1) a) $(u^2-2yu-y^2)u_x + (xy+xu)u_y = xy - xu$

$$\frac{dx}{u^2-2yu-y^2} = \frac{dy}{xy+xu} = \frac{du}{xy-xu}$$

I $\quad \lambda = 0, \quad \mu = \nu = 1$

$$(xy+xu) + (xy-xu) = 2xy$$

$$\Rightarrow \quad \frac{dy+du}{2xy} = \frac{dy}{x(y+u)}$$

$$\Rightarrow \quad \frac{d(y+u)}{2y} = \frac{dy}{y+u}$$

$$\Rightarrow \quad (y+u)\,d(y+u) = 2y\,dy$$

$$\Rightarrow \quad \frac{1}{2}(y+u)^2 = y^2 + C \quad \Rightarrow \quad 2C = \boxed{u^2 + 2yu - y^2 =: \varphi(x,y,u)}$$

II $\quad \lambda = x, \ \mu = y, \ \nu = u$

$$x(u^2-2yu-y^2) + y(xy+xu) + u(xy-xu)$$

$$= \cancel{xu^2} - \cancel{2xyu} - \cancel{xy^2} + \cancel{xy^2} + \cancel{xyu} + \cancel{xyu} - \cancel{xu^2} = 0$$

$$\left.\begin{array}{l} \psi_x = x \\ \psi_y = y \\ \psi_u = u \end{array}\right\} \Rightarrow \ldots \Rightarrow \boxed{\psi(x,y,u) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{u^2}{2}}$$

$$\frac{\partial(\varphi,\psi)}{\partial(x,y,u)} = \begin{bmatrix} 0 & 2(u-y) & 2(u+y) \\ x & y & u \end{bmatrix} \rightsquigarrow \left.\begin{array}{l} \det \dfrac{\partial(\varphi,\psi)}{\partial(x,y)} = -2x(u-y) \\ \det \dfrac{\partial(\varphi,\psi)}{\partial(x,u)} = -2x(u+y) \end{array}\right\}$$

ukoliko $x \neq 0$ tada barem jedna determinanta nije nula

$$\Rightarrow \quad \boxed{F\left(u^2+2yu-y^2, \frac{x^2}{2} + \frac{y^2}{2} + \frac{u^2}{2}\right) = 0} \quad (F \in C^1(\mathbb{R}^2) \text{ proizvoljna})$$

je dano opće rješenje gornje jednadžbe na $(\mathbb{R}\setminus\{0\}) \times \mathbb{R}$.

---

ukoliko je $x=0$ iz jednadžbe vidimo da mora vrijediti

$$(u^2(0,y) - 2yu(0,y) - y^2)\,u_x(0,y) = 0,$$

pa je $\boxed{u^2(0,y) - 2yu(0,y) - y^2 = 0}$ ili $\boxed{u_x(0,y) = 0}$.

1) b) $2y\,u_x + u\,u_y = 2yu^2$

$$\frac{dx}{2y} = \frac{dy}{u} = \frac{du}{2yu^2}$$

(I) $\dfrac{dx}{2y} = \dfrac{du}{2yu^2}$

$\Rightarrow dx = \dfrac{du}{u^2}$ $\quad (u \neq 0)$

$\Rightarrow x = -\dfrac{1}{u} + C \Rightarrow C = \boxed{x + \dfrac{1}{u} =: \varphi(x,y,u)}$

(II) $\dfrac{dy}{u} = \dfrac{du}{2yu^2}$

$2y\,dy = \dfrac{du}{u}$ $\quad (u \neq 0)$

$\Rightarrow y^2 = \ln|u| + C \Rightarrow C = \boxed{y^2 - \ln|u| =: \psi(x,y,u)}$

$$\frac{\partial(\varphi,\psi)}{\partial(x,y,u)} = \begin{bmatrix} 1 & 0 & -\frac{1}{u^2} \\ 0 & 1 & -\frac{1}{u} \end{bmatrix} \rightsquigarrow$$ očito je matrica uvijek ranga 2

