1) $\begin{cases} x u_x - y u_y = u - y \\ u(y^2, y) = y \end{cases}$

$$s(y) = (y^2, y) \Rightarrow \dot{s}(y) = (2y, 1) \Rightarrow \boxed{n(y^2, y) = (1, -2y)}$$

(4) $$\begin{bmatrix} y^2 \\ -y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2y \end{bmatrix} = y^2 + 2y^2 = 3y^2 = 0 \Leftrightarrow y = 0$$

$\Rightarrow$ ishodište je jedina karakt. točka.

(6) $$\begin{cases} \frac{dx}{d\tau} = x \Rightarrow x(\tau) = C_1 e^{\tau} \\ \frac{dy}{d\tau} = -y \Rightarrow y(\tau) = C_2 e^{-\tau} \\ \frac{dz}{d\tau} = z - y \Rightarrow \boxed{\frac{dz}{d\tau} = z - C_2 e^{-\tau}} \end{cases}$$

$$\frac{dz}{d\tau} = z \Rightarrow z = C_3 e^{\tau}$$

VARIJACIJA KONSTANTI: $C_3' e^{\tau} + \cancel{C_3 e^{\tau}} = \cancel{C_3 e^{\tau}} - C_2 e^{-\tau}$

$$C_3' = -C_2 e^{-2\tau}$$

$$\Rightarrow C_3(\tau) = \frac{C_2}{2} e^{-2\tau} + C_3$$

$$\Rightarrow \boxed{z(\tau) = \frac{C_2}{2} e^{-\tau} + C_3 e^{\tau}}$$

$\boxed{\tau \in \langle -\infty, +\infty \rangle}$

(5)
$$y_0^2 = x(0) = C_1 \Rightarrow \boxed{x(\tau) = y_0^2 e^{\tau}}$$
$$y_0 = y(0) = C_2 \Rightarrow \boxed{y(\tau) = y_0 e^{-\tau}}$$
$$y_0 = z(0) = \frac{C_2}{2} + C_3 \Rightarrow C_3 = \frac{y_0}{2} \Rightarrow \boxed{z(\tau) = \frac{y_0}{2}\left(e^{-\tau} + e^{\tau}\right) = y_0 \operatorname{ch} \tau}$$

$(x, y) \in \mathbb{R}^2$, $\quad x = y_0^2 e^{\tau}$ očito $\underline{x \geq 0}$

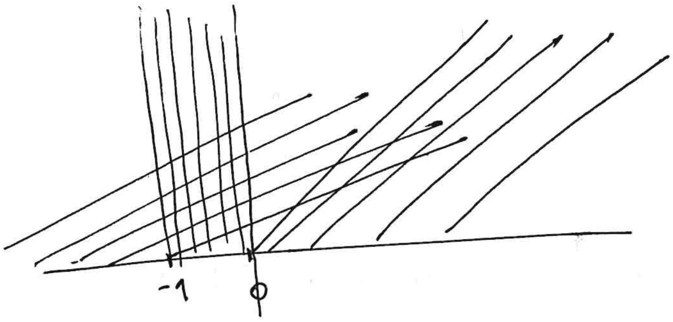
$$y = y_0 e^{-\tau}$$

(5) $$(x, y \neq 0) \quad \frac{x}{y^2} = \cancel{y_0}\, e^{3\tau} \Rightarrow 3\tau = \ln \frac{x}{y^2} \Rightarrow \boxed{\tau = \frac{1}{3} \ln \frac{x}{y^2}}$$

$$y_0 = y e^{\tau} = y \sqrt[3]{\frac{x}{y^2}} = \sqrt[3]{xy}$$

$$\Rightarrow \boxed{u(x, y) = \frac{\sqrt[3]{xy}}{2}\left(\sqrt[3]{\frac{x}{y^2}} + \sqrt[3]{\frac{y^2}{x}}\right) = \frac{1}{2}\left(\sqrt[3]{\frac{x^2}{y}} + y\right)}$$

2)



