# Druga zadaća: Parcijalne diferencijalne jednadžbe I
12.01.2017.

**1.** [7] Lagrangeevom metodom nađite opće rješenje jednadžbe

$$yuu_x + xuu_y = xy.$$

**2.** [8] Izvedite formulu za rješenje rubne zadaće

$$\begin{cases} -\triangle u & = f \\ u(\cdot,\cdot,0) & = 0 \end{cases}$$

na području $\{(x_1,x_2,x_3) \in \mathbb{R}^3 | x_3 > 0\}$ pri čemu je $f(x_1,x_2,x_3) = e^{-x_3}\sin(x_1)\cos(x_2)$.

**3.** [8] Izvedite formulu za rješenje početne zadaće

$$\begin{cases} u_t - \triangle u & = f \\ u(0,\cdot,\cdot) & = g \end{cases}$$

na području $\{(t,x_1,x_2) \in \mathbb{R}^3 | t > 0\}$ pri čemu je $f(t,x_1,x_2) = t\sin(t)$ i $g(x_1,x_2) = e^{-x_1^2}\cos(x_2)$.

**4.** [7] Neka je $u$ rješenje početne zadaće

$$\begin{cases} u_{tt} - u_{xx} & = 0 \\ u(0,\cdot) & = g \\ u_t(0,\cdot) & = h \end{cases}$$

na području $\{(t,x) \in \mathbb{R}^2 | t > 0\}$ pri čemu su $g$ i $h$ glatke funkcije s kompaktnim nosačem. Definiramo kinetičku energiju $k(t) = \frac{1}{2}\int_{\mathbb{R}} u_t^2(x,t)\mathrm{d}x$ i potencijalnu energiju $p(t) = \frac{1}{2}\int_{\mathbb{R}} u_x^2(x,t)\mathrm{d}x$. Dokažite da izraz $k(t)+p(t)$ ne ovisi o vremenu te da postoji $t_0$ takav da je $k(t) = p(t)$ za svaki $t \geq t_0$.

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**1.** Imamo jednakosti

$$\frac{\mathrm{d}x}{yu} = \frac{\mathrm{d}y}{xu} = \frac{\mathrm{d}u}{xy},$$

iz čega slijedi $x\mathrm{d}x - y\mathrm{d}y = 0$ i $u\mathrm{d}u - y\mathrm{d}y = 0$. Dakle imamo implicitno zadano rješenje s $\phi(x, y, u) = x^2 - y^2 = C_1$ i $\psi(x, y, u) = u^2 - y^2 = C_2$, za $C_1, C_2 \in \mathbb{R}$.

Jacobijeva matrica za $(\phi, \psi)$ je

$$\begin{pmatrix} 2x & -2y & 0 \\ 0 & -2y & 2u \end{pmatrix},$$

iz čega vidimo da treba biti zadovoljen barem jedan od uvjeta

$$\begin{vmatrix} 2x & -2y \\ 0 & -2y \end{vmatrix} \neq 0, \quad \begin{vmatrix} 2x & 0 \\ 0 & 2u \end{vmatrix} \neq 0, \quad \begin{vmatrix} -2y & 0 \\ -2y & 2u \end{vmatrix} \neq 0,$$

odnosno $xy \neq 0$, $xu \neq 0$ ili $uy \neq 0$.

**2.** Greenova funkcija za danu domenu je

$$G(x,y)=\frac{1}{4\pi}\Big(\frac{1}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2}}-\frac{1}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3+x_3)^2}}\Big),$$

pa trebamo izračunati integral

$$I=\iiint_{\mathbb{R}^3_+}G(x,y)\sin(y_1)\cos(y_2)e^{-y_3}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3.$$

Prvo napravimo supstitucije $y_1\mapsto y_1+x_1$ i $y_2\mapsto y_2+x_2$ iz čega dobijemo uz adicijske formule za trigonometrijske funkcije

$$I=\frac{1}{4\pi}\iiint_{\mathbb{R}^3_+}\Big(\frac{1}{\sqrt{y_1^2+y_2^2+(y_3-x_3)^2}}-\frac{1}{\sqrt{y_1^2+y_2^2+(y_3+x_3)^2}}\Big)$$
$$(\sin y_1\cos x_1-\sin x_1\cos y_1)(\cos y_2\cos x_2-\sin y_2\sin x_2)e^{-y_3}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3.$$

Članovi koji imaju barem jedan $\sin y_1$ ili $\sin y_2$ nestanu zbog neparnosti pa je

$$I=-\frac{\sin x_1\cos x_2}{4\pi}\iiint_{\mathbb{R}^3_+}\Big(\frac{1}{\sqrt{y_1^2+y_2^2+(y_3-x_3)^2}}-\frac{1}{\sqrt{y_1^2+y_2^2+(y_3+x_3)^2}}\Big)$$
$$\cos y_1\cos y_2e^{-y_3}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3.$$

Sljedeće iskoristimo formulu $2\cos y_1\cos y_2=\cos(y_1+y_2)+\cos(y_1-y_2)$ i uvodimo supstituciju $z_1=y_1+y_2$ i $z_2=y_1-y_2$ čiji je Jacobijan $\frac{1}{2}$.

$$I=-\frac{\sin x_1\cos x_2}{16\pi}\iiint_{\mathbb{R}^3_+}\Big(\frac{1}{\sqrt{(\frac{z_1+z_2}{2})^2+(\frac{z_1-z_2}{2})^2+(y_3-x_3)^2}}-\frac{1}{\sqrt{(\frac{z_1+z_2}{2})^2+(\frac{z_1-z_2}{2})^2+(y_3+x_3)^2}}\Big)$$
$$(\cos z_1+\cos z_2)e^{-y_3}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}y_3,$$

tj.

$$I=-\frac{\sqrt{2}\sin x_1\cos x_2}{16\pi}\iiint_{\mathbb{R}^3_+}\Big(\frac{1}{\sqrt{z_1^2+z_2^2+2(y_3-x_3)^2}}-\frac{1}{\sqrt{z_1^2+z_2^2+2(y_3+x_3)^2}}\Big)$$
$$(\cos z_1+\cos z_2)e^{-y_3}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}y_3.$$

Koristeći simetriju između $z_1$ i $z_2$ zaključujemo

$$I=-\frac{\sqrt{2}\sin x_1\cos x_2}{8\pi}$$
$$\int_{\mathbb{R}_+}e^{-y_3}\int_{\mathbb{R}}\cos z_2\int_{\mathbb{R}}\Big(\frac{1}{\sqrt{z_1^2+z_2^2+2(y_3-x_3)^2}}-\frac{1}{\sqrt{z_1^2+z_2^2+2(y_3+x_3)^2}}\Big)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}y_3.$$

Sada možemo iskoristiti $\int_{-M}^{M}\frac{\mathrm{d}t}{\sqrt{t^2+a^2}}=2\ln(\frac{M}{a}+\sqrt{1+\frac{M^2}{a^2}})$ pa je integral po $z_1$ jednak

$$2\lim_{M\to\infty}\ln\frac{\frac{M}{a}+\sqrt{1+\frac{M^2}{a^2}}}{\frac{M}{b}+\sqrt{1+\frac{M^2}{b^2}}}=2\lim_{M\to\infty}\ln\frac{\frac{1}{a}+\sqrt{\frac{1}{M^2}+\frac{1}{a^2}}}{\frac{1}{b}+\sqrt{\frac{1}{M^2}+\frac{1}{b^2}}}=2\ln\frac{b}{a}=\ln\frac{b^2}{a^2},$$

gdje su $a^2 = z_2^2 + 2(y_3 - x_3)^2$ i $b^2 = z_2^2 + 2(y_3 + x_3)^2$.

Sljedeće računamo pomoću parcijalne integracije i Jordanove leme

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}} \ln \frac{z_2^2 + 2(y_3 + x_3)^2}{z_2^2 + 2(y_3 - x_3)^2} \cos z_2 \mathrm{d}z_2 \\
&= -\int_{\mathbb{R}} \frac{z_2^2 + 2(y_3 - x_3)^2}{z_2^2 + 2(y_3 + x_3)^2} \; \frac{2z_2(z_2^2 + 2(y_3 - x_3)^2 - z_2^2 - 2(y_3 + x_3)^2)}{(z_2^2 + 2(y_3 - x_3)^2)^2} \sin z_2 \mathrm{d}z_2 \\
&= -\int_{\mathbb{R}} \frac{1}{z_2^2 + 2(y_3 + x_3)^2} \; \frac{-16 z_2 x_3 y_3}{z_2^2 + 2(y_3 - x_3)^2} \sin z_2 \mathrm{d}z_2 \\
&= 16 x_3 y_3 \int_{\mathbb{R}} \frac{z_2}{(z_2^2 + 2(y_3 + x_3)^2)(z_2^2 + 2(y_3 - x_3)^2)} \sin z_2 \mathrm{d}z_2 \\
&= -16 i \, x_3 y_3 \int_{\mathbb{R}} \frac{z_2 e^{i z_2}}{(z_2^2 + 2(y_3 + x_3)^2)(z_2^2 + 2(y_3 - x_3)^2)} \mathrm{d}z_2 \\
&= 32\pi \, x_3 y_3 \, (\mathrm{Res}_{\sqrt{2}i|y_3 + x_3|} \, h + \mathrm{Res}_{\sqrt{2}i|y_3 - x_3|} \, h),
\end{aligned}$$

gdje je

$$\begin{aligned}
h(z_2) &= \frac{z_2 e^{i z_2}}{(z_2^2 + 2(y_3 + x_3)^2)(z_2^2 + 2(y_3 - x_3)^2)} \\
&= \frac{z_2 e^{i z_2}}{(z_2 - \sqrt{2}i|y_3 + x_3|)(z_2 + \sqrt{2}i|y_3 + x_3|)(z_2 - \sqrt{2}i|y_3 - x_3|)(z_2 + \sqrt{2}i|y_3 - x_3|)}.
\end{aligned}$$

Reziduume računamo formulom $\mathrm{Res}_{z_0} h = \lim_{z \to z_0} (z - z_0) h(z)$ iz čega lako slijedi

$$\begin{aligned}
\mathrm{Res}_{\sqrt{2}i|y_3 + x_3|} \, h &= -\frac{1}{4} \, \frac{1}{|y_3 + x_3| - |y_3 - x_3|} \, \frac{1}{|y_3 + x_3| + |y_3 - x_3|} e^{-\sqrt{2}|y_3 + x_3|} \\
&= -\frac{1}{4} \, \frac{e^{-\sqrt{2}|y_3 + x_3|}}{(y_3 + x_3)^2 - (y_3 - x_3)^2} = -\frac{1}{16} \, \frac{e^{-\sqrt{2}|y_3 + x_3|}}{x_3 y_3} \\
\mathrm{Res}_{\sqrt{2}i|y_3 - x_3|} \, h &= -\frac{1}{4} \, \frac{1}{|y_3 - x_3| - |y_3 + x_3|} \, \frac{1}{|y_3 - x_3| + |y_3 + x_3|} e^{-\sqrt{2}|y_3 - x_3|} \\
&= -\frac{1}{4} \, \frac{e^{-\sqrt{2}|y_3 - x_3|}}{(y_3 - x_3)^2 - (y_3 + x_3)^2} = \frac{1}{16} \, \frac{e^{-\sqrt{2}|y_3 - x_3|}}{x_3 y_3},
\end{aligned}$$

pa je $I_1 = 2\pi(e^{-\sqrt{2}|y_3 - x_3|} - e^{-\sqrt{2}|y_3 + x_3|})$.

Konačno, preostaje izračunati

$$I = -\frac{\sqrt{2} \sin x_1 \cos x_2}{4} \int_0^{+\infty} (e^{-\sqrt{2}|y_3 - x_3|} - e^{-\sqrt{2}|y_3 + x_3|}) e^{-y_3} \mathrm{d}y_3.$$

Za prvi član imamo (primijetimo $x_3 > 0$)

$$\begin{aligned}
\int_0^{+\infty} e^{-y_3} e^{-\sqrt{2}|y_3 - x_3|} \mathrm{d}y_3 &= \int_0^{x_3} e^{-y_3} e^{-\sqrt{2}(x_3 - y_3)} \mathrm{d}y_3 + \int_{x_3}^{+\infty} e^{-y_3} e^{-\sqrt{2}(y_3 - x_3)} \mathrm{d}y_3 \\
&= e^{-\sqrt{2}x_3} \int_0^{x_3} e^{(\sqrt{2}-1)y_3} \mathrm{d}y_3 + e^{\sqrt{2}x_3} \int_{x_3}^{+\infty} e^{-(1+\sqrt{2})y_3} \mathrm{d}y_3
\end{aligned}$$

$$
\begin{aligned}
&= e^{-\sqrt{2}x_3}\frac{1}{\sqrt{2}-1}(e^{(\sqrt{2}-1)x_3}-1)+e^{\sqrt{2}x_3}\frac{1}{1+\sqrt{2}}e^{-(1+\sqrt{2})x_3}.\\
&= 2\sqrt{2}e^{-x_3}-(\sqrt{2}+1)e^{-\sqrt{2}x_3},
\end{aligned}
$$

a za drugi

$$
\begin{aligned}
\int_0^{+\infty} e^{-y_3}e^{-\sqrt{2}|y_3+x_3|}\mathrm{d}y_3 &= e^{-\sqrt{2}x_3}\int_0^{+\infty} e^{-(\sqrt{2}+1)y_3}\mathrm{d}y_3\\
&= (\sqrt{2}-1)e^{-\sqrt{2}x_3}.
\end{aligned}
$$

Uvrstimo li to u izraz za $I$ lako se dobije

$$
u(x_1,x_2,x_3)=\sin x_1\cos x_2(e^{-\sqrt{2}x_3}-e^{-x_3}).
$$

**3.** Rješenje je dano formulom

$$u(t,x)=\int_{\mathbb{R}^2}\Phi(t,x-y)g(y)\mathrm{d}y+\int_0^t\int_{\mathbb{R}^2}\Phi(t-s,x-y)f(s,y)\mathrm{d}y\mathrm{d}s.$$

Primijetimo da $f$ ne ovisi o varijabli $y$ i $\int_{\mathbb{R}^2}\Phi(t-s,y)\mathrm{d}y=1$ pa je drugi integral zapravo samo $\int_0^t s\sin s\,\mathrm{d}s=-t\cos t+\sin t$.

Prvi integral je

$$\begin{aligned}I_1=&\iint_{\mathbb{R}^2}\frac{1}{4\pi t}e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2}{4t}}e^{-y_1^2}\cos y_2\mathrm{d}y_1\mathrm{d}y_2=\\&=\frac{1}{4\pi t}\int_{\mathbb{R}}e^{-\frac{(x_1-y_1)^2}{4t}}e^{-y_1^2}\mathrm{d}y_1\int_{\mathbb{R}}e^{-\frac{(x_2-y_2)^2}{4t}}\cos y_2\mathrm{d}y_2.\end{aligned}$$

Napravimo supstitucije $y_1\mapsto y_1+x_1$ i $y_2\mapsto y_2+x_2$. U drugom integralu prehodnog izraza dobijemo članove $\cos y_2\cos x_2-\sin y_1\sin x_1$, no član uz $\sin y_1$ nestane zbog neparnosti. Sljedeće iskoristimo formulu $\int_{\mathbb{R}}e^{-ax^2}\cos(bx)\mathrm{d}x=\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}}$ iz tablica

$$\begin{aligned}I_1&=\frac{e^{-x_1^2}\cos x_2}{4\pi t}\int_{\mathbb{R}}e^{-\frac{y_1^2}{4t}}e^{-y_1^2-2y_1x_1}\mathrm{d}y_1\int_{\mathbb{R}}e^{-\frac{y_2^2}{4t}}\cos y_2\mathrm{d}y_2\\&=\frac{e^{-x_1^2-t}\cos x_2}{\sqrt{4\pi t}}\int_{\mathbb{R}}e^{-(1+\frac{1}{4t})y_1^2-2y_1x_1}\mathrm{d}y_1.\end{aligned}$$