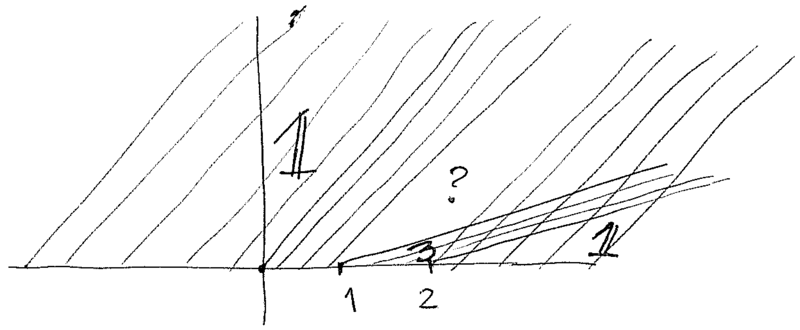
$\Rightarrow$ s $\boxed{F\left(x + \frac{1}{u}, y^2 - \ln|u|\right) = 0}$ $\quad (F \in C^1(\mathbb{R}^2)$ proizvoljna$)$
su dana sva rješenja, osim $u \equiv 0$ (odnosno rješenja koja poprimaju vrijednost 0).

2) $\begin{cases} u_t + uu_x = 0 \\ u(0,\cdot) = g \end{cases}$, $g(x) = \begin{cases} 1, & x<1 \\ 3, & 1 \le x \le 2 \\ 1, & x>2 \end{cases}$

(Projicirane) karakteristike su oblika

$x(t) = g(x_0)t + x_0$, a po karakteristici je rj. jednako $u(t, x(t)) = g(x_0)$,

pa imamo



(I) U području gdje nema karakteristika proširimo rješenje s

$$\boxed{u(t,x) = \frac{x-1}{t}} .$$

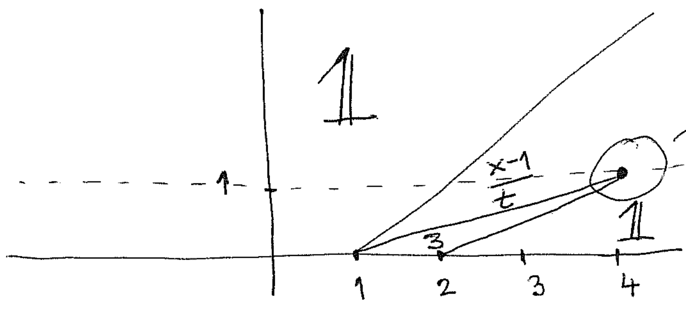
To je dobro jer: • $u(t, t+1) = \frac{t+1-1}{t} = 1$ ✓

• $u(t, 3t+1) = \frac{3t+1-1}{t} = 3$ ✓

(podudara se vrijednost sa susjednim područjima)

(II) Karakteristike se sijeku u točki $(t,x) = (0,2)$.

$$\left.\begin{array}{l} \left.\begin{array}{l} u_l = 3 \\ u_r = 1 \end{array}\right\} \Rightarrow [u] = 2 \\ \left.\begin{array}{l} F_l = \frac{9}{2} \\ F_r = \frac{1}{2} \end{array}\right\} \Rightarrow [F] = 4 \end{array}\right\} \Rightarrow \begin{array}{l} 2\dot{s} = 4 \\ \dot{s} = 2 \Rightarrow s(t) = 2t + C \\ 2 = s(0) = C \\ \Rightarrow \boxed{s(t) = 2t+2} \end{array}$$



područje s vrijednosti 3 je „nestalo" pa sad trebamo razdvojiti područja s vrijednosti $\frac{x-1}{t}$ i 1.

III Krivulja šoka u točki $(t,x) = (1,4)$.

$$\left.\begin{aligned} u_l &= \frac{s(t)-1}{t} \\ u_r &= 1 \end{aligned}\right\} \Rightarrow [u] = \frac{s(t)-1}{t} - 1$$

$$\left.\begin{aligned} F_l &= \frac{1}{2}\left(\frac{s(t)-1}{t}\right)^2 \\ F_r &= \frac{1}{2} \end{aligned}\right\} \Rightarrow [F] = \frac{1}{2}\left(\left(\frac{s(t)-1}{t}\right)^2 - 1\right)$$

$$= \frac{1}{2}\left(\frac{s(t)-1}{t} - 1\right)\left(\frac{s(t)-1}{t} + 1\right)$$

$$\Rightarrow \left(\frac{s-1}{t} - 1\right)\dot{s} = \frac{1}{2}\left(\frac{s-1}{t} - 1\right)\left(\frac{s-1}{t} + 1\right) \quad \Big/ : \left(\frac{s-1}{t} - 1\right)$$

provjerit ćemo i ovaj uvjet

$$\dot{s} = \frac{1}{2}\left(\frac{s-1}{t} + 1\right)$$

$$\boxed{\dot{s} = \frac{s}{2t} - \frac{1}{2t} + \frac{1}{2}} \leftarrow \text{linearna ODJ}$$

