PDJ 1 – KOLOKVIJ 23.1.2018.

– rješenja –

1) a) $u_{tt}u_x^2 - 2u_{xt}u_tu_x + u_t^2u_{xx} = 0 \quad / : u_x^3 \neq 0 \quad (*)$

$$\frac{u_{tt}}{u_x} - 2\frac{u_{xt}u_t}{u_x^2} + \frac{u_t^2u_{xx}}{u_x^3} = 0$$

$$\frac{u_{tt}}{u_x} - \frac{u_tu_{xt}}{u_x^2} = \frac{u_{xt}u_t}{u_x^2} - \frac{u_t^2u_{xx}}{u_x^3}$$

$$\frac{u_{tt}}{u_x} - \frac{u_tu_{xt}}{u_x^2} = \frac{u_t}{u_x}\left(\frac{u_{xt}}{u_x} - \frac{u_tu_{xx}}{u_x^2}\right) \quad (**)$$

b) $v := \frac{u_t}{u_x}$

Pogledajmo $(**)$:

$$\underbrace{\frac{u_{tt}u_x - u_tu_{xt}}{u_x^2}}_{=\partial_t\left(\frac{u_t}{u_x}\right)} = \underbrace{\frac{u_t}{u_x}}_{=v}\underbrace{\left(\frac{u_{tx}u_x - u_tu_{xx}}{u_x^2}\right)}_{=\partial_x\left(\frac{u_t}{u_x}\right)}$$

$$\Rightarrow \boxed{v_t = v\,v_x}$$

c) $\begin{cases} u(0,x) = 3 + e^{2x} \\ u_t(0,x) = 6e^{2x} \end{cases} \quad (***)$

Iz $(***)$ trebamo najprije izvesti početne uvjete za v.

$$v(0,x) = \frac{u_t(0,x)}{u_x(0,x)} = \frac{6e^{2x}}{2e^{2x}} = 3$$

($u_x(0,x)$ dobivamo deriviranjem f-je $x \mapsto u(0,x)$)

Time dobivamo zadaću za v:

$$\begin{cases} v_t - v\,v_x = 0 \\ v(0,x) = 3 \end{cases}, \quad x \in \mathbb{R},\ t > 0.$$

Metodom karakteristika se jednostavno dobiva da je

$$v(t,x) = 3, \quad t > 0,\ x \in \mathbb{R},$$

rješenje gornje zadaće.

$$\Rightarrow \begin{cases} u_t - 3u_x = 0 \\ u(0,x) = 3 + e^{2x} \end{cases}$$

jednadžbu smo dobili iz $v = \frac{u_t}{u_x}$ i uvrštavanja $v \equiv 3$

Dobili smo linearnu jednadžbu 1. reda s konstantnim koef. za koju znamo da je rj. dano s (može se izvesti i npr. s metodom karakteristika)

$$\underline{u(t,x)} = u(0, x+3t) = \underline{3 + e^{2x+6t}}$$

Jednostavnom provjerom vidimo i da je drugi početni uvjet zadovoljen:

$$u_t(t,x) = 6e^{2x+6t} \Rightarrow u_t(0,x) = 6e^{2x}.$$

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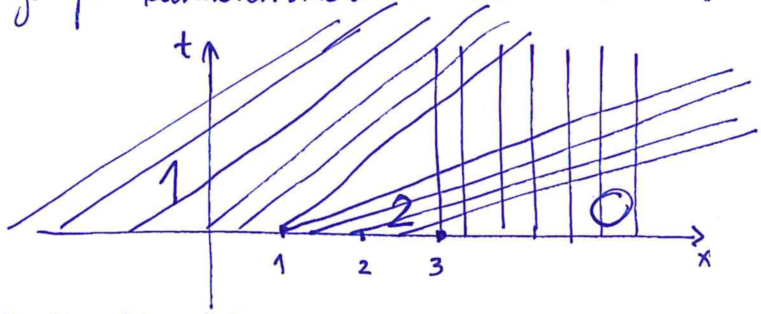
2)
$$\begin{cases} u_t + uu_x = 0 & \text{u } \langle 0,\infty\rangle \times \mathbb{R} \\ u(0,\cdot) = g \end{cases}$$

$$g(x) = \begin{cases} 1, & x<1 \\ 2, & 1\le x<3 \\ 0, & x\ge 3 \end{cases}$$

Riječ je o Burgersovoj jednadžbi.
Projicirane karakteristike su dane s $x(t) = g(x_0)t + x_0$, dok je rj. po karakteristikama dano sa $z(t) = g(x_0)$.



