PRVA ZADAĆA : PDJ 1 2017/2018

RJEŠENJE

1) $\begin{cases} u_t - u_x + v = 0 \\ v_t - v_x + u = 0 \end{cases}$, $u, v : \mathbb{R}^2 \to \mathbb{R}$ nepoznate f-je

Uvodimo supstituciju

$$\boxed{\begin{matrix} \xi = x - t \\ \eta = x + t \end{matrix}} \Rightarrow \boxed{\begin{matrix} x = \frac{\xi+\eta}{2} \\ t = \frac{\eta-\xi}{2} \end{matrix}}$$

te definiramo :

$$\tilde{u}(\xi,\eta) := u(t(\xi,\eta), x(\xi,\eta)) = u\left(\frac{\eta-\xi}{2}, \frac{\xi+\eta}{2}\right)$$

$$\tilde{v}(\xi,\eta) := v(t(\xi,\eta), x(\xi,\eta)) = v\left(\frac{\eta-\xi}{2}, \frac{\xi+\eta}{2}\right)$$

Po pravilu deriviranja kompozicije f-ja imamo :

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial \tilde{u}}{\partial t} = \frac{\partial \tilde{u}}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial \tilde{u}}{\partial \eta}\frac{d\eta}{dt} = -\tilde{u}_\xi + \tilde{u}_\eta$$

$$u_x = \ldots = \tilde{u}_\xi + \tilde{u}_\eta$$

$$\Rightarrow \quad 0 = u_t - u_x + v = (-\tilde{u}_\xi + \tilde{u}_\eta) - (\tilde{u}_\xi + \tilde{u}_\eta) + \tilde{v}$$

$$= -2\tilde{u}_\xi + \tilde{v},$$

te analogno $-2\tilde{v}_\xi + \tilde{u} = 0$. Zakle, f-je $\tilde{u}$ i $\tilde{v}$ zadovoljavaju sustav

$$\begin{cases} -2\tilde{u}_\xi + \tilde{v} = 0 \\ -2\tilde{v}_\xi + \tilde{u} = 0 \end{cases}.$$

Iz prve jednadžbe imamo $\tilde{v} = 2\tilde{u}_\xi$ pa uvrštavanjem u drugu dobivamo

$$\boxed{-4\tilde{u}_{\xi\xi} + \tilde{u} = 0}.$$

Kako se ne javljaju derivacije po η, ovu jednadžbu možemo shvatiti kao ODJ s parametrom η.

Riješimo najprije

$$\left.\begin{matrix} \underline{-4f'' + f = 0} \\ (-4\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm\frac{1}{2}) \end{matrix}\right\} \Rightarrow f(x) = C_1 e^{\frac{x}{2}} + C_2 e^{-\frac{x}{2}}$$

konstante

$$\Rightarrow \quad \tilde{u}(\xi,\eta) = C_1(\eta)\, e^{\frac{\xi}{2}} + C_2(\eta)\, e^{-\frac{\xi}{2}}$$

konstante po ξ, ali mogu ovisiti o parametru η

$$\Rightarrow \quad \tilde{v}(\xi,\eta) = 2\,\tilde{u}_\xi(\xi,\eta) = C_1(\eta)\, e^{\xi/2} - C_2(\eta)\, e^{-\xi/2}$$

$$\Rightarrow \quad \boxed{\begin{aligned} u(t,x) &= C_1(x+t)\, e^{\frac{x-t}{2}} + C_2(x+t)\, e^{-\frac{x-t}{2}} \\ v(t,x) &= C_1(x+t)\, e^{\frac{x-t}{2}} - C_2(x+t)\, e^{-\frac{x-t}{2}} \end{aligned}}$$

Preostalo je odrediti C_1 i C_2 iz početnih uvjeta. Izvedimo najprije formule za općenite početne uvjete

$$\boxed{\begin{aligned} u(0,x) &= u_0(x) \\ v(0,x) &= v_0(x) \end{aligned}}\ .$$

$$u_0(x) = u(0,x) = C_1(x)\, e^{\frac{x}{2}} + C_2(x)\, e^{-\frac{x}{2}}$$
$$v_0(x) = v(0,x) = C_1(x)\, e^{x/2} - C_2(x)\, e^{-x/2}$$

$$\Rightarrow \quad \boxed{\begin{aligned} C_1(x) &= \frac{1}{2}\big(u_0(x) + v_0(x)\big)\, e^{-\frac{x}{2}} \\ C_2(x) &= \frac{1}{2}\big(u_0(x) - v_0(x)\big)\, e^{x/2} \end{aligned}}$$