$+$

$$\boxed{s(1) = 4}$$

homogena: $\dot{s} = \frac{s}{2t}$

$$\int \frac{ds}{s} = \int \frac{dt}{2t}$$

$$\ln|s| = \ln\sqrt{t} + C$$

$$\Rightarrow s = C\sqrt{t}$$

varijacija konstanti:

$$s(t) = C(t)\sqrt{t}$$

$$C'\sqrt{t} + \cancel{\frac{C}{2\sqrt{t}}} = \cancel{\frac{C}{2\sqrt{t}}} - \frac{1}{2t} + \frac{1}{2}$$

$$C' = -\frac{1}{2}t^{-3/2} + \frac{1}{2}t^{-\frac{1}{2}}$$

$$C(t) = t^{-1/2} + t^{1/2} + D$$

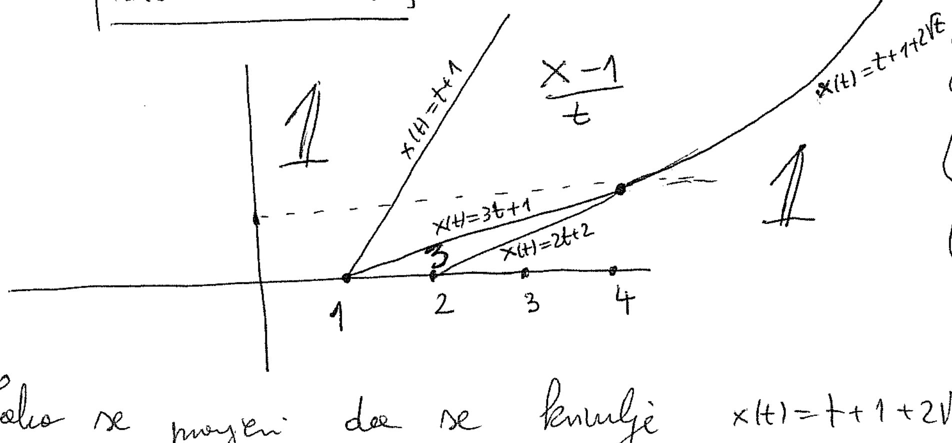
$$= \sqrt{t} + \frac{1}{\sqrt{t}} + D$$

$$\Rightarrow s_p(t) = t + 1$$

$$\Rightarrow s(t) = t + 1 + C\sqrt{t}$$

$$4 = s(1) = 2 + C \Rightarrow C = 2$$

$$\boxed{s(t) = t + 1 + 2\sqrt{t}}$$



ako je $\frac{s-1}{t} - 1 = 0$, onda je $s = t+1$, a ta krivulja ne zadovoljava uvjet $s(1) = 4$.

Lako se provjeri da se krivulje $x(t) = t+1+2\sqrt{t}$ i $x(t) = t+1$ ne sijeku za $t > 0$ pa je gornja skica konačno rješenje.

3) $\begin{cases} u_t + u u_x = 0 \\ u(0,\cdot) = g \end{cases}$

a) $g(x) = -x$

Projicirane karakteristike su oblika

$$x(t;x_0) = -x_0 t + x_0 .$$

Očito sve karakteristike prolaze točkom $(t,x) = (1,0)$, dok se do trenutka $t=1$ ne sijeku.

