Effective relative permeabilities and capillary pressure of immiscible compressible two-phase flow in porous media

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Contents

1 Introduction

In a previous paper [1], we have obtained a homogenization result for immiscible compressible two-phase flow, such as water-gas, in porous media in the case of a single rock-type model, i.e. we scale up the porosity and absolute permeabilities assuming that the capillary pressure and relative permeabilities depend on the saturation solely. We rigorously justified this homogenization process for the problem by using the two-scale convergence. The aim of this paper is to extend these results to a more general model for gas migration through engineered and geological barriers for a deep repository for radioactive waste. More precisely, we consider the case of porous media with several rock types, i.e. capillary pressure and relative permeability curves being different in each type of porous media. This leads to nonlinear transmission conditions representing the continuity of some physical characteristics such as water and gas pressures, at the interfaces that separate different media. The saturation and some other characteristics are then getting discontinuous at the interfaces. It makes the upscaling procedure more complicated. Using the technique of asymptotic expansions, we obtain a global or effective model of an equivalent homogeneous reservoir. It was then shown, that the homogenized model has almost the same form as the initial model and then defining effective parameters makes sense. These methods are based on computing the solution of partial differential equations in a Representative Elementary Volume (REV) with some boundary conditions; for instance periodic, Dirichlet, or Neumann boundary conditions. In this paper, we restrict ourselves to the mathematical homogenization method as described in the books [10] and [11]. For recent developments on the subject, we refer for instance to [1, 5] and the references therein. The problem is formulated in terms of a nonlinear parabolic equation for the non-wetting phase pressure and a nonlinear degenerate parabolic diffusion-convection equation for the wetting saturation phase with rapidly oscillating porosity function, absolute permeability tensor, relative permeabilities and capillary pressure, subject to appropriate transmission, boundary and initial conditions. Let us mention that the main difficulties related to the mathematical analysis of such equations are the coupling, the degeneracy of the diffusion term in the saturation equation and the degeneracy of the temporal term in the pressure equation. Moreover the transmission conditions are nonlinear and the saturation is discontinuous at the interface separating the two media.

2 Microscopic model

We consider a reservoir $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) which is assumed to be a bounded, connected Lipschitz domain with a periodic microstructure. More precisely, we scale this periodic structure by a parameter ε which represents the ratio of the cell size to the whole region and we assume that $0 < \varepsilon \ll 1$ is a small parameter tending to zero. For convenience in notations, we will only consider a field which contains two different rock types. Let $Y = (0, 1)^d$ be a cell consisting of two subdomains (corresponding to two types of rock). Namely, of a connected subdomain Y_1 and of a subdomain Y_2 which is completely surrounded by Y_1. We denote by $\Gamma_{1,2}$ the interface between the two subdomains in Y. Let $\mathbf{1}_2(y)$ be the characteristic function of Y_2 extended $Y-$ periodically to the whole $\mathbb{R}^d$. Thus, the medium Ω contains two subdomains Ω_1^ε and Ω_2^ε, representing two different types of rock and satisfying:

$$\Omega_2^\varepsilon = \left\{ x \in \Omega \, : \, \mathbf{1}_2 \left(\frac{x}{\varepsilon} \right) = 1 \right\} \quad \text{and} \quad \Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}. \tag{2.1}$$

For the sake of simplicity, we assume that $\overline{\Omega_2^\varepsilon} \cap \partial\Omega = \emptyset$. We also introduce the notation:

$$\Gamma_{1,2}^\varepsilon \stackrel{\text{def}}{=} \partial\Omega_1^\varepsilon \cap \partial\Omega_2^\varepsilon \cap \Omega, \tag{2.2}$$

$$\Omega_T \stackrel{\text{def}}{=} \Omega \times (0, T), \quad \Omega_{\ell,T}^\varepsilon \stackrel{\text{def}}{=} \Omega_\ell^\varepsilon \times (0, T), \quad \Sigma_{1,2,T}^\varepsilon \stackrel{\text{def}}{=} \Gamma_{1,2}^\varepsilon \times (0, T), \tag{2.3}$$

where $T > 0$ is fixed, and, from now on, $\ell = 1, 2$.

The water–gas flow in porous reservoirs can be described in terms of the following characteristics:

- $\Phi^\varepsilon(x) = \Phi(\frac{x}{\varepsilon})$ is the porosity of the medium Ω;

- $K^\varepsilon(x) = K(\frac{x}{\varepsilon})$ is the absolute permeability tensor of Ω;

- ϱ_w, ϱ_g are the densities of water and gas, respectively.

- $S^\varepsilon_{\ell,w} = S^\varepsilon_{\ell,w}(x,t)$, $S^\varepsilon_{\ell,g} = S^\varepsilon_{\ell,g}(x,t)$ are the saturations of water and gas in Ω^ε_ℓ, respectively;

- $k^{(\ell)}_{r,w} = k^{(\ell)}_{r,w}(S^\varepsilon_{\ell,w})$, $k^{(\ell)}_{r,g} = k^{(\ell)}_{r,g}(S^\varepsilon_{\ell,g})$ are the relative permeabilities of water and gas in the medium Ω^ε_ℓ, respectively;

- $p^\varepsilon_{\ell,w} = p^\varepsilon_{\ell,w}(x,t)$, $p^\varepsilon_{\ell,g} = p^\varepsilon_{\ell,g}(x,t)$ are the pressures of water and gas in Ω^ε_ℓ, respectively.