Za izračunati posljednji integral jedino još trebamo upotpuniti na kvadrat, $(1+\frac{1}{4t})y_1^2+2y_1x_1=(\sqrt{1+\frac{1}{4t}}y_1+\frac{x_1}{\sqrt{1+\frac{1}{4t}}})^2-\frac{x_1^2}{1+\frac{1}{4t}}$ i supstituirati izraz unutar kvadrata $z=\sqrt{1+\frac{1}{4t}}y_1+\frac{x_1}{\sqrt{1+\frac{1}{4t}}}$. Dakle

$$\begin{aligned}I_1&=\frac{e^{\frac{x_1^2}{1+\frac{1}{4t}}-x_1^2-t}\cos x_2}{\sqrt{4\pi t}}\frac{1}{\sqrt{1+\frac{1}{4t}}}\int_{\mathbb{R}}e^{-z^2}\mathrm{d}z\\&=\frac{e^{\frac{4tx_1^2}{4t+1}-x_1^2-t}\cos x_2}{\sqrt{4\pi t}}\frac{1}{\sqrt{1+\frac{1}{4t}}}\sqrt{\pi}\\&=\frac{e^{\frac{4tx_1^2}{4t+1}-x_1^2-t}\cos x_2}{\sqrt{4t+1}},\end{aligned}$$

i imamo formulu za konačno rješenje

$$u(t,x_1,x_2)=\frac{e^{\frac{4tx_1^2}{4t+1}-x_1^2-t}\cos x_2}{\sqrt{4t+1}}-t\cos t+\sin t.$$

**4.** Zapišimo rješenje naše jednadžbe u obliku $u(t,x) = F(x+t)+G(x-t)$. Iz toga slijedi $u_t(t,x) = F'(x+t)-G'(x-t)$ i $u_x(t,x) = F'(x+t)+G'(x-t)$. Po početnim uvjetima je $g(x) = F(x) + G(x)$ i $h(x) = F'(x) - G'(x)$. Zaključujemo da je $F'(x) = \frac{1}{2}(g'(x) + h(x))$ i $G'(x) = \frac{1}{2}(g'(x) - h(x))$.

Dakle $F'$ i $G'$ imaju kompaktne nosače pa posebno onda i za svaki fiksan $t \geq 0$ funkcije $x \mapsto u_t(t,x)$ i $x \mapsto u_x(t,x)$ imaju kompaktne nosače.

Računamo

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(k(t) - p(t)) &= \int_{\mathbb{R}} u_t\, u_{tt} \mathrm{d}x + \int_{\mathbb{R}} u_x\, u_{xt} \mathrm{d}x \\
&= \int_{\mathbb{R}} u_t\, u_{tt} \mathrm{d}x - \int_{\mathbb{R}} u_{xx}\, u_t \mathrm{d}x \\
&= \int_{\mathbb{R}} u_t\, (u_{tt} - u_{xx}) = 0 \mathrm{d}x,
\end{aligned}$$

pri čemu je prva jednakost slijedila iz parcijalne integracije (nema "rubnog člana" jer funkcija $u_x$ ima kompaktan nosač). Iz ovog slijedi prva tvrdnja.

Za drugu tvrdnju računamo

$$\begin{aligned}
2(k(t) - p(t)) = &\int_{\mathbb{R}} F'(x+t)^2 + G'(x-t)^2 - 2F'(x+t)G'(x-t)\mathrm{d}x \\
&- \int_{\mathbb{R}} F'(x+t)^2 + G'(x-t)^2 + 2F'(x+t)G'(x-t)\mathrm{d}x \\
= &-4 \int_{\mathbb{R}} F'(x+t)G'(x-t)\mathrm{d}x,
\end{aligned}$$

pa sada tvrdnja slijedi jer $F'(x+t)$ ima pomaknut nosač u lijevo za $t$, a $G'(x-t)$ ima pomaknut nosač u desno za $t$, a kako obije funkcije $F'$ i $G'$ imaju kompaktan nosač, onda će nosači $F'(x+t)$ i $G'(x-t)$ biti disjunktni za dovoljno veliki $t$ i posebno će onda podintegralna funkcija biti 0.