$\boxed{t \in [0,1\rangle}$

Za $x \in \mathbb{R}$ iz $x = -x_0 t + x_0$ slijedi $x_0 = \frac{x}{1-t}$, pa je rješenje dano s

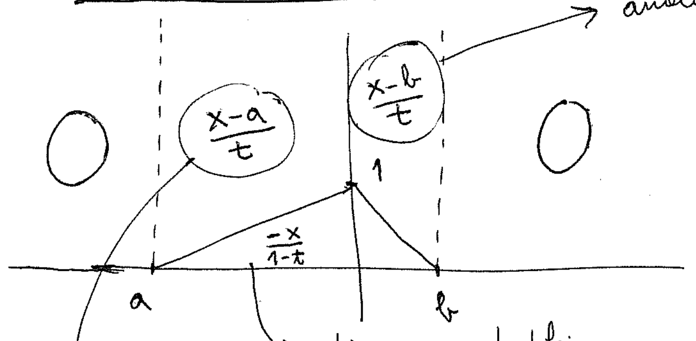
$$\boxed{u(t,x) = g(x_0) = \frac{-x}{1-t}}, \quad x \in \mathbb{R}.$$

Iz gornje formule je jasno da rješenje ne možemo proširiti za $t \geq 1$ jer za fiksan $x \neq 0$ imamo

$$|u(t,x)| \xrightarrow{t \to 1^-} +\infty .$$

b) Nije teško uočiti da se rješenje razlikuje ovisno jesu li a i b istog predznaka ili nisu (kasnije ćemo to i preciznije istaknuti).

I $\quad a < 0 < b$



$\frac{x-b}{t}$ → analogno kao za $\frac{x-a}{t}$

ovdje smo odredili rješenje u a) podzadatku

u ovom području nema karakteristika pa proširujemo rj. t.d. je zadovoljen entropijski uvjet;
$\frac{x-a}{t}$ je dobar odabir jer za $x=a$ dobivamo 0, dok na pravcu $x = -at + a$ dobivamo $-a$ što se podudara s $\frac{-(-at+a)}{1-t} = \frac{at-a}{1-t} = -a\left(\frac{1-t}{1-t}\right) = -a$ ✓.

→ ovim dodefiniranjem rj. dobivamo problem u $(t,x) = (1,0)$ jer se sijeku karakteristike područja $\frac{x-a}{t}$ i $\frac{x-b}{t}$ pa računamo R-H uvjet.

$$\left.\begin{aligned} u_\ell &= \frac{s(t)-a}{t} \\ u_r &= \frac{s(t)-b}{t} \end{aligned}\right\} [u] = \frac{b-a}{t}$$

R-H uvjet u $(t,x)=(1,0)$

$$\left.\begin{aligned} F_\ell &= \frac{1}{2}\left(\frac{s(t)-a}{t}\right)^2 \\ F_r &= \frac{1}{2}\left(\frac{s(t)-b}{t}\right)^2 \end{aligned}\right\} [F] = \frac{1}{2}\cdot\frac{b-a}{t}\cdot\frac{2s(t)-(a+b)}{t}$$

$$\Rightarrow \quad \cancel{\frac{b-a}{t}}\,\dot{s} = \frac{1}{2}\cdot\cancel{\frac{b-a}{t}}\cdot\frac{2s-(a+b)}{t}$$

$$\dot{s} = \frac{s}{t} - \frac{a+b}{2t}$$

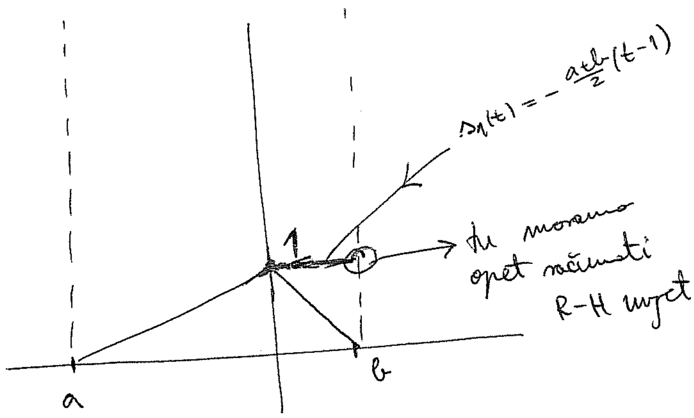
$$\int\frac{ds}{s-\frac{a+b}{2}} = \int\frac{dt}{t}$$

