu oko $(0,0)$ jer je upitno hoćemo li uopće dobiti rj. oko ishodišta.

Tada karakteristike zadovoljavaju jednostavniji sustav ODJ:

$$x' = x \Rightarrow x(\tau) = C_1 e^{\tau}$$

$$y' = -y \Rightarrow y(\tau) = C_2 e^{-\tau}$$ 4

$$z' = \frac{y^2 - x^2}{z} \Rightarrow \frac{1}{2}(z^2)' = C_2^2 e^{-2\tau} - C_1^2 e^{2\tau} \Rightarrow z^2(\tau) = -C_2^2 e^{-2\tau} - C_1^2 e^{2\tau} + C_3$$

Odredimo c_1, c_2, c_3 t.d. $(x(0), y(0), z(0)) = (x_0, x_0, x_0^2+1) \in$ ~~Sx~~ $\{(x,y,u(x,y)) : (x,y) \in S$

$$x_0 = x(0) = c_1$$
$$x_0 = y(0) = c_2$$

$$(x_0^2+1)^2 = z(0)^2 = -x_0^2 - x_0^2 + c_3 \Rightarrow c_3 = 2x_0^2 + (x_0^2+1)^2$$

$$\boxed{\begin{aligned} x(\tau) &= x_0 e^{\tau} \\ y(\tau) &= x_0 e^{-\tau} \\ z^2(\tau) &= -x_0^2\left(e^{2\tau} + e^{-2\tau}\right) + 2x_0^2 + (x_0^2+1)^2 \end{aligned}} \qquad 2$$

Neka je $(x,y) \in \mathbb{R}^2$ proizvoljna. Odredimo parametre x_0 i τ t.d. $(x(\tau), y(\tau)) = (x,y)$

$$\left.\begin{aligned} x &= x_0 e^{\tau} \\ y &= x_0 e^{-\tau} \end{aligned}\right\} \Rightarrow xy = x_0^2 \geq 0$$

$$\Downarrow$$
$$|x_0| = \sqrt{xy}$$

uočavamo da je $xy \geq 0$ nužno da projicirana karak. prolazi kroz (x,y), tj. (x,y) mora biti iz I ili III ~~kvadrata~~ kvadranta

Za $x, y \neq 0$ je $x_0 \neq 0$ pa imamo

$$e^{\tau} = \frac{x}{x_0} \Rightarrow e^{2\tau} = \frac{x^2}{x_0^2} = \frac{x^2}{xy} = \frac{x}{y}$$
$$\Rightarrow e^{-2\tau} = \frac{y}{x}$$

τ se u $z(\tau)$ javlja samo kao $e^{2\tau}$ i $e^{-2\tau}$ pa nam je to dovoljno

$$\boxed{x_0 = \text{sign}(x)\sqrt{xy}}$$

jer iz $x = x_0 e^{\tau}$ slijedi da su x i x_0 istog predznaka.

$x = 0 \Rightarrow x_0 = 0 \Rightarrow y = 0$ pa imamo trivijalnu karak. $(x(\tau), y(\tau)) = (0,0)$.

Dakle, na x-osi i y-osi ~~projicir~~ jedino kroz $(0,0)$ postoji projicirana karak. koja prolazi tom točkom.

$$\Rightarrow u^2(x,y) = z^2(\tau; x_0) = -xy\left(\frac{x}{y} + \frac{y}{x}\right) + 2xy + (xy+1)^2 \qquad 6$$
$$= -x^2 - y^2 + 2xy + (xy+1)^2 = (xy+1)^2 - (x-y)^2$$
$$= \underbrace{(xy+1-x+y)(xy+1+x-y)}$$