The conservation of mass in each phase can be written as (see, e.g., [2, 3, 6, 8, 9]):

$$\begin{cases} \Phi^\varepsilon(x)\dfrac{\partial}{\partial t}(S^\varepsilon_{\ell,w}\,\varrho_w(p^\varepsilon_{\ell,w})) + \operatorname{div}\left(\varrho_w(p^\varepsilon_{\ell,w})\,\vec{q}^{\,\varepsilon}_{\ell,w}\right) = 0 \quad \text{in } \Omega^\varepsilon_{\ell,T}; \\[2mm] \Phi^\varepsilon(x)\dfrac{\partial}{\partial t}(S^\varepsilon_{\ell,g}\,\varrho_g(p^\varepsilon_{\ell,g})) + \operatorname{div}\left(\varrho_g(p^\varepsilon_{\ell,g})\,\vec{q}^{\,\varepsilon}_{\ell,g}\right) = 0 \quad \text{in } \Omega^\varepsilon_{\ell,T}, \end{cases} \tag{2.4}$$

where the velocities of the water and gas $\vec{q}^{\,\varepsilon}_{\ell,w}$, $\vec{q}^{\,\varepsilon}_{\ell,g}$ are defined by the Darcy-Muskat's law:

$$\vec{q}^{\,\varepsilon}_{\ell,w} = -K^\varepsilon(x)\lambda_{\ell,w}(S^\varepsilon_{\ell,w})\left(\nabla p^\varepsilon_{\ell,w} - \varrho_w(p^\varepsilon_{\ell,w})\vec{g}\right), \quad \text{with } \lambda_{\ell,w}(S^\varepsilon_{\ell,w}) = \frac{k^{(\ell)}_{r,w}}{\mu_w}(S^\varepsilon_{\ell,w}); \tag{2.5}$$

$$\vec{q}^{\,\varepsilon}_{\ell,g} = -K^\varepsilon(x)\lambda_{\ell,g}(S^\varepsilon_{\ell,g})\left(\nabla p^\varepsilon_{\ell,g} - \varrho_g(p^\varepsilon_{\ell,g})\vec{g}\right), \quad \text{with } \lambda_{\ell,g}(S^\varepsilon_{\ell,g}) = \frac{k^{(\ell)}_{r,g}}{\mu_g}(S^\varepsilon_{\ell,g}). \tag{2.6}$$

Here $\vec{g}$, μ_w, μ_g are the gravity vector and the viscosities of the water and gas, respectively. From now on we assume that the density of the water is constant, which for the sake of simplicity will be taken equal to one, i.e. $\varrho_w(p^\varepsilon_w) = \text{Const} = 1$, and the gas density ϱ_g is a monotone smooth function such that

$$\varrho_g(p) = \varrho_{\min} \quad \text{for } p \leqslant p_{\min}; \qquad \varrho_g(p) = \varrho_{\max} \quad \text{for } p \geqslant p_{\max};$$
$$\varrho_{\min} < \varrho_g(p) < \varrho_{\max} \quad \text{for } p_{\min} < p < p_{\max}; \tag{2.7}$$

here the pairs of constants $\varrho_{\min}, \varrho_{\max}$ and $p_{\min}, p_{\max}$ satisfy the bounds:

$$0 < \varrho_{\min} < \varrho_{\max} < +\infty \quad \text{and} \quad 0 < p_{\min} < p_{\max} < +\infty. \tag{2.8}$$

In what follows we also make use of the following notation:

$$\varrho^\varepsilon_{\ell,g} = \varrho_g(p^\varepsilon_{\ell,g}). \tag{2.9}$$

The model is completed as follows. By the definition of saturations, one has

$$S^\varepsilon_{\ell,w} + S^\varepsilon_{\ell,g} = 1 \quad \text{with } S^\varepsilon_{\ell,w}, S^\varepsilon_{\ell,g} \geqslant 0. \tag{2.10}$$

To simplify the notation, we denote

$$S^\varepsilon_\ell \stackrel{\text{def}}{=} S^\varepsilon_{\ell,w}. \tag{2.11}$$

Then the curvature of the contact surface between the two fluids links the jump of pressure of two phases to the saturation by the capillary pressure law:

$$P_{\ell,c}(S^\varepsilon_\ell) = p^\varepsilon_{\ell,g} - p^\varepsilon_{\ell,w} \quad \text{with } P'_{\ell,c}(S^\varepsilon_\ell) < 0 \text{ for all } S^\varepsilon_\ell \in [0,1] \text{ and } P_{\ell,c}(1) = 0, \tag{2.12}$$

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where $P'_{\ell,c}(s)$ denotes the derivative of the function $P_{\ell,c}(s)$ with respect to the variable s.

Now due to (2.9), (2.11), and the assumption on the density of the water phase we rewrite (2.4) as follows:

$$
\begin{cases}
\Phi^\varepsilon(x)\dfrac{\partial S^\varepsilon}{\partial t} - \operatorname{div}\left(K^\varepsilon(x)\lambda_w\left(\dfrac{x}{\varepsilon},S^\varepsilon\right)(\nabla p_w^\varepsilon - \vec{g})\right) = 0 \quad \text{in } \Omega_{1,T}^\varepsilon \cup \Omega_{2,T}^\varepsilon; \\[3mm]
\Phi^\varepsilon(x)\dfrac{\partial\Theta^\varepsilon}{\partial t} - \operatorname{div}\left(K^\varepsilon(x)\lambda_g\left(\dfrac{x}{\varepsilon},S^\varepsilon\right)\varrho_g(p_g^\varepsilon)\left(\nabla p_g^\varepsilon - \varrho_g(p_g^\varepsilon)\vec{g}\right)\right) = 0 \quad \text{in } \Omega_{1,T}^\varepsilon \cup \Omega_{2,T}^\varepsilon; \\[3mm]
P_c\left(\dfrac{x}{\varepsilon},S^\varepsilon\right) = p_g^\varepsilon - p_w^\varepsilon \quad \text{in } \Omega_T,
\end{cases}
\tag{2.13}
$$