$$\Rightarrow \quad s - \frac{a+b}{2} = Ct$$

$$\Rightarrow \quad s(t) = Ct + \frac{a+b}{2}$$

$$0 = s(1) = C + \frac{a+b}{2} \Rightarrow C = -\frac{a+b}{2}$$

$$\Rightarrow \boxed{s_1(t) = -\frac{a+b}{2}(t-1)}$$



skica odgovara slučaju $a+b<0$ $(-a>b)$

I_1 $\quad a<0<b$ & $-a>b$

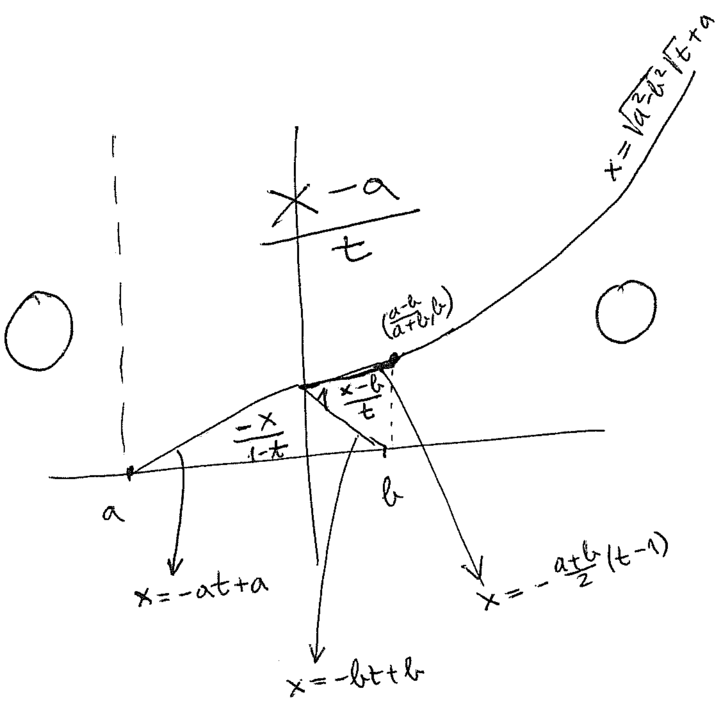
Promatramo sjecište pravaca

$$\begin{cases} x = -\frac{a+b}{2}(t-1) \\ x = b \end{cases}$$

$$b = -\frac{a+b}{2}t + \frac{a+b}{2}$$

$$b-a = -(a+b)t$$

$$t = \frac{a-b}{a+b} > 1$$



R-H uvjet u $(t,x)=\left(\frac{a-b}{a+b}, b\right)$

$$\left.\begin{aligned} u_\ell &= \frac{s(t)-a}{t} \\ u_r &= 0 \end{aligned}\right\} [u] = \frac{s(t)-a}{t}$$

$$\left.\begin{aligned} F_\ell &= \frac{1}{2}\left(\frac{s(t)-a}{t}\right)^2 \\ F_r &= 0 \end{aligned}\right\} [F] = \frac{1}{2}\left(\frac{s(t)-a}{t}\right)^2$$

$$\frac{s-a}{t}\,\dot{s} = \frac{1}{2}\left(\frac{s-a}{t}\right)^2 \Big/ : \frac{s-a}{t} \neq 0$$

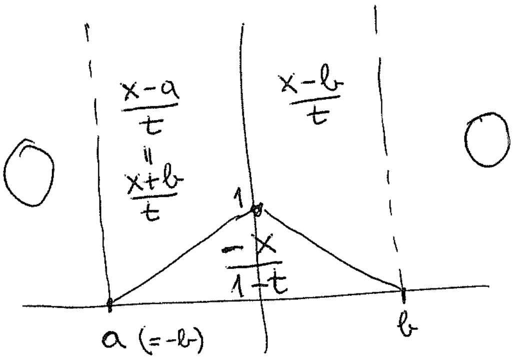
$$\dot{s} = \frac{1}{2t}(s-a)$$

$$\Rightarrow \quad s(t) = C\sqrt{2t} + a$$

$$b = s\left(\frac{a-b}{a+b}\right) = C\sqrt{\frac{2(a-b)}{a+b}} + a \Rightarrow C = \sqrt{\frac{a^2-b^2}{2}} \Rightarrow \boxed{s_2(t) = \sqrt{a^2-b^2}\sqrt{t} + a}$$