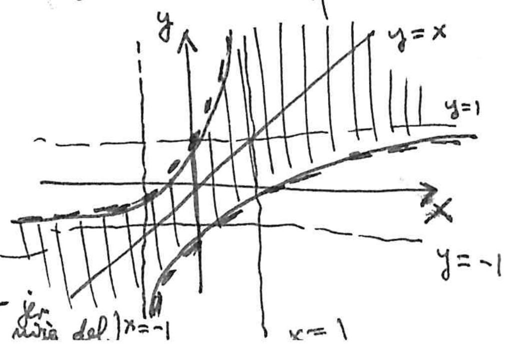
ovo mora biti ≥ 0 da nj. ovom formulom bude dobro def

$$\Rightarrow u(x,y) = \sqrt{(xy+1)^2 - (x-y)^2}$$ za (x,y) na koje je izraz pod $\sqrt{\ }$ ≥ 0.

uzeli smo predznak + jer $u(x,x) = x^2+1$.

lako se provjeri da je nj. početne jednadžbe gdje je dobro def.

ovo je okolina skupa S na kojoj je rješenje dobro def (nb isključujemo jer na nbi der. nije def.) 3

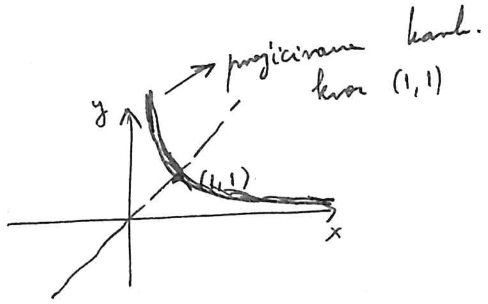


Projicirana karak. kroz $(1,1)$:

$1 \cdot 1 \geq 0$ ✓

$x_0 = 1$

$$\left.\begin{array}{l} x(\tau) = e^{\tau} \\ y(\tau) = e^{-\tau} \end{array}\right\} \Rightarrow y = \frac{1}{x}$$



B

Projicirana karak. kroz $\left(\frac{1}{2}, -\frac{1}{6}\right)$:

$\frac{1}{2} \cdot \left(-\frac{1}{6}\right) < 0$ $\S\S$ ne postoji projicirana karak.

Međutim, u točki $\left(\frac{1}{2}, -\frac{1}{6}\right)$ je definirana y.

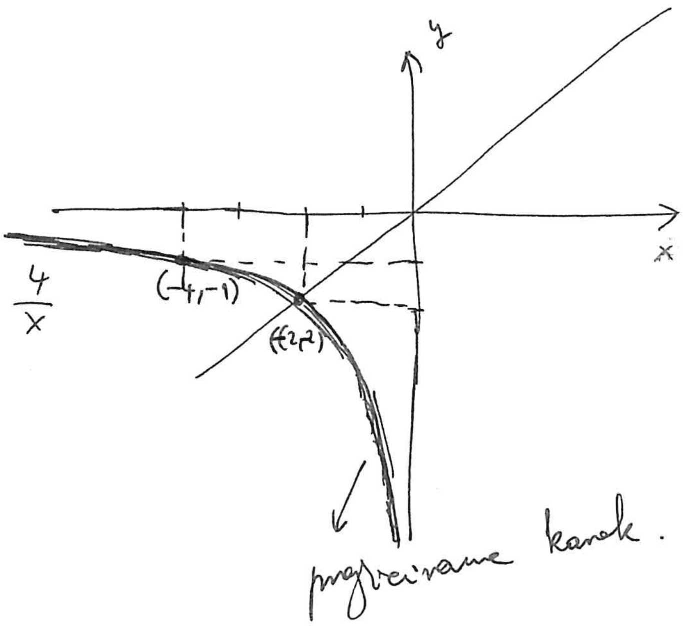
Projicirana karak. kroz $(-4,-1)$:

$x_0 = \operatorname{sign}(-4)\sqrt{4} = -2$

$x(\tau) = -2e^{\tau} \Rightarrow e^{\tau} = -\frac{x}{2}$

$y(\tau) = -2e^{-\tau}$

$y = -2\left(-\frac{2}{x}\right) = \frac{4}{x}$



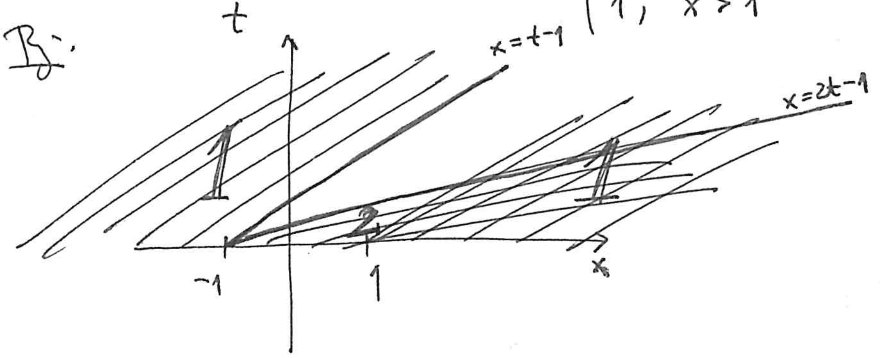
2) Odredite entropijsko rješenje Cauchyjeve zadaće u $\langle 0,\infty\rangle\times\mathbb{R}$

$$\begin{cases} u_t + u u_x = 0 \\ u(0,\cdot) = g \end{cases},$$

gdje je

$$g(x) = \begin{cases} 1, & x < -1 \\ 2, & -1 \leq x \leq 1 \\ 1, & x > 1 \end{cases}.$$

Rj:



projicirane karak. su dane s $\underline{x(t) = g(x_0)t + x_0}$

(I) U području $\{(t,x)\in\mathbb{R}^+\times\mathbb{R} : t-1 < x < 2t-1\}$ nemamo karak. pa postavljamo rj. s

$$u(t,x) = \frac{x+1}{t}.$$

$u(t,x)\big|_{2t-1} = 2$ ✓ , $u(t,x)\big|_{t-1} = 1$ ✓

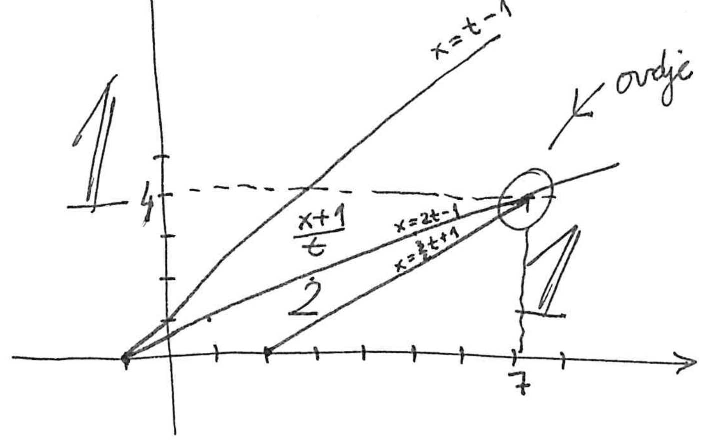
(II) R–H uvjet u $(t,x) = (0,1)$

$$\left.\begin{aligned} u_l &= 2 \\ u_r &= 1 \end{aligned}\right\} [u] = 1 \quad \Rightarrow \quad \dot{s} = \frac{3}{2} \quad \Rightarrow \quad s(t) = \frac{3}{2}t + C$$

$$\left.\begin{aligned} F_l &= 2 \\ F_r &= \frac{1}{2} \end{aligned}\right\} [F] = \frac{3}{2} \qquad 1 = s(0) = C$$

$$\Rightarrow \boxed{x = s(t) = \frac{3}{2}t + 1}$$

Trenutna situacija:



ovdje opet računamo R-H uvjet

III R-H uvjet u $(t,x) = (4,7)$

$$\left.\begin{aligned} u_l &= \frac{s+1}{t} \\ u_r &= 1 \end{aligned}\right\} \quad [u] = \frac{s+1}{t} - 1$$

