TREĆA ZADAĆA: PDJ 1 2017./2018. (rj.)

2) $$\begin{cases} \Delta u = 0 \\ u(-x,x) = 0 \\ u(x,x) = x\chi_{[0,1]}(x) \end{cases}$$

na $\{(x_1,x_2)\in\mathbb{R}^2 : x_1+x_2>0,\ x_1-x_2<0\}$

Rj: **1. NAČIN**

Iskoristit ćemo rotacijsku invarijantnost Laplaceovog operatora i svesti problem na domenu $\{(x_1,x_2)\in\mathbb{R}^2 : x_1,x_2>0\}$, ~~kao~~ što se već radilo na vježbama.

$$R := \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$ ... rotacija za $\frac{\pi}{4}$ u pozitivnom smjeru

$$v(x_1,x_2) := u\big(R(x_1,x_2)\big) = u\left(\tfrac{\sqrt{2}}{2}(x_1-x_2,\ x_1+x_2)\right)$$

Sada,
$$\begin{cases} \Delta v = 0 \quad u\ \{(x_1,x_2)\in\mathbb{R}^2 : x_1,x_2>0\} \\ v(0,\cdot) = 0 \\ v(x,0) = \frac{\sqrt{2}}{2}x\,\chi_{[0,1]}\left(\frac{\sqrt{2}}{2}x\right) = \frac{\sqrt{2}}{2}x\,\chi_{[0,\sqrt{2}]}(x) \end{cases}$$

Ovo je već u neku ruku jednostavniji problem, ali nije identičan zadatku 7 s vježbi pa ćemo ipak promotriti

$$\tilde{v}(x) := v(\sqrt{2}x),$$

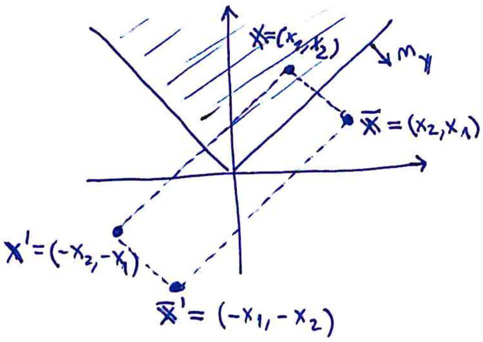
pa
$$\begin{cases} \Delta\tilde{v} = 0 \quad u\ \{(x_1,x_2)\in\mathbb{R}^2 : x_1,x_2>0\} \\ \tilde{v}(0,\cdot) = 0 \\ \tilde{v}(x,0) = x\chi_{[0,1]}(x) \end{cases}.$$

Sada s vježbi znamo:

$$\tilde{v}(x_1,x_2) = \frac{x_2}{\pi}\ln\sqrt{\frac{(x_1-1)^2+x_2^2}{(x_1+1)^2+x_2^2}} - \frac{x_1}{\pi}\left(\operatorname{arctg}\left(\frac{x_1-1}{x_2}\right) - \operatorname{arctg}\left(\frac{x_1+1}{x_2}\right)\right).$$

$$\begin{aligned} u(x_1,x_2) &= v\big(R^T(x_1,x_2)\big) \\ &= \tilde{v}\left(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}(x_1,x_2)\right) \\ &= \tilde{v}\left(\frac{x_1+x_2}{2},\ \frac{x_2-x_1}{2}\right) \\ &= \frac{x_2-x_1}{2\pi}\ln\sqrt{\frac{(x_1+x_2-2)^2+(x_2-x_1)^2}{(x_1+x_2+2)^2+(x_2-x_1)^2}} - \frac{x_1+x_2}{2\pi}\left(\operatorname{arctg}\frac{x_1+x_2-2}{x_2-x_1} - \operatorname{arctg}\frac{x_1+x_2+2}{x_2-x_1}\right) \end{aligned}$$