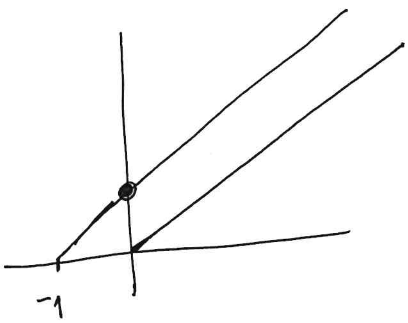
• R-H u $(t,x) = (0,-1)$

$$\left.\begin{array}{l} \left.\begin{array}{l} u_l = 2 \\ u_r = 0 \end{array}\right\} [u] = 2 \\ \\ \left.\begin{array}{l} F_l = 2 \\ F_r = 0 \end{array}\right\} [F] = 2 \end{array}\right\} \Rightarrow \dot{s} = 1 \Rightarrow s(t) = t + C$$

$$-1 = s(0) = C$$

$$\Rightarrow \boxed{s_1(t) = t - 1}$$

• popunimo s $\frac{x}{t}$



$$2(t - \sqrt{t}) = t$$

$$t = 2\sqrt{t} \quad /^2$$

$$t^2 = 4t \quad /:t$$

$$\boxed{t = 4}$$

Dalje, još R-H u $(4,4)$

• R-H uvjet u $(t,x) = (1,0)$

$$\left.\begin{array}{l} u_l = 2 \\ u_r = \frac{s}{t} \end{array}\right\} [u] = 2 - \frac{s}{t}$$

$$\left.\begin{array}{l} F_l = 2 \\ F_r = \frac{1}{2}\left(\frac{s}{t}\right)^2 \end{array}\right\} [F] = 2 - \frac{1}{2}\left(\frac{s}{t}\right)^2 = \frac{1}{2}\left(4 - \left(\frac{s}{t}\right)^2\right) = \frac{1}{2}\left(2 - \frac{s}{t}\right)\left(2 + \frac{s}{t}\right)$$

$$\Rightarrow \left(2 - \frac{s}{t}\right)\dot{s} = \frac{1}{2}\left(2 - \frac{s}{t}\right)\left(2 + \frac{s}{t}\right) \quad /: 2 - \frac{s}{t} \neq 0$$

$$\boxed{\dot{s} = \frac{s}{2t} + 1}$$

$$\frac{ds}{s} = \frac{dt}{2t}$$

$$\ln|s| = \ln\sqrt{t} + C$$

$$s(t) = C\sqrt{t}$$

$$C'\sqrt{t} + \cancel{\frac{C}{2\sqrt{t}}} = \cancel{\frac{C}{2\sqrt{t}}} + 1$$

$$C' = t^{-\frac{1}{2}}$$

$$C = 2t^{\frac{1}{2}} + D$$

$$\left.\begin{array}{l} \Rightarrow s(t) = 2t + D\sqrt{t} \\ 0 = s(1) = 2 + D \Rightarrow D = -2 \end{array}\right\} \Rightarrow \boxed{s_2(t) = 2(t - \sqrt{t})}$$

• R-H ungot $u\ (t,x) = (4,4)$

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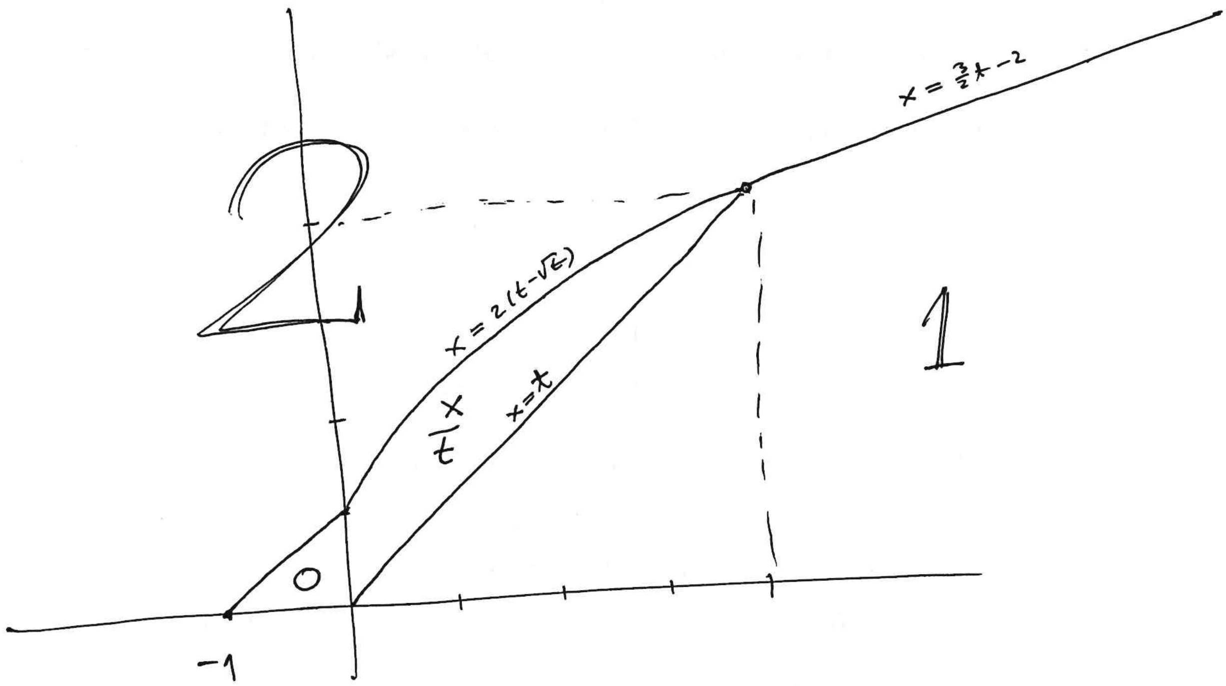
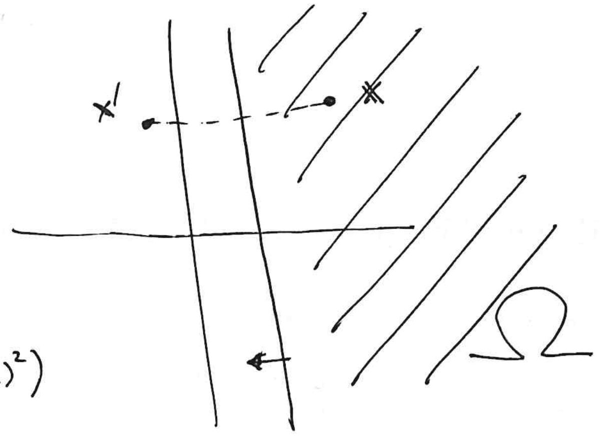
$$\left.\begin{array}{l}\left.\begin{array}{l} u_l = 2 \\ u_r = 1 \end{array}\right\} [u] = 1 \\ \\ \left.\begin{array}{l} F_l = 2 \\ F_r = \frac{1}{2} \end{array}\right\} [F] = \frac{3}{2} \end{array}\right\} \Rightarrow \quad \dot{s} = \frac{3}{2} \Rightarrow s(t) = \frac{3}{2}t + C$$

$$4 = s(4) = 6 + C \Rightarrow C = -2$$

$$\boxed{s_3(t) = \frac{3}{2}t - 2}$$



3) $\begin{cases} \Delta u = 0 \\ u(1,x_2) = \operatorname{sign}(x_2) \end{cases}$



$x = (x_1, x_2)$

$x' = (2-x_1, x_2)$

$$G(x,y) = -\frac{1}{4\pi} \ln\left((x_1-y_1)^2 + (x_2-y_2)^2\right) + \frac{1}{4\pi} \ln\left((x_1+y_1-2)^2 + (x_2-y_2)^2\right)$$

$\vec{n}(1,x_2) = (-1,0)$

$$u(x_1,x_2) = \int_{-\infty}^{+\infty} \frac{\partial G}{\partial \vec{n}_y}(x_1,x_2;1,y)\, \operatorname{sign}(y)\, dy$$

$$\partial_{\vec{n}_y} G = -\partial_{y_1} G = -\frac{1}{4\pi} \frac{2(y_1-x_1)}{(x_1-y_1)^2+(x_2-y_2)^2} + \frac{1}{4\pi} \frac{2(x_1+y_1-2)}{(x_1+y_1-2)^2+(x_2-y_2)^2}$$

$$\partial_{\vec{n}_y} G(x_1,x_2;1,y) = \frac{1}{4\pi} \frac{2x_1-2+2x_1-2}{(x_1-1)^2+(x_2-y)^2} = \frac{x_1-1}{\pi} \frac{1}{(x_1-1)^2+(x_2-y)^2}$$

$$\Rightarrow u(x_1,x_2) = -\frac{x_1-1}{\pi} \int_{-\infty}^{0} \frac{1}{(x_1-1)^2+(x_2-y)^2}\, dy + \frac{x_1-1}{\pi} \int_{0}^{\infty} \frac{1}{(x_1-1)^2+(x_2-y)^2}\, dy$$

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$$= \frac{x_1-1}{\pi}\left(-\frac{1}{x_1-1} \operatorname{arctg} \frac{z}{x_1-1} \Big|_{-\infty}^{-x_2} + \frac{1}{x_1-1} \operatorname{arctg} \frac{z}{x_1-1} \Big|_{-x_2}^{\infty} \right)$$

$$= \frac{1}{\pi}\left(-\operatorname{arctg} \frac{-x_2}{x_1-1} - \cancel{\frac{\pi}{2}} + \cancel{\frac{\pi}{2}} - \operatorname{arctg} \frac{-x_2}{x_1-1} \right)$$

$$= \boxed{\frac{2}{\pi} \operatorname{arctg} \frac{x_2}{x_1-1}}$$

4) $\begin{cases} u_t - \Delta u = e^t \\ u(0,x) = \sin(x_1 - x_2 - x_3) \end{cases}$

$$u(t,x) = \underbrace{\int_{\mathbb{R}^3} \phi(t, x-y) g(y)\, dy}_{\text{I}} + \underbrace{\int_0^t \int_{\mathbb{R}^3} \phi(t-s, x-y)\, f(s,y)\, dy\, ds}_{\text{II}}$$

$$\text{I} = \int_{\mathbb{R}^3} \phi(t, x-y) \sin(y_1 - y_2) \cos y_3 - \sin y_3 \cos(y_1 - y_2)\, dy_1\, dy_2\, dy_3$$

$$= \int_{\mathbb{R}^3} \phi(t, x-y) (\sin y_1 \cos y_2 - \cos y_1 \sin y_2) \cos y_3\, dy_1\, dy_2\, dy_3$$

$$- \int_{\mathbb{R}^3} \phi(t, x-y) (\cos y_1 \cos y_2 + \sin y_1 \sin y_2) \sin y_3\, dy_1\, dy_2\, dy_3$$

$$F(x) := \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \sin y\, dy = \left\{ \begin{matrix} z = y - x \\ dz = dy \end{matrix} \right\} = \int_{\mathbb{R}} e^{-\frac{z^2}{4t}} (\underbrace{\sin z \cos x}_{=0} + \sin x \cos z)\, dz$$

$$= \sin x \sqrt{4\pi t}\, e^{-t}$$

$$G(t,x) := \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \cos y\, dy = \cos x \int_{\mathbb{R}} e^{-\frac{z^2}{4t}} \cos z\, dz$$

$$= \cos x \sqrt{4\pi t}\, e^{-t}$$

$$\text{I} = (4\pi t)^{-3/2} \Big(F(t,x_1) G(t,x_2) G(t,x_3) - G(t,x_1) F(t,x_2) G(t,x_3)$$

$$- G(t,x_1) G(t,x_2) F(t,x_3) - F(t,x_1) F(t,x_2) F(t,x_3) \Big)$$

$$= e^{-3t} \left(\sin x_1 \cos x_2 \cos x_3 - \cos x_1 \sin x_2 \cos x_3 + (\quad) \sin x_3 \right)$$

$$= e^{-3t} \left(\sin(x_1 - x_2) \cos x_3 - \cos(x_1 - x_2) \sin x_3 \right) = e^{-3t} \sin(x_1 - x_2 - x_3)$$

$$\mathrm{II} = \int_0^t e^s \underbrace{\int_{\mathbb{R}^3} \phi(t-s, x-y)\, dy}_{=1}\, ds = e^t - 1$$

$$\Rightarrow \boxed{u(t,x) = e^t - 1 + e^{-3t} \sin(x_1 - x_2 - x_3)}$$