where

$$
S^\varepsilon(x,t) \stackrel{\text{def}}{=} S_1^\varepsilon(x,t)\,\mathbf{1}_1^\varepsilon(x) + S_2^\varepsilon(x,t)\,\mathbf{1}_2^\varepsilon(x);
\tag{2.14}
$$

$$
p_g^\varepsilon(x,t) \stackrel{\text{def}}{=} p_{1,g}^\varepsilon(x,t)\,\mathbf{1}_1^\varepsilon(x) + p_{2,g}^\varepsilon(x,t)\,\mathbf{1}_2^\varepsilon(x) \text{ and } p_w^\varepsilon(x,t) \stackrel{\text{def}}{=} p_{1,w}^\varepsilon(x,t)\,\mathbf{1}_1^\varepsilon(x) + p_{2,w}^\varepsilon(x,t)\,\mathbf{1}_2^\varepsilon(x); \tag{2.15}
$$

$$
\Theta^\varepsilon \stackrel{\text{def}}{=} \varrho_g(p_g^\varepsilon)(1 - S^\varepsilon) = \Theta_1^\varepsilon\,\mathbf{1}_1^\varepsilon(x) + \Theta_2^\varepsilon(x,t)\,\mathbf{1}_2^\varepsilon(x) \quad \text{with } \Theta_\ell^\varepsilon \stackrel{\text{def}}{=} \varrho_g(p_{\ell,g}^\varepsilon)(1 - S_\ell^\varepsilon); \tag{2.16}
$$

$$
P_c\left(\frac{x}{\varepsilon},S^\varepsilon\right) \stackrel{\text{def}}{=} P_{1,c}\left(S_1^\varepsilon\right)\mathbf{1}_1^\varepsilon(x) + P_{2,c}\left(S_2^\varepsilon\right)\mathbf{1}_2^\varepsilon(x); \tag{2.17}
$$

$$
\lambda_w\left(\frac{x}{\varepsilon},S^\varepsilon\right) \stackrel{\text{def}}{=} \lambda_{1,w}(S_1^\varepsilon)\,\mathbf{1}_1^\varepsilon(x) + \lambda_{2,w}(S_2^\varepsilon)\,\mathbf{1}_2^\varepsilon(x) \text{ and } \lambda_g\left(\frac{x}{\varepsilon},S^\varepsilon\right) \stackrel{\text{def}}{=} \lambda_{1,g}(S_1^\varepsilon)\,\mathbf{1}_1^\varepsilon(x) + \lambda_{2,g}(S_2^\varepsilon)\,\mathbf{1}_2^\varepsilon(x), \tag{2.18}
$$

where

$$
\lambda_{\ell,g}(S_\ell^\varepsilon) := \lambda_{\ell,g}(1 - S_\ell^\varepsilon). \tag{2.19}
$$

Here $\mathbf{1}_\ell^\varepsilon(x) = \mathbf{1}_\ell(\frac{x}{\varepsilon})$ is the characteristic function of the subdomain Ω_ℓ^ε.

The phase fluxes and pressures are to be continuous on the interface $\Gamma_{1,2}$. Namely,

$$
\begin{cases}
\vec{q}_{1,w}^{\,\varepsilon} \cdot \vec{\nu} = \vec{q}_{2,w}^{\,\varepsilon} \cdot \vec{\nu} \text{ and } \vec{q}_{1,g}^{\,\varepsilon} \cdot \vec{\nu} = \vec{q}_{2,g}^{\,\varepsilon} \cdot \vec{\nu} \quad \text{on } \Sigma_{1,2,T}^\varepsilon; \\[2mm]
p_{1,w}^\varepsilon = p_{2,w}^\varepsilon \text{ and } p_{1,g}^\varepsilon = p_{2,g}^\varepsilon \quad \text{on } \Sigma_{1,2,T}^\varepsilon,
\end{cases}
\tag{2.20}
$$

where $\Sigma_{1,2,T}^\varepsilon$ is defined in (2.3), $\vec{\nu}$ is the unit outer normal on $\Gamma_{1,2}$, and the fluxes $\vec{q}_{\ell,w}^{\,\varepsilon}, \vec{q}_{\ell,g}^{\,\varepsilon}$, in the notation (2.9), (2.11) are given by:

$$
\vec{q}_{\ell,w}^{\,\varepsilon} = -K^\varepsilon(x)\lambda_{\ell,w}(S_\ell^\varepsilon)\left(\nabla p_{\ell,w}^\varepsilon - \vec{g}\right) \quad \text{and} \quad \vec{q}_{\ell,g}^{\,\varepsilon} = -K^\varepsilon(x)\lambda_{\ell,g}(S_\ell^\varepsilon)\left(\nabla p_{\ell,g}^\varepsilon - \varrho_{\ell,g}^\varepsilon\vec{g}\right).
$$

Remark 1 *It is important to notice that in contrast to the functions $p_g^\varepsilon, p_w^\varepsilon$, the function S^ε has a jump on the interface $\Gamma_{1,2}$. Namely, it is easy to see from interface conditions (2.20) for the phase pressures that*

$$
P_{1,c}(S_1^\varepsilon) = P_{2,c}(S_2^\varepsilon) \quad \text{on } \Sigma_{1,2,T}^\varepsilon. \tag{2.21}
$$

This means that $S_1^\varepsilon \neq S_2^\varepsilon$ on $\Sigma_{1,2,T}^\varepsilon$.