I_2 $a<0<b$ & $-a=b$

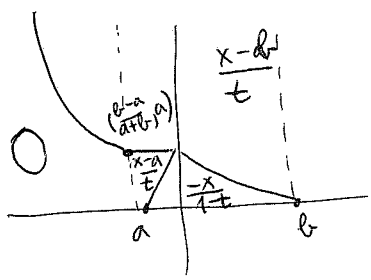
U ovom slučaju je $s_1(t)=0$ pa više nema presjeka karakteristika i rješenje je dano s



I_3 $a<0<b$ & $-a<b$

Analogno kao I_1 samo što promatramo presjek pravaca

$$\begin{cases} x = -\frac{a+b}{2}(t-1) \\ x = a \end{cases}$$



R-H uvjet u $(t,x) = \left(\frac{b-a}{a+b}, a\right)$

$$\left.\begin{aligned} u_l &= 0 \\ u_r &= \frac{s(t)-b}{t} \\ F_l &= 0 \\ F_r &= \frac{1}{2}\left(\frac{s(t)-b}{t}\right)^2 \end{aligned}\right\} \Rightarrow \dot{s} = \frac{1}{2t}(s-b)$$

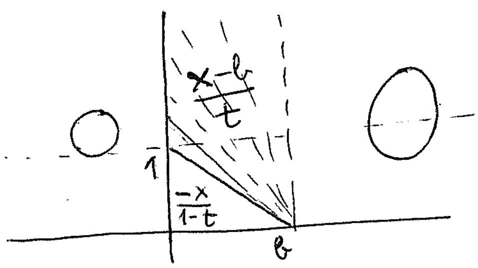
$$\Rightarrow s(t) = C\sqrt{2t} + b$$

$$a = s\left(\frac{b-a}{a+b}\right) = C\sqrt{\frac{2(b-a)}{a+b}} + b$$

$$\Rightarrow \boxed{s_2(t) = -\sqrt{b^2-a^2}\sqrt{t} + b}$$

II $a=0$ ili $b=0$

Ukoliko je $a=0$ tada imamo



karakteristike se sijeku u $(t,x)=(1,0)$

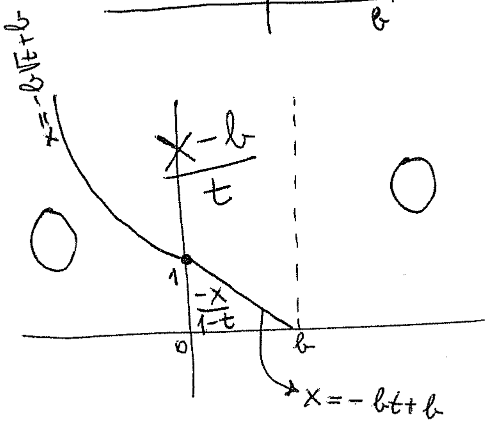
$$\left.\begin{aligned} u_l &= 0 \\ u_r &= \frac{s(t)-b}{t} \\ F_l &= 0 \\ F_r &= \frac{1}{2}\left(\frac{s(t)-b}{t}\right)^2 \end{aligned}\right\} \Rightarrow \dot{s} = \frac{1}{2t}(s-b)$$

$\Rightarrow$ $s(t) = C\sqrt{2t} + b$

(račun kao u I_3)

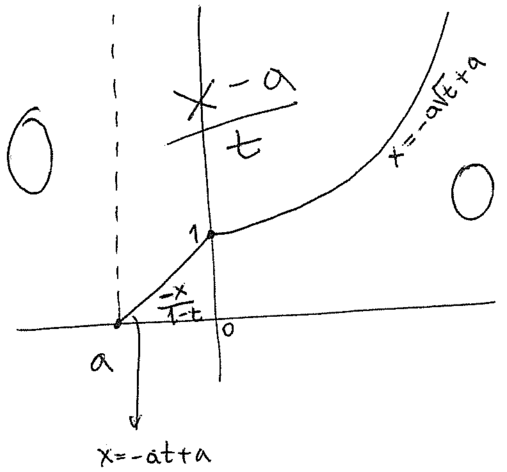
$$0 = s(1) = C\sqrt{2} + b \Rightarrow C = -\frac{b}{\sqrt{2}}$$