U ovom zadatku imamo

$$u_0(x) = x^2 - 1$$
$$v_0(x) = x^2 + 1 \quad ,$$

pa je

$$C_1(x) = x^2 e^{-\frac{x}{2}}$$
$$C_2(x) = -e^{x/2} \quad ,$$

odnosno

$$\boxed{\begin{aligned} u(t,x) &= (x+t)^2\, \cancel{e^{-\frac{x}{2}}}\, e^{-\frac{t}{2}}\, \cancel{e^{\frac{x}{2}}}\, e^{-\frac{t}{2}} - e^{\frac{x}{2}} e^{\frac{t}{2}} e^{-\frac{x-t}{2}} \\ &= (x+t)^2 e^{-t} - e^{t} \\ v(t,x) &= (x+t)^2 e^{-t} + e^{t} \end{aligned}}$$

2) $\begin{cases} u_x + x u_y = 0 \\ u(x,-1) = e^x \end{cases}$

- $\vec{a}(x,y,z) = \begin{bmatrix} 1 \\ x \end{bmatrix}$, $b(x,y,z) = 0$
- $S = \{(x,-1) : x \in \mathbb{R}\} \Rightarrow \vec{\nu}(x,-1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow \vec{a}(x,-1,z) \cdot \vec{\nu}(x,-1) = \begin{bmatrix} 1 \\ x \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x \Rightarrow$ jednadžba je karakteristična samo u točki $(0,-1)$.

Karakteristike

$$\frac{dx}{d\tau} = 1 \Rightarrow x(\tau) = \tau + C_1$$

$$\frac{dy}{d\tau} = x \Rightarrow \frac{dy}{d\tau} = \tau + C_1 \Rightarrow y(\tau) = \frac{\tau^2}{2} + C_1\tau + C_2$$

$$\frac{dz}{d\tau} = 0 \Rightarrow z(\tau) = C_3$$

Neka je $x_0 \in \mathbb{R}$ po volji odabran. Odredimo C_1 i C_2 t.d. $(x(0), y(0)) = (x_0, -1) \in S$.

$$\left.\begin{aligned} x_0 &= x(0) = C_1 \Rightarrow C_1 = x_0 \\ -1 &= y(0) = C_2 \Rightarrow C_2 = -1 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x(\tau; x_0) &= \tau + x_0 \\ y(\tau; x_0) &= \frac{\tau^2}{2} + x_0\tau - 1 \end{aligned}} \quad (2.1)$$

$z(0)$ mora biti jednako $u(x_0,-1) = e^{x_0}$

$$\Rightarrow e^{x_0} = z(0) = C_3 \Rightarrow C_3 = e^{x_0} \Rightarrow \boxed{z(\tau; x_0) = e^{x_0}} \quad (2.2)$$

Neka je $(x,y) \in \mathbb{R}^2$ proizvoljna. Ispitajmo postoji li x_0 t.d. projicirana karakteristika prolazi točkom (x,y).

$$x = \tau + x_0 \Rightarrow \tau = x - x_0$$

$$y = \frac{\tau^2}{2} + x_0\tau - 1$$

$$\begin{aligned} y &= \frac{(x-x_0)^2}{2} + x_0(x-x_0) - 1 \\ &= \frac{x^2}{2} - \cancel{x x_0} + \frac{x_0^2}{2} + \cancel{x_0 x} - x_0^2 - 1 \\ &= \frac{x^2}{2} - 1 - \frac{x_0^2}{2} \end{aligned}$$

$$\Rightarrow \boxed{y = \frac{x^2}{2} - 1 - \frac{x_0^2}{2}} \quad (2.3) \Rightarrow \boxed{x_0^2 = x^2 - 2 - 2y} \quad (2.4)$$