2. NAČIN   Odrediti Greenovu f-ju koristeći refleksije



$$\Rightarrow G(\mathbb{x},\mathbb{y}) = \Phi(|\mathbb{x}-\mathbb{y}|) - \Phi(|\bar{\mathbb{x}}-\mathbb{y}|) - \Phi(|\mathbb{x}'-\mathbb{y}|) + \Phi(|\bar{\mathbb{x}}'-\mathbb{y}|)$$

$$= -\frac{1}{4\pi}\ln\left((x_1-y_1)^2+(x_2-y_2)^2\right) + \frac{1}{4\pi}\ln\left((x_2-y_1)^2+(x_1-y_2)^2\right) + \frac{1}{4\pi}\ln\left((x_2+y_1)^2+(x_1+y_2)^2\right) - \frac{1}{4\pi}\ln\left((x_1+y_1)^2+(x_2+y_2)^2\right)$$

$$\Rightarrow u(x_1,x_2) = -\int_{\{(x_1,x_2):\, x_1=x_2,\, x_1>0\}} \frac{\partial G}{\partial n_{\mathbb{y}}}(\mathbb{x},\mathbb{y})\, \overbrace{g(\mathbb{y})}^{\text{vrijednost f-je na rubu}}\, dS_{\mathbb{y}}$$

$$= \left\{ \begin{array}{l} \gamma : [0,\infty) \to \{(x_1,x_2): x_1=x_2,\, x_1>0\} \\ \gamma(y) = (y,y) \\ \gamma'(y) = (1,1) \Rightarrow |\gamma'(y)| = \sqrt{2} \end{array} \right\}$$

$$= -\int_0^\infty \frac{\partial G}{\partial n_{\mathbb{y}}}(x_1,x_2,\gamma(y))\, g(\gamma(y))\, |\gamma'(y)|\, dy$$

$$= -\sqrt{2}\int_0^1 \frac{\partial G}{\partial n_{\mathbb{y}}}(x_1,x_2,y,y)\, y\, dy$$

> Ako je $\tilde{G}$ Greenova f-ja za $\{(x_1,x_2)\in\mathbb{R}^2 : x_1,x_2>0\}$ tada se lako pokaže da je
> $$G(\mathbb{x},\mathbb{y}) = \tilde{G}(R^T\mathbb{x}, R^T\mathbb{y}),$$
> pri čemu je
> $$R = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (\text{rot. za } \tfrac{\pi}{4})$$

$n_{(y,y)} = \frac{\sqrt{2}}{2}(1,-1)$ ... jedinična normala

$$\Rightarrow \frac{\partial G}{\partial n_{\mathbb{y}}}(x_1,x_2,y,y) = \frac{\sqrt{2}}{2}\frac{\partial G}{\partial y_1}(\mathbb{x},y,y) - \frac{\sqrt{2}}{2}\frac{\partial G}{\partial y_2}(\mathbb{x},y,y)$$

$$\Rightarrow u(x_1,x_2) = \int_0^1 \frac{\partial G}{\partial y_2}(x_1,x_2;y,y)\, y\, dy - \int_0^1 \frac{\partial G}{\partial y_1}(x_1,x_2;y,y)\, y\, dy$$

$$\vdots$$

3) $\begin{cases} u_t - \Delta u = 0 & u \ \mathbb{R}^+ \times \mathbb{R}^2 \\ u(0,x_1,x_2) = \sin(2x_1 - x_2) \end{cases}$

Rj: Po formuli za rješenje imamo

$$u(t,x_1,x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(t, x_1-y_1, x_2-y_2)\, \overbrace{\sin(2y_1-y_2)}^{=\sin 2y_1 \cos y_2 - \cos 2y_1 \sin y_2}\, dy_1\, dy_2$$

$$= \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x_1-y_1)^2}{4t}} e^{-\frac{(x_2-y_2)^2}{4t}} \left(\sin 2y_1 \cos y_2 - \cos 2y_1 \sin y_2\right) dy_1\, dy_2$$

$$= \frac{1}{4\pi t} \cancel{\cos} \int_{-\infty}^{\infty} \cos y_2\, e^{-\frac{(x_2-y_2)^2}{4t}} \left(\int_{-\infty}^{\infty} e^{-\frac{(x_1-y_1)^2}{4t}} \sin 2y_1\, dy_1 \right) dy_2$$

$$- \frac{1}{4\pi t} \int_{-\infty}^{\infty} \sin y_2\, e^{-\frac{(x_2-y_2)^2}{4t}} \left(\int_{-\infty}^{\infty} e^{-\frac{(x_1-y_1)^2}{4t}} \cos 2y_1\, dy_1 \right) dy_2$$

$$\left\{ \begin{array}{l} z = \cancel{y_1} - \cancel{x_1} \Rightarrow dz = dy_1 \\ y_1 = z + x_1 \end{array} \right\}$$

$$= \frac{1}{4\pi t} \int_{-\infty}^{\infty} \cos y_2\, e^{-\frac{(x_2-y_2)^2}{4t}} \left(\int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}} \left(\overbrace{\sin 2z \cos 2x_1}^{\text{ovaj dio bude } 0} + \sin 2x_1 \cos 2z \right) dz \right) dy_2$$