Now we specify the boundary and initial conditions. We suppose that the boundary $\partial\Omega$ consists of two parts Γ_{inj} and Γ_{imp} such that $\Gamma_{inj} \cap \Gamma_{imp} = \emptyset$, $\partial\Omega = \overline{\Gamma}_{inj} \cup \overline{\Gamma}_{imp}$. The boundary conditions are given by:

$$
\begin{cases}
p_{1,g}^\varepsilon(x,t) = p_{1,w}^\varepsilon(x,t) = 0 \quad \text{on } \Gamma_{inj} \times (0,T); \\[2mm]
\vec{q}_{1,w}^{\,\varepsilon} \cdot \vec{\nu} = \vec{q}_{1,g}^{\,\varepsilon} \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_{imp} \times (0,T).
\end{cases}
\tag{2.22}
$$

Finally, the initial conditions read:

$$
p_w^\varepsilon(x,0) = p_w^o(x) \quad \text{and} \quad p_g^\varepsilon(x,0) = p_g^o(x) \quad \text{in } \Omega. \tag{2.23}
$$

3 Upscaling of an immiscible compressible two-phase flow model

In this section, we use the method of two-length-scale asymptotic expansions, see e.g. [4, 12], to derive the limiting equations in the case of a periodic structure.

We are looking for an asymptotic expansion of $(p_w^\varepsilon, p_g^\varepsilon, S^\varepsilon)$ of the form

$$p_w^\varepsilon = p_w^{(0)}(x,t) + \varepsilon\, p_w^{(1)}(x,y,t) + \varepsilon^2\, p_w^{(2)}(x,y,t) + \dots ; \tag{3.1}$$

$$p_g^\varepsilon = p_g^{(0)}(x,t) + \varepsilon\, p_g^{(1)}(x,y,t) + \varepsilon^2\, p_g^{(2)}(x,y,t) + \dots ; \tag{3.2}$$

$$S^\varepsilon = S^{(0)}(x,y,t) + \varepsilon\, S^{(1)}(x,y,t) + \varepsilon^2\, S^{(2)}(x,y,t) + \dots \tag{3.3}$$

where $y = x/\varepsilon$. We note first that the capillary pressure law $(2.13)_3$ implies the form of macroscopic capillary pressure law:

$$P_c\left(y, S^{(0)}\right) = p_g^{(0)} - p_w^{(0)}. \tag{3.4}$$

From the assumption that $p_w^{(0)}$ and $p_g^{(0)}$ are are independent of the fast variable y it follows that the dependence of $S^{(0)}(x,y,t)$ on the fast variable comes only in the from of the characteristic functions of the sets Y_1, Y_2, namely

$$S^{(0)}(x,y,t) = S_1^{(0)}(x,t)\mathbf{1}_1(y) + S_2^{(0)}(x,t)\mathbf{1}_2(y). \tag{3.5}$$

The values $S_1^{(0)}$ and $S_2^{(0)}$ are uniquely determined by the capillary preesure $p_g^{(0)} - p_w^{(0)}$ through the capillary pressure functions:

$$P_{\ell,c}(S_\ell^{(0)}) = p_g^{(0)} - p_w^{(0)}. \tag{3.6}$$

Remark 2 *As in [7] (p. 201), we claim that the independence of $p_w^{(0)}, p_g^{(0)}$ of the fast variable y is due to the continuity of the phase pressures on the interface $\Sigma_{1,2,T}^\varepsilon$. In fact, this assumption is confirmed for the case of incompressible immiscible flows because the functions $p_w^\varepsilon, p_g^\varepsilon$ are uniformly bounded in the space $L^r(0,T; W^{1,r}(\Omega))$, where $1 < r < 2$ (see [13]). In our case we can prove the similar result for the function p_w^ε, only. The assumption (3.5) follows from the compactness result for the extensions of the saturation function S_ℓ^ε.*

The functions λ_w, λ_g, and ϱ_g depend in a nonlinear way on the saturation S^ε and the gas pressure p_g^ε, respectively. Then we use the Taylor expansions for both functions $\lambda_w(\frac{x}{\varepsilon}, S^\varepsilon), \lambda_g(\frac{x}{\varepsilon}, S^\varepsilon)$, namely:

$$\lambda_w\left(\frac{x}{\varepsilon}, S^\varepsilon\right) = \lambda_w\left(y, S^{(0)}\right) + \varepsilon\, \lambda_w'\left(y, S^{(0)}\right) S^{(1)} + \dots ; \tag{3.7}$$

$$\lambda_g\left(\frac{x}{\varepsilon}, S^\varepsilon\right) = \lambda_g\left(y, S^{(0)}\right) + \varepsilon\, \lambda_g'\left(y, S^{(0)}\right) S^{(1)} + \dots \tag{3.8}$$

In the same way, taking into account (3.2), we get:

$$\varrho_g\left(p_g^\varepsilon\right) = \varrho_g\left(p_g^{(0)}\right) + \varepsilon\, \varrho_g'\left(p_g^{(0)}\right) p_g^{(1)} + \dots \tag{3.9}$$

3.1 Upscaling of the saturation equation

First we notice that the derivatives obey the law:

$$\frac{\partial}{\partial x_i} := \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i} \quad (i = 1, .., d). \tag{3.10}$$