Očito je nužan i dovoljan uvjet

$$\boxed{x^2 - 2 - 2y \geq 0}$$

$$\Updownarrow$$

$$\boxed{y \leq \frac{x^2}{2} - 1} \quad (2.5)$$

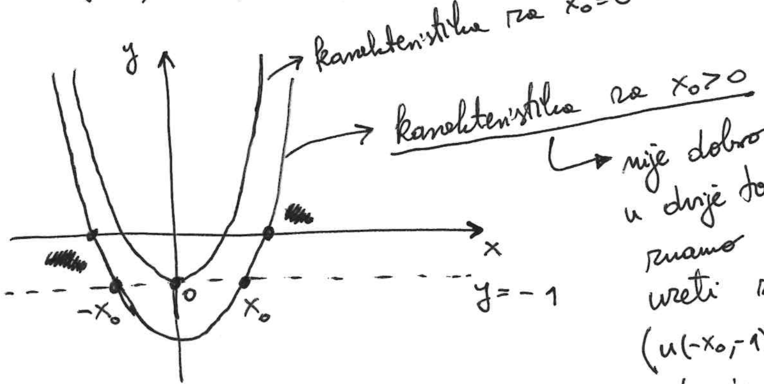
Ukoliko je zadovoljen uvjet (2.5) trebamo još odabrati predznak x_0 jer iz (2.4) slijedi:

$$x_0 = \pm\sqrt{x^2-2-2y}\,.$$

Pogledajmo najprije poseban slučaj $y=-1$. Točka $(x,-1)$ zadovoljava (2.5), a kako se ona nalazi na pravcu S zapravo bismo htjeli $x_0 = x$, dok iz gornje formule slijedi $x_0 = \pm|x|$, pa očito uzimamo $x_0 = \operatorname{sign}(x)\,|x| = x$, odnosno općenito

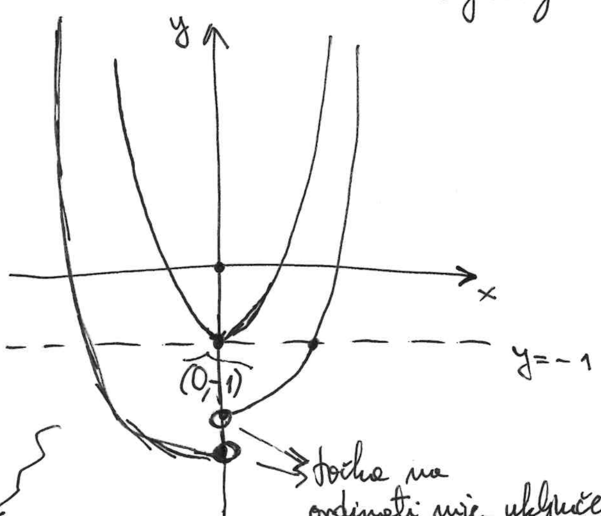
$$\boxed{x_0 = \operatorname{sign}(x)\sqrt{x^2-2-2y}} \qquad (2.6)$$

Gornje razmatranje možemo opravdati i geometrijski. Nakon eliminiranja parametra τ u (2.1) dobivamo da su projicirane karakteristike dane krivuljom (parabolom) (2.3). Ta parabola siječe pravac $y=-1$ u dvije točke: $(-x_0,-1)$ i $(x_0,-1)$



nije dobro da siječe pravac S u dvije točke jer onda ne znamo koji ćemo vrijednost uzeti za početni uvjet ($u(-x_0,-1)$ ili $u(x_0,-1)$) pa razdvajamo karakteristiku na dva dijela

Sada karakteristike ovako izgledaju



točka na ordinati nije uključena

① $x_0 > 0$

$$x(\tau;x_0) = \tau + x_0, \quad y(\tau;x_0) = \frac{\tau^2}{2} + x_0\tau - 1, \quad \tau \in \langle -x_0, +\infty\rangle$$

② $x_0 < 0$

$$x(\tau;x_0) = \tau + x_0, \quad y(\tau;x_0) = \frac{\tau^2}{2} + x_0\tau - 1, \quad \tau \in \langle -\infty, -x_0\rangle$$

na slici je očito da cijela karakteristika leži s jedne strane y-osi; dakle, x i x_0 su istog predznaka!

jer na polupravcu $\{(0,y) : y < -1\}$ imamo (općenito) singularitete. Naravno, ovaj polupravac se dobije za naš izbor karakteristika, dok se drugim izborom može dobiti druga krivulja s početkom u točki $(0,-1)$.