$$\left.\begin{aligned} F_l &= \frac{1}{2}\left(\frac{s+1}{t}\right)^2 \\ F_r &= \frac{1}{2} \end{aligned}\right\} \quad [F] = \frac{1}{2}\left(\frac{s+1}{t} - 1\right)\left(\frac{s+1}{t} + 1\right)$$

$$\Rightarrow \left(\frac{s+1}{t} - 1\right)\dot{s} = \frac{1}{2}\left(\frac{s+1}{t} - 1\right)\left(\frac{s+1}{t} + 1\right) \quad /: \frac{s+1}{t} - 1 \neq 0$$

$$\dot{s} = \frac{s}{2t} + \frac{1}{2t} + \frac{1}{2}$$

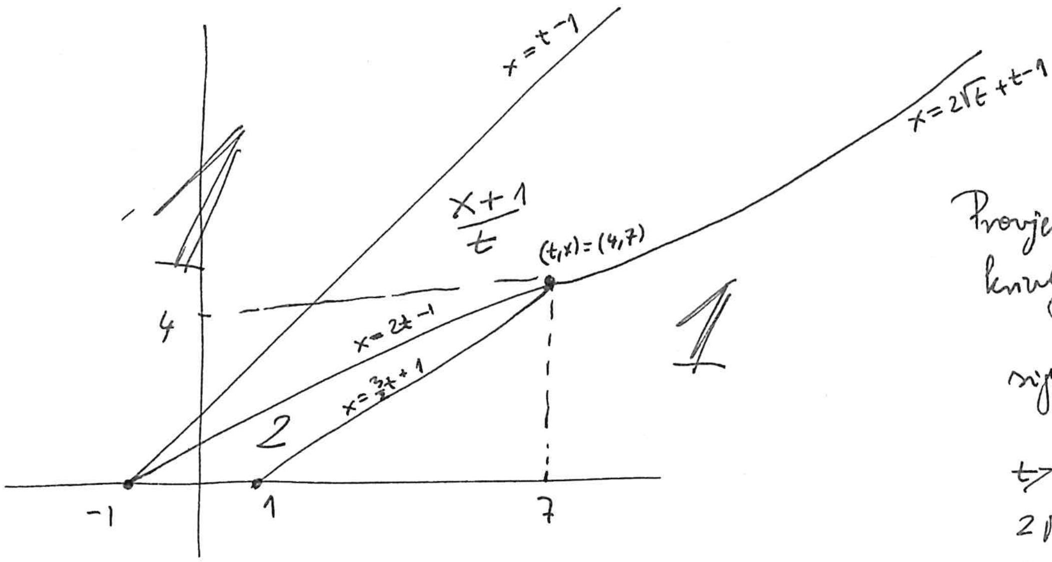
HOMOGENA: $\dot{s} = \frac{s}{2t} \Rightarrow s(t) = C\sqrt{t}$

PARTIKULARNO RJ.: $t - 1$

$$\Rightarrow s(t) = C\sqrt{t} + t - 1$$

$$7 = s(4) = 2C + 3 \Rightarrow \underline{C = 2} \quad \Rightarrow \boxed{x = 2\sqrt{t} + t - 1}$$

Konačno:



Provjera da se krivulje $x = t - 1$ i $x = 2\sqrt{t} + t - 1$ ne sijeku za $t > 4$:

$$\cancel{t-1} = 2\sqrt{t} + \cancel{t-1}$$

$$2\sqrt{t} = 0$$

$$\Rightarrow t = 0 \quad \checkmark$$

4) Dana je početno-rubna zadaća na $\mathbb{R}^+ \times \mathbb{R}^+$

$$\begin{cases} u_t - u_{xx} = 0 \\ u(0,x) = u_0(x) \\ u(t,0) = 0 \end{cases},$$

gdje je $u_0 : [0,+\infty\rangle \to \mathbb{R}$ ~~dan~~ jednaka

a) $\cos x - x$

b) $\sin x - x$.

Ispitajte u kojem slučaju su rubni i početni uvjet kompatibilni, te u tom slučaju odredite rj.

Rj: U (a) uvjeti nisu kompatibilni jer $u_0(0) = 1 \neq 0$, dok pod (b) jesu.

Također, pod (b) proširujemo u_0 po neparnosti:

$$\tilde{u}_0 : \mathbb{R} \to \mathbb{R}, \quad \tilde{u}_0(x) = \sin x - x .$$

Ako je u rj. početne zadaće, a s $\tilde{u}$ označimo proširenje po neparnosti, tada $\tilde{u}$ zadovoljava

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} = 0 \\ \tilde{u}(0,x) = \tilde{u}_0 \end{cases}.$$

Sada odredimo $\tilde{u}$ i onda je $u = \tilde{u}\big|_{[0,+\infty\rangle}$.

Lako se izravnim računom dobiva da je $\tilde{u}(t,x) = e^{-t}\sin x - x$,
pa je onda $\underline{u(t,x) = e^{-t}\sin x - x, \quad t \geq 0,\ x \geq 0}$.

5) Za $f \in C^2([0,\infty\rangle \times \mathbb{R}^3)$, $g \in C^3(\mathbb{R}^3)$, $h \in C^2(\mathbb{R}^3)$ i $a \in \mathbb{R}$ pokažite da zadaća

$$(*)\begin{cases} u_{tt} - \Delta u + a u_t - a u_{x_1} = f & \text{u } \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0,\cdot) = g \\ u_t(0,\cdot) = h \end{cases}$$

ima jedinstveno rj. i izvedite eksplicitnu formulu za rješenje.

UPUTA: Svedite jed. na valnu jednadžbu.

Rj. Odredimo $A, B \in \mathbb{R}$ t.d. $v(t,x) := u(t,x)\, e^{At + Bx_1}$ zadovoljava valnu jednadžbu (mogli smo v tražiti u obliku $u(t,x)\, e^{At + \vec{B}\cdot x}$, ali bi se dobilo da $\vec{B}$ ima samo prvu komponentu netrivijalnu), gdje u zadovoljava početnu jednadžbu.

$$v_{tt}(t,x) = u_{tt}(t,x)\, e^{At+Bx_1} + 2A u_t(t,x)\, e^{At+Bx_1} + A^2 u(t,x)\, e^{At+Bx_1}$$

$$v_{x_1x_1}(t,x) = u_{x_1x_1}(t,x)\, e^{At+Bx_1} + 2B u_{x_1}(t,x)\, e^{At+Bx_1} + B^2 u(t,x)\, e^{At+Bx_1}$$

$$v_{x_kx_k}(t,x) = u_{x_kx_k}(t,x)\, e^{At+Bx_1}, \quad k \neq 1$$

$$\Rightarrow \Delta v(t,x) = \Delta u(t,x)\, e^{At+Bx_1} + 2B u_{x_1}(t,x)\, e^{At+Bx_1} + B^2 u(t,x)\, e^{At+Bx_1}$$

$$\Rightarrow v_{tt} - \Delta v = e^{At+Bx_1}\left(u_{tt} - \Delta u + 2A u_t - 2B u_{x_1} + (A^2 - B^2) u\right)$$

$$\left.\begin{aligned} 2A &= a \\ -2B &= -a \end{aligned}\right\} \Rightarrow A = B = \frac{a}{2} \Rightarrow A^2 - B^2 = 0$$

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$$\Rightarrow \boxed{v(t,x) = u(t,x)\, e^{\frac{a}{2}(t+x_1)}}$$

Ako u zadovoljava $(*)$, tada v zadovoljava

$$(**)\begin{cases} v_{tt} - \Delta v = e^{\frac{a}{2}(t+x_1)} f \\ v(0,\cdot) = e^{\frac{a}{2}x_1} g \\ v_t(0,\cdot) = e^{\frac{a}{2}x_1} h + \frac{a}{2} e^{\frac{a}{2}x_1} g. \end{cases}$$

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10

Dokažimo najprije da $(*)$ ima jedinstveno rj.