Then due to (3.10), from the first equation in (2.13), we have:

$$\Phi(y)\frac{\partial}{\partial t}\left(S^{(0)}(x, y, t) + \varepsilon\, S^{(1)}(x, y, t) + ...\right) -$$

$$-\left\{\mathrm{div}_x + \frac{1}{\varepsilon}\,\mathrm{div}_y\right\}\left[K(y)\left(\lambda_w\left(y, S^{(0)}\right) + \varepsilon\,\lambda'_w\left(y, S^{(0)}\right)S^{(1)} + ...\right)\times\right.$$

$$\left.\times\left(\left\{\nabla_x + \frac{1}{\varepsilon}\nabla_y\right\}\left(p_w^{(0)}(x, t) + \varepsilon\, p_w^{(1)}(x, y, t) + ...\right) - \vec{g}\right)\right] = 0. \tag{3.11}$$

The divergence operator in (3.11) we rewrite as follows:

$$\left\{\mathrm{div}_x + \frac{1}{\varepsilon}\,\mathrm{div}_y\right\}\left[K(y)\left(\lambda_w\left(y, S^{(0)}\right) + ...\right)\left(\left\{\nabla_x + \frac{1}{\varepsilon}\nabla_y\right\}\left(p_w^{(0)}(x, t) + ...\right) - \vec{g}\right)\right] =$$

$$= \left\{\mathrm{div}_x + \frac{1}{\varepsilon}\,\mathrm{div}_y\right\}\left[K(y)\left(\lambda_w\left(y, S^{(0)}\right) + ...\right)\left(\nabla_x\left(p_w^{(0)}(x, t) + ...\right) - \vec{g} + \frac{1}{\varepsilon}\nabla_y\left(p_w^{(0)}(x, t) + ...\right)\right)\right] =$$

$$= \mathrm{div}_x\left[K(y)\left(\lambda_w\left(y, S^{(0)}\right) + ...\right)\left(\nabla_x\left(p_w^{(0)}(x, t) + ...\right) - \vec{g}\right)\right] +$$

$$+ \frac{1}{\varepsilon}\,\mathrm{div}_x\left[K(y)\left(\lambda_w\left(y, S^{(0)}\right) + ...\right)\nabla_y\left(p_w^{(0)}(x, t) + ...\right)\right] +$$

$$+ \frac{1}{\varepsilon}\,\mathrm{div}_y\left[K(y)\left(\lambda_w\left(y, S^{(0)}\right) + ...\right)\left(\nabla_x\left(p_w^{(0)}(x, t) + ...\right) - \vec{g}\right)\right] +$$

$$+ \frac{1}{\varepsilon^2}\,\mathrm{div}_y\left[K(y)\left(\lambda_w\left(y, S^{(0)}\right) + ...\right)\nabla_y\left(p_w^{(0)}(x, t) + ...\right)\right]. \tag{3.12}$$

Now in a standard way we have:

Terms of order ε^{-2}. From the fourth term on the right hand side of (3.12) we get the following equation:

$$-\mathrm{div}_y\left(K(y)\lambda_w\left(y, S^{(0)}\right)\nabla_y p_w^{(0)}\right) = 0. \tag{3.13}$$

It is evidently satisfied because $p_w^{(0)}$ does not depend on y.

Terms of order ε^{-1}. From the second, third, and fourth terms on the right hand side of (3.12) we get:

$$-\mathrm{div}_x\left(K(y)\lambda_w\left(y, S^{(0)}\right)\nabla_y p_w^{(0)}\right) - \mathrm{div}_y\left(K(y)\lambda_w\left(y, S^{(0)}\right)\left(\nabla_x p_w^{(0)} - \vec{g}\right)\right) -$$

$$-\mathrm{div}_y\left(K(y)\lambda'_w\left(y, S^{(0)}\right)S^{(1)}\nabla_y p_w^{(0)} + K(y)\lambda_w\left(y, S^{(0)}\right)\nabla_y p_w^{(1)}\right) = 0.$$

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Taking into account that $p_w^{(0)} = p_w^{(0)}(x, t)$, from this equation we get:

$$-\text{div}_y \left(K(y)\lambda_w \left(y, S^{(0)} \right) \left(\nabla_x p_w^{(0)} - \vec{g} \right) \right) - \text{div}_y \left(K(y)\lambda_w \left(y, S^{(0)} \right) \nabla_y p_w^{(1)} \right) = 0. \tag{3.14}$$

In a standard way we set:

$$p_w^{(1)}(x, y, t) = \sum_j \xi_{j,w}(x, t, y) \left(\frac{\partial p_w^{(0)}}{\partial x_j}(x, t) - g_j \right), \tag{3.15}$$

where g_j is the j–th component of $\vec{g}$. Then taking into account definitions (3.5) and (3.15), from equation (3.14) we get the cell problem:

$$\boxed{\begin{cases} -\text{div}_y \left(K(y)\lambda_w \left(y; S^{(0)} \right) \left(\nabla_y \xi_{j,w} + \vec{e}_j \right) \right) = 0 \quad \text{in } Y; \\ y \mapsto \xi_{j,w}(y) \quad Y - \text{periodic} \end{cases}} \tag{3.16}$$

with $\vec{e}_j$ being the j–th coordinate vector. Here

$$\lambda_w \left(y; S^{(0)} \right) \stackrel{\text{def}}{=} \lambda_{1,w} \left(S_1^{(0)} \right) \mathbf{1}_1(y) + \lambda_{2,w} \left(S_2^{(0)} \right) \mathbf{1}_2(y). \tag{3.17}$$

As we noted before, the macro-scale saturation variables $S_1^{(0)}(x, t)$ and $S_2^{(0)}(x, t)$ are completely determined by the value of macro-scale capillary pressure $p_g^{(0)}(x, t) - p_w^{(0)}(x, t)$. Therefore, the solution $\xi_{j,w}$ of the cell problem (3.16) depends on the macro-scale capillary pressure as a parameter, and theoretically, the cell problem (3.16) has to be solved for all values of the macro-scale capillary pressures. In practice, it will be solved only for a finite number of capillary pressure values.

