Sveučilište u Dubrovniku
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Logika, skupovi i diskretna matematika

Exercise 5

The assignments are due to 19.12.2005.

Tutorial 5.1

1. Determine which of the given relations is an equivalence relation on the set of all people.
   (a) $\{(x, y) : x$ and y have the same height$\}$,
   (b) $\{(x, y) : x$ and y have, at some time, lived in the same country$\}$,
   (c) $\{(x, y) : x$ and y have the same first name$\}$,
   (d) $\{(x, y) : x$ is taller than $y\}$,
   (e) $\{(x, y) : x$ and y have the same parents$\}$.
2. Determine which of the given relations is an equivalence relation on $\{1, 2, 3, 4, 5\}$.
   (a) $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)\}$,
   (b) $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (3, 4), (4, 3)\}$,
   (c) $\{(x, y) : 1 \leq x \leq 5$ and $1 \leq y \leq 5\}$,
   (d) $\{(x, y) : 4$ divides $x - y\}$.
3. Let $X = \{1, 2, 3, 4, 5\}$, $Y = \{3, 4\}$, and $C = \{1, 3\}$. Define the relation R on 2^X, the set of all subsets of X, as
   $$A\,R\,B \qquad \text{if } A \cup Y = B \cup Y.$$
   (a) Show that R is an equivalence relation.
   (b) List the elements of $[C]$, the equivalence class containing C.
   (c) How many distinct equivalence classes are there?

Tutorial 5.2

1. Write the matrix of the relation R from $X = \{1, 2, 3\}$ to $Y = \{\alpha, \beta, \gamma, \delta\}$ given by

$$R = \{(1, \delta), (2, \alpha), (2, \Sigma), (3, \beta), (3, \Sigma)\}.$$

2. How can one quickly determine whether a relation R is antisymmetric by examining the matrix of R?

3. Test whether the relation on $\{1, 2, 3, 4\}$ represented by the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is reflexive, symmetric, antisymmetric, and/or transitive.

4. Let $R_1 = \{(1, x), (1, y), (2, x), (3, x)\}$ be a relation from $\{1, 2, 3\}$ to $\{x, y\}$ and let $R_2 = \{(x, b), (y, b), (y, a), (y, c)\}$ be a relation from $\{x, y\}$ to $\{a, b, c\}$.

   (a) Find the matrix A_1 of the relation R_1.

   (b) Find the matrix A_2 of the relation R_2.

   (c) Compute the matrix product $A_1 A_2$.

   (d) Use the result of part (c) to find the matrix of the relation $R_2 \circ R_1$.

   (e) Use the result of part (d) to find the relation $R_2 \circ R_1$ as a set of ordered pairs.

Tutorial 5.3

1. Use mathematical induction to prove the equality $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every positive integer n.

2. Use mathematical induction to prove the equality $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(n+2)}{6}$ for every positive integer n.

3. Use mathematical induction to prove the inequality $2n + 1 \leq 2^n$ for every $n = 3, 4, \ldots$.

4. Use mathematical induction to prove that $7^n - 1$ is divisible by 6 for all $n \geq 1$.

Tutorial 5.4

1. Write out a proof (using induction on n) of the fact that of $A_1, A_2, \ldots, A_n$ are mutually disjoint sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

2. How many national flags can be constructed from three equal vertical strips, using the colors red, white, blue, and green?

3. A committee of nine people must elect a chairman, secretary and treasurer. In how many ways can this be done?

4. Calculate the total number of permutations of $[1, 6]$ which satisfy $\sigma^2 = \mathrm{id}$ and $\sigma \neq \mathrm{id}$.

Homework assignment 5.1 **5 points**

1. Verify all three conditions to check whether the following relations are equivalence relations on $\{1, 2, 3, 4, 5\}$.
   (a) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$,
   (b) $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 5), (5, 1), (3, 5), (5, 3), (1, 3), (3, 1)\}$,
   (c) $\{(x, y) : 3 \text{ divides } x + y\}$.
2. Let
$$X = \{\text{Zagreb}, \text{Berlin}, \text{Stockholm}, \text{Dresden}, \text{Dubrovnik}\}.$$
   Define a relation R on X as $x\,R\,y$ if x and y are in the same country.
   (a) Show that R is an equivalence relation.
   (b) List the equivalence classes of X.
3. Show that $\{(x, y) : k \text{ divides } x - y\}$ is an equivalence relation on $\mathbb{Z}$ for any $k \in \mathbb{N}$. How many equivalence classes are there? (This number will depend on k, try for small values such as $k = 3$ or $k = 4$ first.)

Homework assignment 5.2 **3 points**

1. Write the matrix of the relation R from $X = \{x, y, z\}$ to $Y = \{a, b, c, d\}$ given by
$$R = \{(x, a), (x, c), (y, a), (y, b), (z, d)\}.$$
2. Let $R_1 = \{(x, y) : x \text{ divides } y\}$ be a relation from $\{2, 3, 4, 5\}$ to $\{2, 3, 4, 5\}$ and let $R_2 = \{(y, z) : y > z\}$ be a relation from $\{2, 3, 4, 5\}$ to $\{1, 2, 3, 4\}$.
   (a) Find the matrix A_1 of the relation R_1.
   (b) Find the matrix A_2 of the relation R_2.
   (c) Compute the matrix product A_1A_2.
   (d) Use the result of part (c) to find the matrix of the relation $R_2 \circ R_1$.
   (e) Use the result of part (d) to find the relation $R_2 \circ R_1$ as a set of ordered pairs.

Homework assignment 5.3 **5 points**

1. Use mathematical induction to prove the equality $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for every positive integer n.
2. Use mathematical induction to prove the equality
$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$
   for every positive integer n.
3. Use mathematical induction to prove that $11^n - 6$ is divisible by 5 for all $n \geq 1$.

Homework assignment 5.4 **5 points**

1. Keys are made by cutting incisions of various depths in a number of positions on a blank key. If there are eight possible depths, how many positions are required to make one million different keys? (Hint: to facilitate the calculation use the fact that 2^{10} is slightly greater than 10^3.)

2. In the usual set of dominos each domino may be represented by the symbol $[x \mid y]$, where x and y are members of the set $\{0, 1, 2, 3, 4, 5, 6\}$. The numbers x and y may be equal. Explain as to why the total number of dominos is 28 rather than 49.

3. How many five-digit telephone numbers have a digit which occurs more than once?

Programming assignment 5.1 **18 points**

This assignment can be solved instead of the homework assignments 5.1–5.4.

Implement C functions which test wether a relation on a set $X = \{0, 1, 2, \ldots, n\}$ given by a matrix (implemented as two-dimensional array) is

1. reflexive;

2. symmetric;

3. transitive.

Write a C program to test the functionality of your functions for the relation from Example 4.23 in the lecture notes.