(I) Karakteristike se sijeku u $(t,x) = (0,3)$ pa računamo R-H uvjet

$$\left.\begin{array}{l} \left.\begin{array}{l} u_l = 2 \\ u_r = 0 \end{array}\right\} [u] = 2 \\ \left.\begin{array}{l} F_l = \frac{1}{2}4 = 2 \\ F_r = 0 \end{array}\right\} [F] = 2 \end{array}\right\} \Rightarrow 2\dot{s} = 2$$

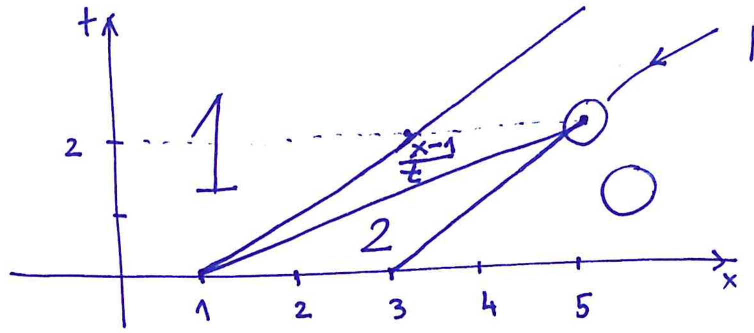
$$\Rightarrow \dot{s} = 1$$

$$\left.\begin{array}{l} \Rightarrow s(t) = t + C \\ 3 = s(0) = C \Rightarrow C = 3 \end{array}\right\} \quad x(t) = \boxed{s_1(t) = t+3} \quad \text{prva krivulja šoka}$$

(II) U području $\{(t,x) \in \mathbb{R}^+ \times \mathbb{R} : t+1 < x < 2t+1, 1<x<3\}$ nemamo karakteristike pa definiramo rj. t.d. je zadovoljen entropijski uvjet

$$u(t,x) = \frac{x-1}{t}$$

Trenutna situacija:



područje s vrijednosti 2 ovdje završava i moramo opet računati R-H uvjet

III R-H uvjet u točki $(t,x)=(2,5)$

$$\left.\begin{array}{l} u_l = \frac{s(t)-1}{t} \\ u_r = 0 \end{array}\right\} [u] = \frac{s(t)-1}{t}$$

$$\left.\begin{array}{l} F_l = \frac{1}{2}\left(\frac{s(t)-1}{t}\right)^2 \\ F_r = 0 \end{array}\right\} [F] = \frac{1}{2}\left(\frac{s(t)-1}{t}\right)^2$$

$$\Rightarrow \frac{s-1}{t}\dot{s} = \frac{1}{2}\left(\frac{s-1}{t}\right)^2 \quad /: \frac{s-1}{t} \neq 0$$