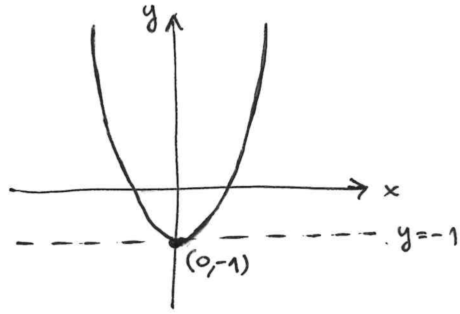
Koristeći uvjet (2.5) uočavamo da kroz točku $(0,0)$ ne prolazi niti jedna projicirana karakteristika, dok kroz točke $(0,-1)$ i $(-3,1)$ prolazi:

$(0,-1)$

$$x_0 = \sqrt{0^2-2+2} = 0$$

$$\left.\begin{aligned} x(\tau) &= \tau \\ y(\tau) &= \frac{\tau^2}{2} - 1 \end{aligned}\right\} \Rightarrow y = \frac{x^2}{2} - 1$$

pa je projicirana karakteristika dana cijelom parabolom $y = \frac{x^2}{2} - 1$



~~$(-2,-1)$~~

~~$x_0 = -\sqrt{4-2+2} = -2$~~

~~$x(\tau) = \tau - 2$~~

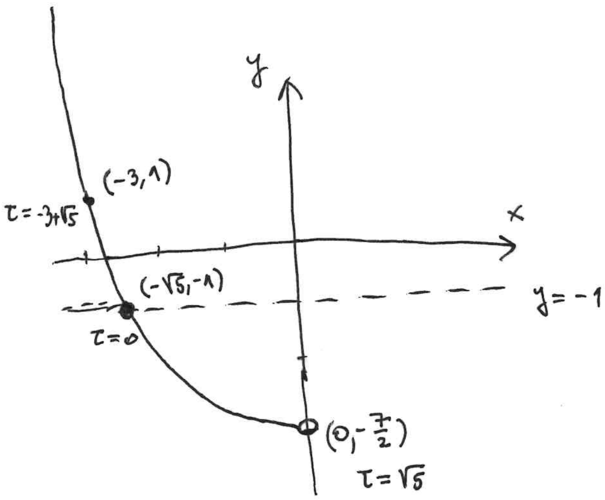
~~$y(\tau) = \frac{\tau^2}{2} - 2\tau - 1$, $\tau \in \langle -\infty, 2\rangle$~~

$(-3,1)$

$$x_0 = -\sqrt{9-2-2} = -\sqrt{5}$$

$$x_0(\tau) = \tau - \sqrt{5}$$

$$y(\tau) = \frac{\tau^2}{2} - \sqrt{5}\tau - 1, \quad \tau \in \langle -\infty, \sqrt{5}\rangle$$



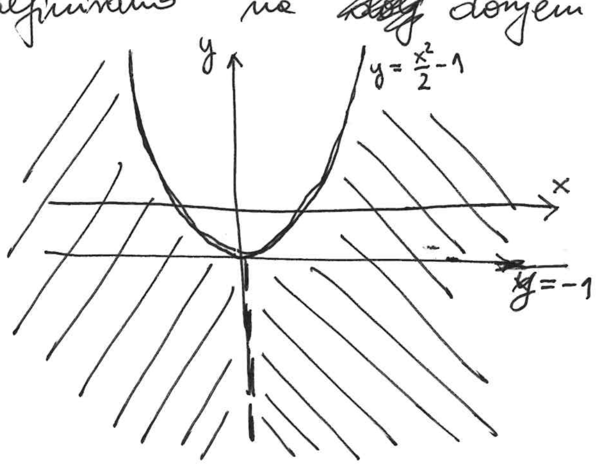
Vratimo se određivanju rješenja.

imamo singularitet za $a=0$ i $b \leq -1$

Za $(x,y) \in \left\{(a,b) \in \mathbb{R}^2 : b \leq \frac{a^2}{2} - 1,\ a \neq 0\right\}$ imamo

$$\boxed{u(x,y) = z(\tau; x_0) = e^{x_0} = e^{\operatorname{sign}(x)\sqrt{x^2-2-2y}}}$$

Rješenje je definirano na donjem skupu



3.) $\begin{cases} 2u_x + u_y = u \\ u(x, ax+b) = x \end{cases}$ 6

- $\vec{a}(x,y,z) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $b(x,y,z) = z$
- $S = \{(x, ax+b) : x \in \mathbb{R}\} \Rightarrow \vec{\nu}(x, ax+b) = \begin{bmatrix} a \\ -1 \end{bmatrix}$

$\Rightarrow \vec{a}(x, ax+b) \cdot \vec{\nu}(x, ax+b) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ -1 \end{bmatrix} = 2a - 1 \Rightarrow$ za $a = \frac{1}{2}$ je jednadžba karakteristična u svim točkama pravca S, dok za $a \neq \frac{1}{2}$ jednadžba nije karakteristična u niti jednoj točki.