Terms of order ε^0. Consider the equation (3.12). Except the evident terms of order ε^0 we also have the following ones. The third term in (3.12), i.e.,

$$\frac{1}{\varepsilon} \text{div}_y \left[K(y) \left(\lambda_w \left(y, S^{(0)} \right) + ... \right) \left(\nabla_x \left(p_w^{(0)}(x, t) + ... \right) - \vec{g} \right) \right],$$

gives:

$$\text{div}_y \left(K(y)\lambda_w' \left(y, S^{(0)} \right) S^{(1)} \left(\nabla_x p_w^{(0)} - \vec{g} \right) \right) + \text{div}_y \left(K(y)\lambda_w \left(y, S^{(0)} \right) \nabla_x p_w^{(1)} \right). \tag{3.18}$$

The fourth term in (3.12), i.e.,

$$\frac{1}{\varepsilon^2} \text{div}_y \left[K(y) \left(\lambda_w \left(y, S^{(0)} \right) + ... \right) \nabla_y \left(p_w^{(0)}(x, t) + ... \right) \right],$$

gives:

$$\frac{1}{2} \text{div}_y \left(K(y)\lambda_w'' \left(y, S^{(0)} \right) \left(S^{(2)} \right)^2 \nabla_y p_w^{(0)} \right) + \text{div}_y \left(K(y)\lambda_w \left(y, S^{(0)} \right) \nabla_y p_w^{(2)} \right) =$$

$$= \text{div}_y \left(K(y)\lambda_w \left(y, S^{(0)} \right) \nabla_y p_w^{(2)} \right). \tag{3.19}$$

Now taking into account these terms, from (3.11), (3.12) we have:

$$\Phi(y)\frac{\partial S^{(0)}}{\partial t} - \text{div}_x \left(K(y)\lambda_w \left(y, S^{(0)} \right) \left(\nabla_x p_w^{(0)} - \vec{g} \right) \right) - \text{div}_x \left(K(y)\lambda_w' \left(y, S^{(0)} \right) S^{(1)} \nabla_y p_w^{(0)} \right) -$$

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$$-\mathrm{div}_x\left(K(y)\lambda_w\left(y,S^{(0)}\right)\nabla_y p_w^{(1)}\right)-\mathrm{div}_y Z(x,y,t)=0,$$

where

$$Z\overset{\mathrm{def}}{=}K(y)\lambda_w'\left(y,S^{(0)}\right)S^{(1)}\left(\nabla_x p_w^{(0)}-\vec{g}\right)+K(y)\lambda_w\left(y,S^{(0)}\right)\nabla_x p_w^{(1)}+K(y)\lambda_w\left(y,S^{(0)}\right)\nabla_y p_w^{(2)}.$$

Notice that Z is an Y–periodic function. Since $\nabla_y p_w^{(0)}=0$, then the last equation yields:

$$\Phi(y)\frac{\partial S^{(0)}}{\partial t}-\mathrm{div}_x\left(K(y)\lambda_w\left(y;S^{(0)}\right)\left(\nabla_x p_w^{(0)}-\vec{g}+\nabla_y p_w^{(1)}\right)\right)-\mathrm{div}_y Z(x,y,t)=0. \qquad (3.20)$$

Now we integrate (3.20) over the cell Y. Taking into account that Z is Y–periodic we have that

$$\int_Y \mathrm{div}_y Z\,dy=0.$$

Consider the first term on the left hand side of (3.20). Due to (3.5), we have:

$$\int_Y \Phi(y)\frac{\partial S^{(0)}}{\partial t}\,dy=\frac{\partial}{\partial t}\int_Y \Phi(y)\left\{S_1^{(0)}(x,t)\mathbf{1}_1(y)+S_2^{(0)}(x,t)\mathbf{1}_2(y)\right\}\,dy=$$

$$=\frac{\partial}{\partial t}\left[S_1^{(0)}(x,t)\int_{Y_1}\Phi(y)\,dy+S_2^{(0)}(x,t)\int_{Y_2}\Phi(y)\,dy\right]=\langle\Phi\rangle\frac{\partial S^*}{\partial t}, \qquad (3.21)$$

where $\langle\,\cdot\,\rangle$ denotes the mean value of the corresponding function over the cell Y. The variable

$$S^*(x,t)=S^*\left(S^{(0)}\right)\overset{\mathrm{def}}{=}\frac{1}{\langle\Phi\rangle}\left[S_1^{(0)}(x,t)\int_{Y_1}\Phi(y)\,dy+S_2^{(0)}(x,t)\int_{Y_2}\Phi(y)\,dy\right], \qquad (3.22)$$

is the correct macro-scale saturation while the variables $S_\ell^{(0)}$ represent only *partial macro-scale saturations*. Then from (3.20), (3.21) we get:

$$\langle\Phi\rangle\frac{\partial S^*}{\partial t}-\mathrm{div}_x\mathcal{J}_w=0, \qquad (3.23)$$

where

$$\mathcal{J}_w\overset{\mathrm{def}}{=}\int_Y K(y)\lambda_w\left(y;S^{(0)}\right)\left(\nabla_x p_w^{(0)}-\vec{g}+\nabla_y p_w^{(1)}\right)\,dy. \qquad (3.24)$$

Taking into account the definition of the function $p_w^{(1)}$ (see (3.15)), we obtain that

$$\mathcal{J}_w=\int_Y K(y)\lambda_w\left(y;S^{(0)}\right)[\mathbb{I}+\nabla_y\xi_w\left(y;S^{(0)}\right)]\,dy\left(\nabla_x p_w^{(0)}-\vec{g}\right), \qquad (3.25)$$

where $\mathbb{I}$ is the unit tensor, $\nabla_y\xi_w\left(y;S^{(0)}\right)$ is the tensor with the components $\partial\xi_{j,w}/\partial y_i$ and $\xi_{j,w}$ is the solution of (3.16). Thus from (3.23) and (3.25) we obtain the first homogenized equation:

$$\boxed{\langle\Phi\rangle\frac{\partial S^*}{\partial t}-\mathrm{div}_x\left(\Lambda_w(S^{(0)})\left(\nabla_x p_w^{(0)}-\vec{g}\right)\right)=0,} \qquad (3.26)$$

where the homogenized permeability tensor Λ_w of the water phase is given by:

$$\Lambda_w(S^{(0)})\overset{\mathrm{def}}{=}\left\langle K(y)\lambda_w\left(y;S^{(0)}\right)[\mathbb{I}+\nabla_y\xi_w\left(y;S^{(0)}\right)]\right\rangle. \qquad (3.27)$$

We note finally that the function $S^{(0)}(x,t,y)$ is uniquely defined by the macro-scale capillary pressure $p_g^{(0)}(x,t)-p_w^{(0)}(x,t)$ which makes $\Lambda_w(S^{(0)})$ a nonlinear function of $p_g^{(0)}(x,t)-p_w^{(0)}(x,t)$.

3.2 Upscaling of the pressure equation

We recall that this equation reads:

$$\Phi^\varepsilon(x)\frac{\partial\Theta^\varepsilon}{\partial t} - \mathrm{div}\left(K^\varepsilon(x)\lambda_g\left(\frac{x}{\varepsilon},S^\varepsilon\right)\varrho_g(p_g^\varepsilon)\left(\nabla p_g^\varepsilon - \varrho_g(p_g^\varepsilon)\vec{g}\right)\right) = 0 \quad \text{in } \Omega_T. \tag{3.28}$$

First, from (3.3), (3.9) we get:

$$\Phi\left(\frac{x}{\varepsilon}\right)\frac{\partial\Theta^\varepsilon}{\partial t} = \Phi\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial t}\left(\varrho_g(p_g^\varepsilon)\left(1-S^\varepsilon\right)\right) =$$

$$= \Phi(y)\frac{\partial}{\partial t}\left[\left(\varrho_g\left(p_g^{(0)}\right) + \varepsilon\,\varrho_g'\left(p_g^{(0)}\right)p_g^{(1)} + ...\right)\left(1 - S^{(0)} - \varepsilon\,S^{(1)} + ...\right)\right] =$$

$$= \Phi(y)\frac{\partial}{\partial t}\left[\Theta^{(0)} + \varepsilon\left\{-\varrho_g\left(p_g^{(0)}\right)S^{(1)} + \varrho_g'\left(p_g^{(0)}\right)p_g^{(1)}\left(1-S^{(0)}\right)\right\} + ...\right], \tag{3.29}$$

where the function $\Theta^{(0)}$ is given by:

$$\Theta^{(0)} \stackrel{\text{def}}{=} \varrho_g\left(p_g^{(0)}\right)\left(1-S^{(0)}\right). \tag{3.30}$$

Then due to (3.10), from (3.28), we have:

$$\Phi(y)\frac{\partial}{\partial t}\left[\Theta^{(0)} + \varepsilon\left\{-\varrho_g\left(p_g^{(0)}\right)S^{(1)} + \varrho_g'\left(p_g^{(0)}\right)p_g^{(1)}\left(1-S^{(0)}\right)\right\} + ...\right] -$$

$$-\left\{\mathrm{div}_x + \frac{1}{\varepsilon}\,\mathrm{div}_y\right\}\left[K(y)\left(\lambda_g\left(y,S^{(0)}\right) + \varepsilon\,\lambda_g'\left(y,S^{(0)}\right)S^{(1)} + ...\right)\left(\varrho_g\left(p_g^{(0)}\right) + \varepsilon\,\varrho_g'\left(p_g^{(0)}\right)p_g^{(1)} + ...\right) \times \right.$$

$$\left. \times\left(\left\{\nabla_x + \frac{1}{\varepsilon}\nabla_y\right\}\left(p_g^{(0)}(x,t) + \varepsilon\,p_g^{(1)}(x,y,t) + ...\right) - \left(\varrho_g\left(p_g^{(0)}\right) + ...\right)\vec{g}\right)\right] = 0. \tag{3.31}$$