$$- \frac{1}{4\pi t} \int_{-\infty}^{\infty} \sin y_2\, e^{-\frac{(x_2-y_2)^2}{4t}} \left(\int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}} \left(\cos 2z \cos 2x_1 - \underbrace{\sin 2z \sin 2x_1}_{\text{ovaj dio bude } 0} \right) dz \right) dy_2$$

$$= \frac{1}{4\pi t} \int_{-\infty}^{\infty} \cos y_2\, e^{-\frac{(x_2-y_2)^2}{4t}} \left(\int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}} \cos 2z\, dz \right) \sin 2x_1\, dy_2$$

$$- \frac{1}{4\pi t} \int_{-\infty}^{\infty} \sin y_2\, e^{-\frac{(x_2-y_2)^2}{4t}} \underbrace{\left(\int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}} \cos 2z\, dz \right)}_{= \sqrt{4\pi t}\, e^{-4t} \text{ (po formuli)}} \cos 2x_1\, dy_2$$

$$= \frac{\sin 2x_1}{\sqrt{4\pi t}} e^{-4t} \int_{-\infty}^{\infty} e^{-\frac{(x_2-y_2)^2}{4t}} \cos y_2\, dy_2 - \frac{\cos 2x_1}{\sqrt{4\pi t}} e^{-4t} \int_{-\infty}^{\infty} e^{-\frac{(x_2-y_2)^2}{4t}} \sin y_2\, dy_2$$

$$\left\{ \begin{array}{l} z = y_2 - x_2 \Rightarrow dz = dy_2 \\ y_2 = z + x_2 \end{array} \right\}$$

$$= \frac{\sin 2x_1}{\sqrt{4\pi t}} e^{-4t} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}} \left(\cos z \cos x_2 - \underbrace{\sin z \sin x_2}_{=0}\right) dz$$

$$- \frac{\cos 2x_1}{\sqrt{4\pi t}} e^{-4t} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}} \left(\underbrace{\sin z \cos x_2}_{=0} + \cos z \sin x_2\right) dz$$

$$= \frac{\sin 2x_1 \cos x_2}{\sqrt{4\pi t}} e^{-4t} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}} \cos z \, dz - \frac{\cos 2x_1 \sin x_2}{\sqrt{4\pi t}} e^{-4t} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}} \cos z \, dz$$

$$= \sin 2x_1 \cos x_2 \, e^{-4t} e^{-t} - \cos 2x_1 \sin x_2 \, e^{-4t} e^{-t}$$

$$= e^{-5t} \left(\sin 2x_1 \cos x_2 - \cos 2x_1 \sin x_2\right)$$

$$= \underline{e^{-5t} \sin(2x_1 - x_2)}$$

__NAP.__ Naravno, ovo rješenje se moglo i pogoditi pa bi samo bilo potrebno provjeriti da zadovoljava početnu zadaću.

4) $$\begin{cases} u_{tt} - \Delta u = t+1 & u \ \mathbb{R}^+ \times \mathbb{R}^2 \\ u(0,x_1,x_2) = 0 \\ u_t(0,x_1,x_2) = x_1^2 \end{cases}$$

Rj: Rastavit ćemo problem na dva:

$$① \begin{cases} u_{tt} - \Delta u = 0 \\ u(0,x_1,x_2) = 0 \\ u_t(0,x_1,x_2) = x_1^2 \end{cases} , \qquad ② \begin{cases} u_{tt} - \Delta u = t+1 \\ u(0,\cdot) = 0 \\ u_t(0,\cdot) = 0 \end{cases}$$

Po Poissonovoj formuli za ① imamo:

$$u_1(t,x_1,x_2) = \frac{1}{2\pi \cancel{t^2}} \int_{K(x,t)} \frac{\cancel{t^2} y_1^2}{(t^2 - |y-x|^2)^{1/2}} \, dy$$

$$= \left\{ \begin{array}{l} y_1 = r\cos\varphi + x_1 \\ y_2 = r\sin\varphi + x_2 \end{array} \rightsquigarrow \text{Jacobijan je } r \right\}$$

$$= \frac{1}{2\pi} \int_0^t \int_0^{2\pi} r \frac{(r\cos\varphi + x_1)^2}{(t^2 - r^2)^{1/2}} \, d\varphi \, dr$$

$$= \frac{1}{2\pi} \int_0^t \frac{r}{(t^2-r^2)^{1/2}} \left(\int_0^{2\pi} r^2\cos^2\varphi + 2r x_1 \cos\varphi + x_1^2 \, d\varphi \right) dr$$