Karakteristike:

$$\frac{dx}{d\tau} = 2 \Rightarrow x(\tau) = 2\tau + C_1$$

$$\frac{dy}{d\tau} = 1 \Rightarrow y(\tau) = \tau + C_2$$

$$\frac{dz}{d\tau} = z \Rightarrow z(\tau) = C_3 e^{\tau}$$

$x_0 \in \mathbb{R}$,

$$\left.\begin{aligned} x_0 = x(0) = C_1 &\Rightarrow C_1 = x_0 \\ ax_0 + b = y(0) = C_2 &\Rightarrow C_2 = ax_0 + b \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x(\tau) &= 2\tau + x_0 \\ y(\tau) &= \tau + ax_0 + b \end{aligned}}$$

$$x_0 = z(0) = C_3 \Rightarrow C_3 = x_0 \Rightarrow \boxed{z(\tau) = x_0 e^{\tau}}$$

$(x,y) \in \mathbb{R}^2$,

$$x = 2\tau + x_0 \Rightarrow x_0 = x - 2\tau$$

$$y = \tau + ax_0 + b$$

$$y = \tau + a(x - 2\tau) + b$$

$$= (1-2a)\tau + ax + b$$

1 $\boxed{a \neq \frac{1}{2}}$

$$\tau = \frac{y - ax + b}{1-2a}$$

$$\Rightarrow x_0 = x - \frac{2y - 2ax + 2b}{1-2a}$$

$$= \frac{x - \cancel{2a}x - 2y + \cancel{2a}x - 2b}{1-2a}$$

$$= \frac{x - 2y - 2b}{1-2a}$$

$$\Rightarrow \boxed{\begin{aligned} u(x,y) &= z(\tau; x_0) \\ &= \frac{x - 2y - 2b}{1-2a} e^{\frac{y - ax + b}{1-2a}} \end{aligned}}$$

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② $\boxed{a = \frac{1}{2}}$

$y = ax + b \Rightarrow$ projicirane karakteristike prolaze samo kroz točke oblika $(x, ax+b)$, tj. samo kroz točke na pravcu S

$\Rightarrow$ po karakteristikama ne možemo proširiti rješenje van pravca S

Međutim, čak ni na S jednadžba i početni uvjet nisu kompatibilni.

Naime, uvedimo $\boxed{\begin{array}{l} \xi = x + 2y \\ \eta = x - 2y \end{array}} \Rightarrow \boxed{\begin{array}{l} x = \frac{\xi + \eta}{2} \\ y = \frac{\xi - \eta}{4} \end{array}}$

$\tilde{u}(\xi, \eta) := u(x, y)$

Pokaže se da $2u_x + u_y = u \iff 4\tilde{u}_\xi = \tilde{u}$

Na pravcu S ima da je $\underline{\eta = x - 2(\frac{1}{2}x + b) = -2b}$, pa

$$\tilde{u}(\xi, -2b) = u\left(\frac{\xi - 2b}{2}, \frac{\xi + 2b}{4}\right)$$

$$= u\left(\frac{\xi - 2b}{2}, \frac{1}{2}\left(\frac{\xi - 2b}{2}\right) + b\right)$$

$$= \frac{\xi - 2b}{2}$$

$$\Rightarrow \tilde{u}_\xi(\xi, -2b) = \frac{1}{2} \neq \frac{1}{4} \cdot \frac{\xi - 2b}{2} = \frac{1}{4}\tilde{u}(\xi, -2b)$$

$\Rightarrow$ jednadžba $4\tilde{u}_\xi = \tilde{u}$ nije zadovoljena.

Dakle, u ovom slučaju rješenje ne postoji, odnosno Cauchyjeva zadaća nije dobro postavljena.