$$\dot{s} = \frac{1}{2}\cdot\frac{s-1}{t}$$

$$\int\frac{ds}{s-1} = \int\frac{dt}{2t}$$

$$\ln|s-1| = \ln\sqrt{t} + C$$

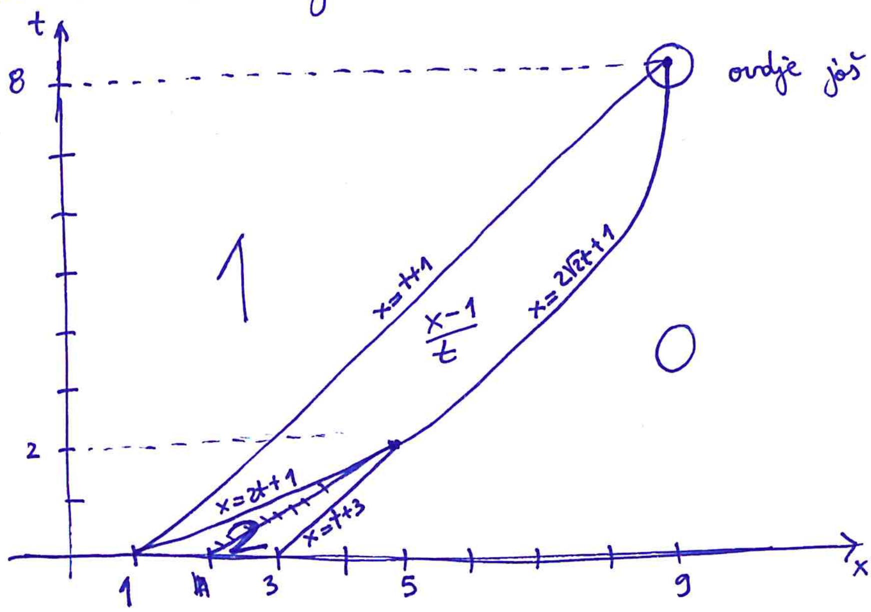
$$s-1 = C\sqrt{t}$$

$$s(t) = C\sqrt{t}+1$$

$$5 = s(2) = C\sqrt{2}+1 \Rightarrow C = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\Rightarrow x = \boxed{s_2(t) = 2\sqrt{2t}+1}$$ druga krivulja šoka

Trenutna situacija:



ovdje još treba provjeriti R-H uvjet

$$t+1 = 2\sqrt{2t}+1$$

$$t = \sqrt{8t} \quad /: \sqrt{t} \neq 0$$

$$\sqrt{t} = \sqrt{8}$$

$$\boxed{t=8}$$

IV R-H uvjet u točki $(t,x)=(8,9)$

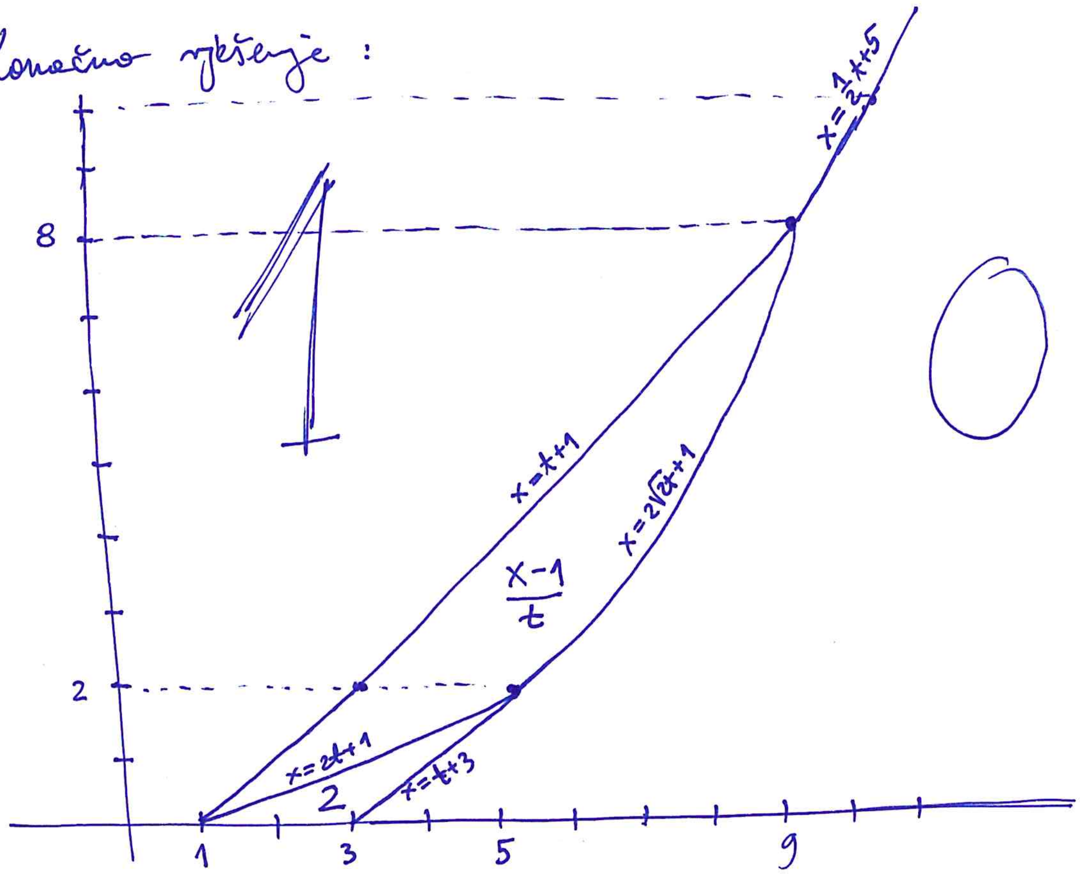
$$\left.\begin{array}{l} u_l = 1 \\ u_r = 0 \\ F_l = \frac{1}{2} \\ F_r = 0 \end{array}\right\} \Rightarrow \dot{s} = \frac{1}{2} \Rightarrow s(t) = \frac{1}{2}t + C$$