$$\Rightarrow \boxed{s(t) = -b\sqrt{t} + b}$$



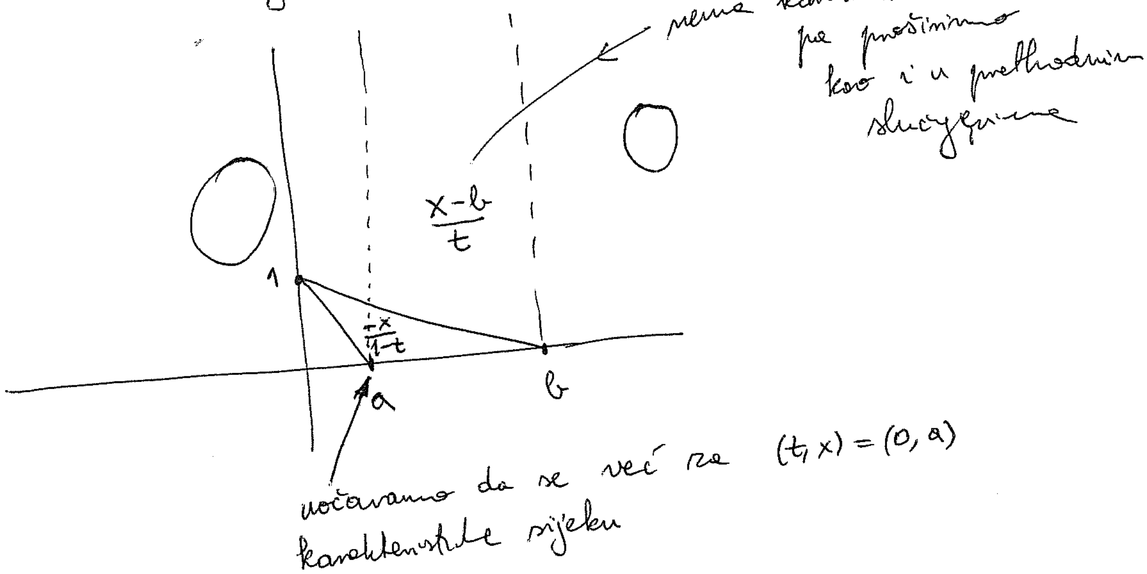
rješenje za $a=0$

Analogno za $b=0$ imamo:



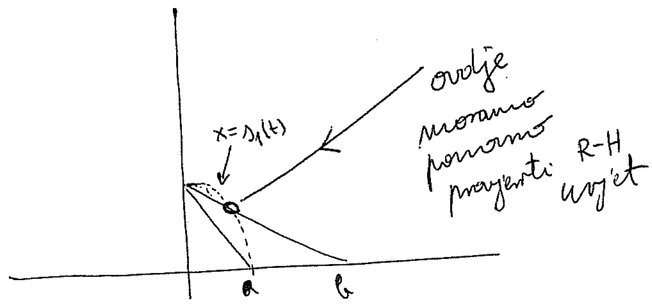
III $0<a<b$ ili $a<b<0$

Promotrimo slučaj $0<a<b$, dok će slučaj $a<b<0$ biti u potpunosti analogan.



R-H uvjet u $(t,x)=(0,a)$

$$\left.\begin{aligned} u_l &= 0 \\ u_r &= -\frac{s(t)}{1-t} \end{aligned}\right\} \quad [u] = \frac{s(t)}{1-t}$$



$$\left.\begin{aligned} F_l &= 0 \\ F_r &= \frac{1}{2}\left(\frac{s(t)}{1-t}\right)^2 \end{aligned}\right\} \quad [F] = -\frac{1}{2}\left(\frac{s(t)}{1-t}\right)^2$$

$$\Rightarrow \quad \frac{s}{1-t}\dot{s} = -\frac{1}{2}\left(\frac{s}{1-t}\right)^2 \quad \Big/ : \frac{s}{1-t} \neq 0$$

$$\dot{s} = -\frac{s}{2(1-t)}$$

$$\left.\begin{aligned} \int \frac{ds}{s} = -\int \frac{dt}{2(1-t)} \Rightarrow \ln|s| = \ln\sqrt{1-t} + C \Rightarrow s(t) &= C\sqrt{1-t} \\ a = s(0) &= C \end{aligned}\right\} \Rightarrow \boxed{s_1(t) = a\sqrt{1-t}}$$