Neka su u_1, u_2 dva rj. $(*)$ i definirajmo $\tilde{u} := u_1 - u_2$.

Tada $\tilde{v}(t,x) := \tilde{u}(t,x)\, e^{\frac{a}{2}(t+x_1)}$ zadovoljava

$$\begin{cases} \tilde{v}_{tt} - \Delta\tilde{v} = 0 \\ v(0,\cdot) = 0 \\ v_t(0,\cdot) = 0 \end{cases} \Rightarrow \tilde{v} \equiv 0 \Rightarrow \tilde{u} \equiv 0 \quad ✓$$

Odredimo eksplicitno rj. za (**). Rastavimo problem na dva:

$$(**_1)\quad \begin{cases} v_{tt} - \Delta v = 0 \\ v(0,\cdot) = e^{\frac{a}{2}x_1} g \\ v_t(0,\cdot) = e^{\frac{a}{2}x_1} h + \frac{a}{2} e^{\frac{a}{2}x_1} g \end{cases} \qquad\qquad (**_2)\quad \begin{cases} v_{tt} - \Delta v = e^{\frac{a}{2}(t+x_1)} f \\ v(0,\cdot) = 0 \\ v_t(0,\cdot) = 0 \end{cases}$$

Kirchhoffova formula — Duhamelovo načelo

Rješenje $(**_1)$:

$$v_1(t,x) = \fint_{S(x,t)} \Big(t e^{\frac{a}{2}y_1} h(y) + t \tfrac{a}{2} e^{\frac{a}{2}y_1} g(y) + e^{\frac{a}{2}y_1} g(y) + e^{\frac{a}{2}y_1} \nabla g(y)\cdot(y-x) + \tfrac{a}{2} e^{\frac{a}{2}y_1} g(y)\,(y_1 - x_1) \Big)\, dS_y$$

Rješenje $(**_2)$:

$$v_2(t,x) = \int_0^t w(t,x;s)\,ds \qquad \begin{cases} w_{tt}(\cdot;s) - \Delta w(\cdot;s) = 0 \\ w(s,\cdot;s) = 0 \\ w_t(s,\cdot;s) = f(s,\cdot)\, e^{\frac{a}{2}(s+x_1)} \end{cases}$$

$$= \int_0^t w^s(t-s,x)\,ds \qquad \begin{cases} w(t,x;s) = w^s(t-s,x), \text{ gdje} \\ w^s_{tt} - \Delta w^s = 0 \\ w^s(0,\cdot) = 0 \\ w^s_t(0,x) = f(s,x)\, e^{\frac{a}{2}x_1 + \frac{a}{2}s} \end{cases}$$

Kirchhoff

$$= \int_0^t \fint_{S(x,t-s)} (t-s)\, e^{\frac{a}{2}(s+y_1)} f(s,y)\, dS_y$$

Rješenje (**): $v = v_1 + v_2$.

Konačno rj. je dano s:

$$\begin{aligned} u(t,x) &= e^{-\frac{a}{2}(t+x_1)}\, v(t,x) \\ &= e^{-\frac{a}{2}(t+x_1)} \fint_{S(x,t)} \Big(t e^{\frac{a}{2}y_1} h(y) + t\tfrac{a}{2} e^{\frac{a}{2}y_1} g(y) + e^{\frac{a}{2}y_1} g(y) + e^{\frac{a}{2}y_1} \nabla g(y)\cdot(y-x) + \tfrac{a}{2} e^{\frac{a}{2}y_1} g(y)(y_1-x_1) \Big)\, dS_y \\ &\quad + e^{-\frac{a}{2}(t+x_1)} \int_0^t \fint_{S(x,t-s)} (t-s)\, e^{\frac{a}{2}(s+y_1)} f(s,y)\, dS_y \end{aligned}$$