$$9 = s(8) = 4 + C \Rightarrow C = 5$$

$$\Rightarrow x = \boxed{s_3(t) = \frac{1}{2}t+5}$$

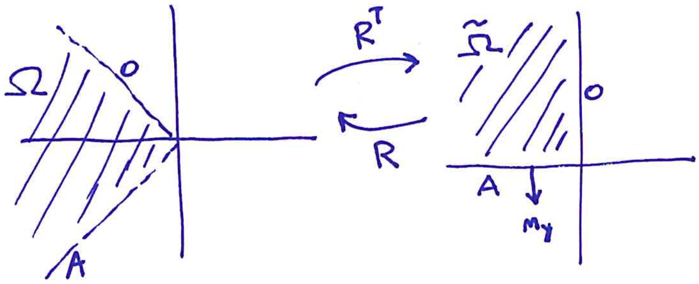
treća krivulja šoka

Konačno rješenje :



3) $\begin{cases} \Delta u = 0 & u \quad \Omega = \{(x_1,x_2)\in\mathbb{R}^2 : x_1+x_2<0,\ x_1-x_2<0\} \\ u(-x,x) = 0, & x \geq 0 \\ u(-x,-x) = A, & x\geq 0 \end{cases}$

Rj: Rotirat ćemo koordinatni sustav t.d. se svedemo na skup $\tilde{\Omega} = \{(x_1,x_2)\in\mathbb{R}^2 : x_1<0,\ x_2>0\}$



$$v(x_1,x_2) := u\left(R(x_1,x_2)\right) \qquad , \qquad R = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1\end{pmatrix}$$

$$= u\left(\frac{\sqrt{2}}{2}x_1 - \frac{\sqrt{2}}{2}x_2,\ \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_2\right)$$

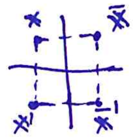
$x \geq 0$, $\quad v(-x,0) = u\left(-\frac{\sqrt{2}}{2}x, -\frac{\sqrt{2}}{2}x\right) = A$

$$v(0,x) = u\left(-\frac{\sqrt{2}}{2}x,\ \frac{\sqrt{2}}{2}x\right) = 0$$

$\Rightarrow$ v zadovoljava sljedeću zadaću

$$\begin{cases} \Delta v = 0 & u \quad \tilde{\Omega} \\ v(-x,0) = A & , \ x\geq 0 \\ v(0,x) = 0 & \end{cases}$$

Odredimo Greenovu f-ju za $\tilde{\Omega}$:



$$G(x,y) = \Phi(|x-y|) - \Phi(|\bar{x}-y|) - \Phi(|x'-y|) + \Phi(|\bar{x}'-y|)$$
$$= -\frac{1}{4\pi}\ln\left((x_1-y_1)^2+(x_2-y_2)^2\right) + \frac{1}{4\pi}\ln\left((x_1+y_1)^2+(x_2-y_2)^2\right)$$
$$+\frac{1}{4\pi}\ln\left((x_1-y_1)^2+(x_2+y_2)^2\right) - \frac{1}{4\pi}\ln\left((x_1+y_1)^2+(x_2+y_2)^2\right)$$

$x = (x_1,x_2)$
$\bar{x} = (-x_1,x_2)$
$x' = (x_1,-x_2)$
$\bar{x}' = (-x_1,-x_2)$

$$v(x_1,x_2) = -\int_{\partial\tilde{\Omega}} \frac{\partial G}{\partial n_y}(x,y)\, v|_{\partial\tilde{\Omega}}(y)\, dS_y$$
$$= -\int_0^\infty \frac{\partial G}{\partial n_y}(x,-y,0)\, A\, dy$$

$$\frac{\partial G}{\partial n_y} = -\frac{\partial G}{\partial y_2} \quad \Rightarrow \quad v(x_1,x_2) = A\int_0^\infty \frac{\partial G}{\partial y_2}(x_1,x_2,-y,0)\,dy$$