Odredimo najprije presjek krivulje

$$\begin{cases} x = -bt+b \\ x = a\sqrt{1-t} \end{cases} \qquad (t \leq 1)$$

$$a\sqrt{1-t} = b(1-t) \quad \Big/ : \sqrt{1-t}$$

$$a = b\sqrt{1-t}$$

$$1-t = \frac{a^2}{b^2}$$

$$t = 1-\frac{a^2}{b^2} \Rightarrow x = \frac{a^2}{b} \left(< \frac{a^2}{a} = a\right)$$

$$\left[t = 1-\frac{a^2}{b^2} > 1-\frac{b^2}{b^2} = 0 \Rightarrow \underline{\underline{t > 0}} \right]$$

R-H uvjet u $(t,x) = \left(1-\frac{a^2}{b^2}, \frac{a^2}{b}\right)$

$$\left.\begin{array}{l} \left.\begin{array}{l} u_l = 0 \\ u_r = \frac{s(t)-b}{t} \end{array}\right\} [u] = -\frac{s(t)-b}{t} \\ \left.\begin{array}{l} F_l = 0 \\ F_r = \frac{1}{2}\left(\frac{s(t)-b}{t}\right)^2 \end{array}\right\} [F] = -\frac{1}{2}\left(\frac{s(t)-b}{t}\right)^2 \end{array}\right\} \Rightarrow \frac{s-b}{t}\,\dot{s} = \frac{1}{2}\left(\frac{s-b}{t}\right)^2 \quad \Big/ : \frac{s-b}{t} \neq 0$$

$$\dot{s} = \frac{1}{2}\,\frac{s-b}{t}$$

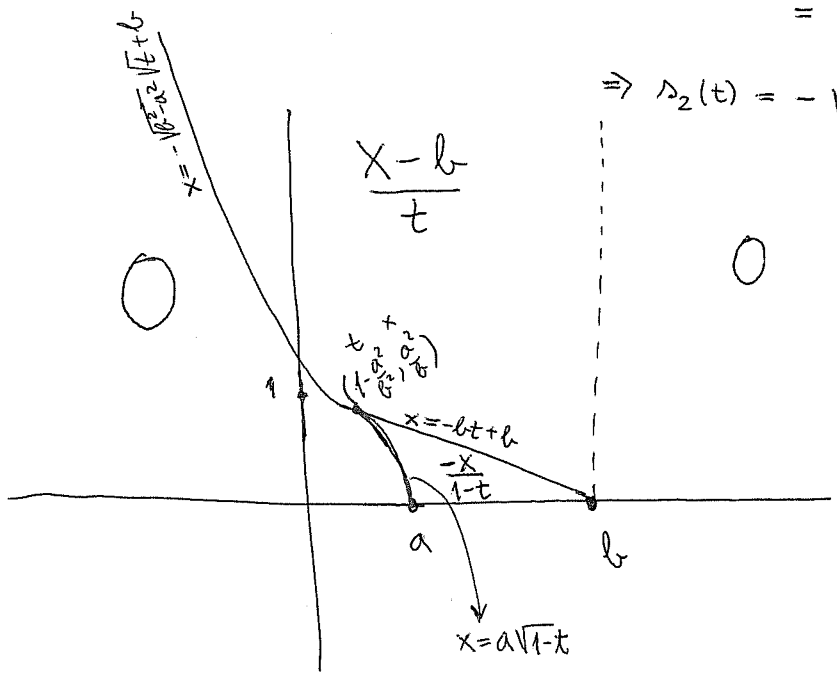
$$\int \frac{ds}{s-b} = \int \frac{dt}{2t}$$

$$\Rightarrow s(t) = C\sqrt{2t} + b$$

$$\frac{a^2}{b} = s\left(1-\frac{a^2}{b^2}\right) = C\sqrt{2\left(1-\frac{a^2}{b^2}\right)} + b$$

$$\Rightarrow C = \frac{\frac{a^2}{b} - b}{\sqrt{2\left(1-\frac{a^2}{b^2}\right)}} = \frac{-\frac{1}{\cancel{b}}(b^2-a^2)}{\frac{\sqrt{2}}{\cancel{b}}\sqrt{b^2-a^2}} = -\frac{\sqrt{b^2-a^2}}{\sqrt{2}}$$

$$\Rightarrow s_2(t) = -\sqrt{b^2-a^2}\,\sqrt{t} + b$$



Analogno za slučaj $a<b<0$ dobivamo: