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Logika, skupovi i diskretna matematika

Exercise 2

The assignments are due to 07.11.2005.

Tutorial 2.1

1. Which of the following sentences can be regarded as logical statements?
   (a) Zagreb is the capital of Italy.
   (b) In the afternoon I will probably drink some coffee.
   (c) If the dog bites the cat, an iron will hit the mouse.
   (d) $A \wedge B$, where A and B are statements.
   (e) $\wedge B \neg C$, where A, B, and C are statements.
2. Give the truth value of each of the following statements.
   (a) $\neg(2 > 1 \wedge 2 > 4)$
   (b) $\neg(2 > 1 \vee 2 > 4)$
   (c) $2 > 1 \Rightarrow$ There is an integer x with $x < 0$.
3. Use the convention to remove as many parentheses as possible.
   (a) $(((\neg A) \vee B) \Rightarrow (A \wedge C)) \Rightarrow (B \wedge C)$
   (b) $\neg(A \wedge B)$
   (c) $((A \wedge D) \Rightarrow (\neg B)) \Leftrightarrow C$
4. Show the following logical equivalences using truth tables.
   (a) $(A \Leftrightarrow B) \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$
   (b) $A \veebar B \equiv \neg(A \Leftrightarrow B)$
   (c) $\neg(A \wedge B) \equiv \neg A \vee \neg B$
   (d) $(A \vee B) \vee C \equiv A \vee (B \vee C)$
5. Prove that $\{\downarrow\}$ is a set of generators using the fact that $\{\neg, \vee\}$ is a set of generators.
6. Show $A \Rightarrow B \equiv \neg B \Rightarrow \neg A$ using the relation $A \Rightarrow B \equiv \neg A \vee B$ and the rules of propositional calculus.

Tutorial 2.2

1. Prove $A, A \Rightarrow B \models B$ for two statements A and B. (modus ponens)
2. Show $A \Rightarrow B, A \vee C, C \Rightarrow \neg D, D \models B$ using the rules of propositional calculus.
3. Show that $\sqrt{2}$ is not rational.

Tutorial 2.3

1. Show that the set $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ supplied with the operations

$$\begin{aligned} a \cdot b &= \mathrm{gcd}(a, b), \\ a + b &= \mathrm{lcm}(a, b), \\ \overline{a} &= 30/a, \end{aligned}$$

   is a Boolean algebra.

2. Show that an isomorphism f between two Boolean algebras always satisfies $f(0) = 0$.

Tutorial 2.4 Construct a CNF for the following Boolean functions:

1. $F(p, q, r, s) = \overline{\overline{p} \cdot ((p \cdot \overline{q}) + r) \cdot (q + s)} + (p + \overline{r} + s)$,
2. $F(p, q) = (p \cdot q) + \overline{\overline{p} + q}$.

Which of these functions satisfies $F \equiv 1$?

Tutorial 2.5 Draw a logic circuit for the Boolean function

$$F(x, y, z) = \overline{x \cdot y} + ((x \cdot y) \cdot \overline{z}).$$

Homework assignment 2.1 **8 points**

1. Negate the following two statements.
   (a) Zagreb is the largest town in Croatia.
   (b) The wall is black.
2. Give the truth value of each of the following statements.
   (a) $2+2=5 \Rightarrow 2+2=4$
   (b) $(\neg(1=0 \wedge 0=0)) \Rightarrow 2=3$
   (c) $(\neg(1=0 \vee 0=0)) \Rightarrow 2=3$
3. Give conditions under which $A \Rightarrow B$ is true but its converse, $B \Rightarrow A$, is false.
4. Use the convention to remove as many parentheses as possible.
   (a) $((A \Rightarrow C) \wedge ((\neg B) \Rightarrow C)) \Rightarrow ((A \vee B) \Rightarrow C)$
   (b) $A \Rightarrow (B \Leftrightarrow (A \Leftrightarrow B))$
   (c) $(\neg A) \Rightarrow ((\neg A) \vee B)$
5. Show the following logical equivalences using truth tables.
   (a) $A \uparrow B \equiv \neg(A \wedge B)$ (this justifies the name NAND – “not and”)
   (b) $A \downarrow B \equiv \neg(A \vee B)$ (this justifies the name NOR – “not or”)
   (c) $\neg(A \vee B) \equiv \neg A \wedge \neg B$
   (d) $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$
6. How many rows are in a truth table for a formula consisting of $k \geq 1$ variables?
7. Prove that $\{\uparrow\}$ is a set of generators using the fact that $\{\neg, \wedge\}$ is a set of generators.

Homework assignment 2.2 **3 points**

1. Prove $A \Rightarrow B, \neg B \models \neg A$ for two statements A and B. (modus tollens)
2. Identify the deduction rule used in each of the following arguments (syllogism, modus ponens, modus tollens, or law of exluded middle).
   (a) Buy these shoes or don’t buy these shoes; there is no other possibility!
   (b) Rich people buy boats. Anna is rich; she should therefore buy a boat.
   (c) Good programmers write clean code. John’s code is a mess; he can’t be a good programmer.

Homework assignment 2.3 **3 points**

1. Show that the element 1 of a Boolean algebra is uniquely defined.
2. Show that an isomorphism f between two Boolean algebras always satisfies $f(1) = 1$.
3. Let B_1 and B_2 be Boolean algebras and let f be an isomorphism between B_1 and B_2. Show that if A is a subalgebra of B_1 then $f(A)$ is a subalgebra of B_2.

Homework assignment 2.4 **6 points**

1. Construct a CNF for the following Boolean functions:

   (a) $F(p, q, r) = \overline{\overline{p} + q} + \left(\overline{\overline{q} + r} + (\overline{p} + r)\right)$,

   (b) $F(p, q, r) = \overline{(\overline{p} + q) \cdot (r + p)} + (r + q)$.

   Which of these functions satisfies $F \equiv 1$?

2. Explain why the construction presented in the lecture applied to

$$(x_1 \cdot y_1) + (x_2 \cdot y_2) + \cdots + (x_n \cdot y_n)$$

   leads to a CNF that consists of 2^n terms. This shows that CNFs may also suffer from a "combinatorial explosion". (You can earn extra points by providing a formal proof.)

Homework assignment 2.5 **4 points**

1. Is it always possible to represent a Boolean function by a logic circuit which contains only NAND gates? Explain your answer.

2. A multiplexer is a circuit with three inputs (x, y, and z) and one output, which has the following behavior: If z is 1, then the output is equal to x; but if z is 0, then the output is equal to y.

   (a) Write down a Boolean function that describes the output of this circuit in terms of its inputs x, y, and z.

   (b) Draw a logic circuit using your expression from part (a) as a "blueprint".