$$\frac{\partial G}{\partial y_2}(x_1,x_2,y_1,y_2) = -\frac{1}{\underset{2}{\cancel{4}}\pi}\cdot\frac{\cancel{2}(y_2-x_2)}{(x_1-y_1)^2+(x_2-y_2)^2} + \frac{1}{2\pi}\,\frac{y_2-x_2}{(x_1+y_1)^2+(x_2-y_2)^2}$$

$$+\frac{1}{2\pi}\,\frac{y_2+x_2}{(x_1-y_1)^2+(x_2+y_2)^2} - \frac{1}{2\pi}\,\frac{y_2+x_2}{(x_1+y_1)^2+(x_2+y_2)^2}$$

$$\frac{\partial G}{\partial y_2}((x_1,x_2),-y,0) = \frac{x_2}{\pi}\,\frac{1}{(x_1+y)^2+x_2^2} - \frac{x_2}{\pi}\,\frac{1}{(x_1-y)^2+x_2^2}$$

$$\Rightarrow\ v(x_1,x_2) = \frac{Ax_2}{\pi}\int_0^\infty \frac{dy}{(x_1+y)^2+x_2^2} - \frac{Ax_2}{\pi}\int_0^\infty \frac{dy}{(x_1-y)^2+x_2^2}$$

- $$\int_0^\infty \frac{dy}{(x_1+y)^2+x_2^2} = \frac{1}{x_2^2}\int_0^\infty \frac{dy}{\left(\frac{y+x_1}{x_2}\right)^2+1} = \left|\begin{matrix} z = \frac{y+x_1}{x_2} \\ dz = \frac{1}{x_2}\,dy\end{matrix}\right|$$

$$= \frac{1}{x_2}\int_{\frac{x_1}{x_2}}^{+\infty} \frac{dz}{z^2+1} = \frac{1}{x_2}\,\mathrm{arctg}\, z\Big|_{\frac{x_1}{x_2}}^{+\infty}$$

$$\left(\lim_{y\to+\infty}\frac{y+x_1}{x_2}=+\infty \ \text{ pa } \ x_2>0\right)$$

$$= \frac{1}{x_2}\left(\frac{\pi}{2} - \mathrm{arctg}\,\frac{x_1}{x_2}\right)$$

- $$\int_0^\infty \frac{dy}{(y-x_1)^2+x_2^2} = \frac{1}{x_2^2}\int_0^\infty \frac{dy}{\left(\frac{y-x_1}{x_2}\right)^2+1} = \left|\begin{matrix} z = \frac{y-x_1}{x_2} \\ dz = \frac{1}{x_2}\,dy\end{matrix}\right|$$

$$= \frac{1}{x_2}\,\mathrm{arctg}\, z\Big|_{-\frac{x_1}{x_2}}^{+\infty} = \frac{1}{x_2}\left(\frac{\pi}{2} - \mathrm{arctg}\left(-\frac{x_1}{x_2}\right)\right)$$

$$= \frac{1}{x_2}\left(\frac{\pi}{2} + \mathrm{arctg}\,\frac{x_1}{x_2}\right)$$

$$\Rightarrow\ v(x_1,x_2) = \frac{A\cancel{x_2}}{\pi}\cdot\frac{1}{\cancel{x_2}}\left(\cancel{\frac{\pi}{2}} - \mathrm{arctg}\,\frac{x_1}{x_2} - \cancel{\frac{\pi}{2}} - \mathrm{arctg}\,\frac{x_1}{x_2}\right)$$

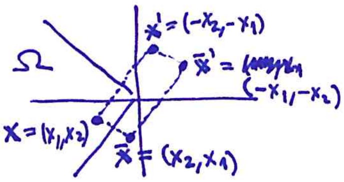
$$= -\frac{2A}{\pi}\,\mathrm{arctg}\left(\frac{x_1}{x_2}\right)$$

$$\Rightarrow\ \boxed{u(x_1,x_2)} = v(R^T(x_1,x_2)) = v\left(\frac{\sqrt{2}}{2}x_1+\frac{\sqrt{2}}{2}x_2,\ -\frac{\sqrt{2}}{2}x_1+\frac{\sqrt{2}}{2}x_2\right)$$

$$= \boxed{-\frac{2A}{\pi}\,\mathrm{arctg}\left(\frac{x_2+x_1}{x_2-x_1}\right)}$$