$$\left\{ \begin{array}{l} \bullet \displaystyle\int_0^{2\pi} \cos^2\varphi \, d\varphi = \int_0^{2\pi} \frac{\cos(2\varphi)+1}{2} \, d\varphi \\ \qquad = \underbrace{\left.\frac{\sin(2\varphi)}{4}\right|_0^{2\pi}}_{=0} + \frac{1}{2} \cdot 2\pi = \pi \\ \bullet \displaystyle\int_0^{2\pi} \cos\varphi \, d\varphi = \sin\varphi \Big|_0^{2\pi} = 0 \end{array} \right.$$

$$= \frac{1}{2} \int_0^t \frac{r^3}{(t^2-r^2)^{1/2}} \, dr + x_1^2 \int_0^t \frac{r}{(t^2-r^2)^{1/2}} \, dr$$

$$= \left\{ \begin{array}{l} s = t^2 - r^2 \Rightarrow r^2 = t^2 - s \\ ds = -2r\,dr \\ r = 0 \Rightarrow s = t^2, \quad r = t \Rightarrow s = 0 \end{array} \right\}$$

$$= \frac{1}{4} \int_0^{t^2} \frac{(t^2 - s)}{\sqrt{s}} \, ds + \frac{x_1^2}{2} \int_0^{t^2} \frac{ds}{\sqrt{s}} = \left(\frac{x_1^2}{2} + \frac{t^2}{4} \right) \int_0^{t^2} \frac{ds}{\sqrt{s}} - \frac{1}{4} \int_0^{t^2} s^{1/2} \, ds$$

$$= \left(x_1^2 + \frac{t^2}{2}\right)\sqrt{s}\,\Big|_0^{t^2} - \frac{1}{4_2}\cdot\frac{2}{3}\, s^{3/2}\Big|_0^{t^2}$$

$$= t x_1^2 + \frac{t^3}{2} - \frac{1}{6}t^3 = \boxed{t x_1^2 + \frac{t^3}{3}}$$

Za ② koristimo Duhamelovo načelo pa promatramo

$$\begin{cases} v_{tt}(\cdot,\cdot;s) - \Delta v(\cdot,\cdot;s) = 0 \\ v(s,\cdot;s) = 0 \\ v_t(s,\cdot;s) = s+1 \end{cases}$$

Onda $\tilde{v}(t,x) := v(t+s, x; s)$ zadovoljava

$$\begin{cases} \tilde{v}_{tt} - \Delta\tilde{v} = 0 \\ \tilde{v}(0,\cdot) = 0 \\ \tilde{v}(0,\cdot) = s+1 \end{cases}$$

Po Poissonovoj formuli imamo

$$\tilde{v}(t,x_1,x_2) = \frac{s+1}{2\pi}\int_{K(x,t)} \frac{dy}{(t^2-|y-x|^2)^{1/2}} = \left\{ \begin{array}{l} y_1 = r\cos\varphi + x_1 \\ y_2 = r\sin\varphi + x_2 \end{array} \right\}$$

$$= \frac{s+1}{2\pi}\left(\int_0^t \frac{r}{(t^2-r^2)^{1/2}}\,dr\right)\left(\int_0^{2\pi} d\varphi\right)$$

$$= (s+1)\int_0^t \frac{r}{(t^2-r^2)^{1/2}}\,dr = t(s+1)$$

$$\Rightarrow v(t,x_1,x_2;s) = \tilde{v}(t-s,x_1,x_2) = (t-s)(s+1) = (t-1)s - s^2 + t$$

$$\Rightarrow u_2(t,x_1,x_2) = \int_0^t v(t,x_1,x_2;s)\,ds = -\int_0^t s^2 ds + (t-1)\int_0^t s\,ds + t\int_0^t ds$$

$$= -\frac{t^3}{3} + (t-1)\frac{t^2}{2} + t^2$$

$$= -\frac{t^3}{3} + \frac{t^3}{2} - \frac{t^2}{2} + t^2 = \boxed{\frac{t^3}{6} + \frac{t^2}{2}}$$

$$\Rightarrow u(t,x_1,x_2) = t x_1^2 + \frac{t^3}{3} + \frac{t^3}{6} + \frac{t^2}{2}$$

$$= t x_1^2 + \frac{t^3}{2} + \frac{t^2}{2}$$

NAP. Naravno, rj. ①, ② ili cijelog problema se moglo i pogoditi.