2. NAČIN Računati bez rotacije koordinatnog sustava.

Greenova f-ja skupa Ω:



$$\Rightarrow G(x,y) = \phi(|x-y|) - \phi(|\bar{x}-y|) - \phi(|x'-y|) + \phi(|\bar{x}'-y|)$$

$$= -\frac{1}{4\pi}\ln\left((x_1-y_1)^2+(x_2-y_2)^2\right)$$

$$+\frac{1}{4\pi}\ln\left((x_2-y_1)^2+(x_1-y_2)^2\right)$$

$$+\frac{1}{4\pi}\ln\left((x_2+y_1)^2+(x_1+y_2)^2\right)$$

$$-\frac{1}{4\pi}\ln\left((x_1+y_1)^2+(x_2+y_2)^2\right)$$

$$u(x_1,x_2) = -\int_{\partial\Omega}\frac{\partial G}{\partial \nu_y}(x,y)\, u|_{\partial\Omega}(y)\, dS_y$$

netrivijalan je samo integral po $\{(x_1,x_2)\in\mathbb{R}^2 : x_1=x_2,\ x_1<0\}$.

Parametriziramo tu krivulju γ

$$\gamma : [0,+\infty\rangle \to$$

$$\gamma(y) = (-y,-y) \quad\Rightarrow\quad |\gamma'(y)| = |(-1,-1)| = \sqrt{2}$$

$$\Rightarrow u(x_1,x_2) = -\int_\gamma \frac{\partial G}{\partial\nu_y}\, u|_{\partial\Omega}\, ds_y = -\int_0^{+\infty}\frac{\partial G}{\partial\nu_y}(x_1,x_2,-y,-y)\, A\cdot\sqrt{2}\, dy$$

$$= -\sqrt{2}A\int_0^{+\infty}\frac{\partial G}{\partial\nu_y}(x_1,x_2,-y,-y)\, dy$$

U točki $(-y,-y)$ vanjska normala je jednaka

$$\nu_{(-y,-y)} = \frac{1}{\sqrt{2}}(1,-1)$$

$$\Rightarrow \frac{\partial G}{\partial\nu_y} = \frac{1}{\sqrt{2}}\frac{\partial G}{\partial y_1} - \frac{1}{\sqrt{2}}\frac{\partial G}{\partial y_2}$$

$$\Rightarrow u(x_1,x_2) = A\int_0^{+\infty}\frac{\partial G}{\partial y_2}(x_1,x_2,-y,-y)\, dy - A\int_0^{+\infty}\frac{\partial G}{\partial y_1}(x_1,x_2,-y,-y)\, dy$$

$$\frac{\partial G}{\partial y_2}(x,y) = \frac{1}{2\pi}\frac{x_2-y_2}{(x_1-y_1)^2+(x_2-y_2)^2} - \frac{1}{2\pi}\frac{x_1-y_2}{(x_2-y_1)^2+(x_1-y_2)^2} + \frac{1}{2\pi}\frac{x_1+y_2}{(x_2+y_1)^2+(x_1+y_2)^2}$$

$$-\frac{1}{2\pi}\frac{x_2+y_2}{(x_1+y_1)^2+(x_2+y_2)^2}$$

$$\frac{\partial G}{\partial y_2}(x_1,x_2,-y,-y) = \frac{1}{2\pi}\frac{x_2+\cancel{y}-x_1-\cancel{y}}{(x_1+y)^2+(x_2+y)^2} - \frac{1}{2\pi}\frac{x_2-x_1}{(x_1-y)^2+(x_2-y)^2} = \frac{x_2-x_1}{2\pi}\left(\frac{1}{(x_1+y)^2+(x_2+y)^2} - \frac{1}{(x_1-y)^2+(x_2-y)^2}\right),$$

$$\frac{\partial G}{\partial y_1}(x,y) = \frac{1}{2\pi}\frac{x_1-y_1}{(x_1-y_1)^2+(x_2-y_2)^2} - \frac{1}{2\pi}\frac{x_2-y_1}{(x_2-y_1)^2+(x_1-y_2)^2} + \frac{1}{2\pi}\frac{x_2+y_1}{(x_2+y_1)^2+(x_1+y_2)^2}$$

$$- \frac{1}{2\pi}\frac{x_1+y_1}{(x_1+y_1)^2+(x_2+y_2)^2}$$

$$\Rightarrow \frac{\partial G}{\partial y_1}(x_1,x_2,-y,-y) = \frac{x_1-x_2}{2\pi}\frac{1}{(x_1+y)^2+(x_2+y)^2} - \frac{x_1-x_2}{2\pi}\frac{1}{(x_1-y)^2+(x_2-y)^2}$$

$$= -\frac{\partial G}{\partial y_2}(x_1,x_2,-y,-y)$$

$$\Rightarrow \boxed{u(x_1,x_2)} = A\frac{x_2-x_1}{\pi}\left(\int_0^\infty \frac{dy}{(x_1+y)^2+(x_2+y)^2} - \int_0^\infty \frac{dy}{(x_1-y)^2+(x_2-y)^2}\right)$$

$$= A\frac{x_2-x_1}{\pi}\left(\int_0^\infty \frac{dy}{2\left(y+\frac{x_1+x_2}{2}\right)^2+\frac{(x_1-x_2)^2}{2}} - \int_0^\infty \frac{dy}{2\left(y-\frac{x_1+x_2}{2}\right)^2+\frac{(x_1-x_2)^2}{2}}\right)$$

$$= A\frac{\cancel{x_2-x_1}}{\cancel{2}\pi}\left(\frac{\cancel{2}}{\cancel{x_2-x_1}}\operatorname{arctg}\left(\frac{2y}{x_2-x_1}+\frac{x_1+x_2}{x_2-x_1}\right)\Bigg|_0^\infty - \frac{\cancel{2}}{\cancel{x_2-x_1}}\operatorname{arctg}\left(\frac{2y}{x_2-x_1}-\frac{x_1+x_2}{x_2-x_1}\right)\Bigg|_0^\infty\right)$$

$$= \frac{A}{\pi}\left(\cancel{\frac{\pi}{2}} - \operatorname{arctg}\left(\frac{x_1+x_2}{x_2-x_1}\right) - \cancel{\frac{\pi}{2}} + \operatorname{arctg}\left(-\frac{x_1+x_2}{x_2-x_1}\right)\right)$$

$$= \boxed{\frac{-2A}{\pi}\operatorname{arctg}\left(\frac{x_2+x_1}{x_2-x_1}\right)}$$

4) $\begin{cases} u_t - \Delta u = 4t^3 - 2 & \text{u } \mathbb{R}^+ \times \mathbb{R}^2 \\ u(0,x_1,x_2) = e^{-2x_1^2}(1 - \cos x_2) \end{cases}$

R:
$$u(t,x_1,x_2) = \underbrace{\int_0^t \int_{\mathbb{R}^2} \Phi(t-s, x-y)(4s^3-2)\,dy\,ds}_{\mathrm{I}} + \underbrace{\int_{\mathbb{R}^2} \Phi(t, x-y)\, e^{-2y_1^2}(1-\cos y_2)\,dy_1 dy_2}_{\mathrm{II}}$$

$$\mathrm{I} = \int_0^t (4s^3-2)\Big(\underbrace{\int_{\mathbb{R}^2} \Phi(t-s,x-y)\,dy}_{=1}\Big)ds = \int_0^t (4s^3-2)\,ds = s^4\Big|_0^t - 2s\Big|_0^t$$
$$= t^4 - 2t$$

$$\mathrm{II} = \frac{1}{4\pi t} \underbrace{\int_{\mathbb{R}} e^{-\frac{(x_1-y_1)^2}{4t}} e^{-2y_1^2}\,dy_1}_{\mathrm{II}_1} \underbrace{\int_{\mathbb{R}} e^{-\frac{(x_2-y_2)^2}{4t}}(1-\cos y_2)\,dy_2}_{\mathrm{II}_2}$$

$$\mathrm{II}_1 = \int_{\mathbb{R}} e^{-\frac{y_1^2 - 2x_1y_1 + x_1^2 + 8ty_1^2}{4t}}\,dy_1 = \int_{\mathbb{R}} e^{-\frac{(8t+1)\left(y_1 - \frac{x_1}{8t+1}\right)^2 + \frac{8tx_1^2}{8t+1}}{4t}}\,dy_1$$
$$= e^{-\frac{2x_1^2}{8t+1}} \int_{\mathbb{R}} e^{-\frac{8t+1}{4t}\left(y_1 - \frac{x_1}{8t+1}\right)^2} dy_1 = \left\{ \begin{array}{l} z = y_1 - \frac{x_1}{8t+1} \\ dz = dy_1 \end{array} \right\}$$
$$= e^{-\frac{2x_1^2}{8t+1}} \int_{\mathbb{R}} e^{-\frac{8t+1}{4t}z^2}\,dz \overset{\text{formula}}{=} \sqrt{\frac{4t\pi}{8t+1}}\, e^{-\frac{2x_1^2}{8t+1}}$$

$$\mathrm{II}_2 = \underbrace{\int_{\mathbb{R}} e^{-\frac{(x_2-y_2)^2}{4t}}\,dy_2}_{=\sqrt{4\pi t}} - \int_{\mathbb{R}} e^{-\frac{(x_2-y_2)^2}{4t}} \cos y_2\,dy_2 \qquad \left\{ \begin{array}{l} z = y_2 - x_2 \\ dz = dy_2 \end{array} \right\}$$
$$= \sqrt{4\pi t} - \int_{\mathbb{R}} e^{-\frac{z^2}{4t}} \cos z \cos x_2\,dz + \underbrace{\int_{\mathbb{R}} e^{-\frac{z^2}{4t}} \sin z \sin x_2\,dz}_{=0}$$
$$= \sqrt{4\pi t} - \sqrt{4\pi t}\, e^{-t} \cos x_2$$

$$\Rightarrow \mathrm{II} = \frac{1}{\cancel{4\pi t}} \frac{\cancel{\sqrt{4\pi t}}}{\sqrt{8t+1}} e^{-\frac{2x_1^2}{8t+1}} \cancel{\sqrt{4\pi t}}\left(1 - e^{-t}\cos x_2\right) = \frac{1}{\sqrt{8t+1}} e^{-\frac{2x_1^2}{8t+1}}\left(1 - e^{-t}\cos x_2\right)$$

$$\Rightarrow \boxed{u(t,x_1,x_2) = t^4 - 2t + \frac{1}{\sqrt{8t+1}} e^{-\frac{2x_1^2}{8t+1}}\left(1 - e^{-t}\cos x_2\right)}$$

5) $u \in C^{1,2}([0,T]\times[0,1])$, $T>0$

$$\begin{cases} u_t - u_{xx} + u = 0 \\ u(0,x) = u_0(x) \\ u(t,0) = g(t) \\ u(t,1) = h(t) \end{cases} \qquad \boxed{\begin{matrix} u_0 \leq M \\ g, h \leq M \end{matrix}}$$

R: Uvedimo supstituciju t.d. "eliminiramo" član u iz jednadžbe.

$$v(t,x) = u(t,x)e^{at}$$

$$v_t = u_t e^{at} + a u e^{at}$$

$$v_{xx} = u_{xx} e^{at}$$

$$v_t - v_{xx} = (u_t - u_{xx} + au)e^{at}$$ Uzmimo $\underline{a=1}$!

Dakle, $\boxed{v(t,x) := u(t,x)e^{t}}$ zadovoljava:

$$\begin{cases} v_t - v_{xx} = 0 \\ v(0,x) = u(0,x) = u_0(x) \\ v(t,0) = g(t)e^{t} \\ v(t,1) = h(t)e^{t} \end{cases}$$

v zadovoljava princip maksimuma pa imamo

$$\max_{[0,T]\times[0,1]} v = \max\Big\{ \underbrace{\max_{x\in[0,1]} u_0(x)}_{\leq M}, \underbrace{\max_{t\in[0,T]} g(t)e^{t}}_{\leq Me^T}, \underbrace{\max_{t\in[0,T]} h(t)e^{t}}_{\leq Me^T} \Big\} \leq Me^T$$

Sada imamo:

$$\max_{(t,x)\in[0,T]\times[0,1]} u(t,x) = \max_{(t,x)\in[0,T]\times[0,1]} v(t,x)e^{-t} \leq \Big(\max_{(t,x)\in[0,T]\times[0,1]} v(t,x)\Big)\Big(\max_{t\in[0,T]} e^{-t}\Big)$$

$$\leq Me^T \cdot e^0 = Me^T$$