The differential operator can be rewritten as follows:

$$\left\{\mathrm{div}_x + \frac{1}{\varepsilon}\,\mathrm{div}_y\right\}\left[K(y)\left(\lambda_g\left(y,S^{(0)}\right) + \varepsilon\,\lambda_g'\left(y,S^{(0)}\right)S^{(1)} + ...\right)\left(\varrho_g\left(p_g^{(0)}\right) + \varepsilon\,\varrho_g'\left(p_g^{(0)}\right)p_g^{(1)} + ...\right) \times \right.$$

$$\left. \times\left(\left\{\nabla_x + \frac{1}{\varepsilon}\nabla_y\right\}\left(p_g^{(0)}(x,t) + \varepsilon\,p_g^{(1)}(x,y,t) + ...\right) - \left(\varrho_g\left(p_g^{(0)}\right) + ...\right)\vec{g}\right)\right] =$$

$$= \left\{\mathrm{div}_x + \frac{1}{\varepsilon}\,\mathrm{div}_y\right\}\left[K(y)\left(\lambda_g\left(y,S^{(0)}\right) + \varepsilon\,\lambda_g'\left(y,S^{(0)}\right)S^{(1)} + ...\right)\left(\varrho_g\left(p_g^{(0)}\right) + \varepsilon\,\varrho_g'\left(p_g^{(0)}\right)p_g^{(1)} + ...\right) \times \right.$$

$$\left. \times\left(\nabla_x\left(p_g^{(0)}(x,t) + \varepsilon\,p_g^{(1)}(x,y,t) + ...\right) - \left(\varrho_g\left(p_g^{(0)}\right) + ...\right)\vec{g} + \frac{1}{\varepsilon}\nabla_y\left(p_g^{(0)}(x,t) + \varepsilon\,p_g^{(1)}(x,y,t) + ...\right)\right)\right] =$$

$$= \mathrm{div}_x\left[K(y)\left(\lambda_g\left(y,S^{(0)}\right) + ...\right)\left(\varrho_g\left(p_g^{(0)}\right) + ...\right)\left(\nabla_x\left(p_g^{(0)}(x,t) + ...\right) - \left(\varrho_g\left(p_g^{(0)}\right) + ...\right)\vec{g}\right)\right] +$$

$$+ \frac{1}{\varepsilon}\,\mathrm{div}_x\left[K(y)\left(\lambda_g\left(y,S^{(0)}\right) + ...\right)\left(\varrho_g\left(p_g^{(0)}\right) + ...\right)\nabla_y\left(p_g^{(0)} + ...\right)\right] +$$

$$+ \frac{1}{\varepsilon}\,\mathrm{div}_y\left[K(y)\left(\lambda_g\left(y,S^{(0)}\right) + ...\right)\left(\varrho_g\left(p_g^{(0)}\right) + ...\right)\left(\nabla_x\left(p_g^{(0)}(x,t) + ...\right) - \left(\varrho_g\left(p_g^{(0)}\right) + ...\right)\vec{g}\right)\right] +$$

$$+ \frac{1}{\varepsilon^2}\,\mathrm{div}_y\left[K(y)\left(\lambda_g\left(y,S^{(0)}\right) + ...\right)\left(\varrho_g\left(p_g^{(0)}\right) + ...\right)\nabla_y\left(p_g^{(0)} + ...\right)\right]. \tag{3.32}$$

Now in a standard way we have:

Terms of order ε^{-2}. From the fourth term on the right hand side of (3.32) we have the following equation:

$$-\mathrm{div}_y\left(K(y)\lambda_g\left(y,S^{(0)}\right)\varrho_g\left(p_g^{(0)}\right)\nabla_y p_g^{(0)}\right) = 0. \tag{3.33}$$

It is evidently satisfied because $p_g^{(0)}$ does not depend on y.

Terms of order ε^{-1}. From the second, third, and fourth terms on the right hand side of (3.32) we get:

$$-\mathrm{div}_x\left(K(y)\lambda_g\left(y,S^{(0)}\right)\varrho_g\left(p_g^{(0)}\right)\nabla_y p_g^{(0)}\right)-$$

$$-\mathrm{div}_y\left(K(y)\lambda_g\left(y,S^{(0)}\right)\varrho_g\left(p_g^{(0)}\right)\left(\nabla_x p_g^{(0)}-\varrho_g\left(p_g^{(0)}\right)\vec{g}\right)\right)-$$

$$-\mathrm{div}_y\left(K(y)\lambda_g'\left(y,S^{(0)}\right)S^{(1)}\varrho_g\left(p_g^{(0)}\right)\nabla_y p_g^{(0)}\right)-\mathrm{div}_y\left(K(y)\lambda_g\left(y,S^{(0)}\right)\varrho_g'\left(p_g^{(0)}\right)p_g^{(1)}\nabla_y p_g^{(0)}\right)-$$

$$-\mathrm{div}_y\left(K(y)\lambda_g\left(y,S^{(0)}\right)\varrho_g\left(p_g^{(0)}\right)\nabla_y p_g^{(1)}\right) = 0.$$

Taking into account that $p_g^{(0)}=p_g^{(0)}(x,t)$, from this equation we get:

$$-\mathrm{div}_y\left(K(y)\lambda_g\left(y,S^{(0)}\right)\varrho_g\left(p_g^{(0)}\right)\left(\nabla_x p_g^{(0)}-\varrho_g\left(p_g^{(0)}\right)\vec{g}\right)\right)- \tag{3.34}$$

$$-\mathrm{div}_y\left(K(y)\lambda_g\left(y,S^{(0)}\right)\varrho_g\left(p_g^{(0)}\right)\nabla_y p_g^{(1)}\right) = 0.$$

As in the previous case, in a standard way we set:

$$p_g^{(1)}(x,y,t) = \sum_j \xi_{j,g}(x,y)\left(\frac{\partial\, p_w^{(0)}}{\partial x_j}(x,t)-\varrho_g\left(p_g^{(0)}\right)g_j\right). \tag{3.35}$$

Then the equation (3.34) reduces to the following equation:

$$\boxed{\begin{cases} -\mathrm{div}_y\left(K(y)\lambda_g\left(y;S^{(0)}\right)(\nabla_y\xi_{j,g}+\vec{e}_j)\right) = 0 \quad \text{in } Y; \\ y\mapsto\xi_{j,g}(y) \quad Y-\text{periodic}, \end{cases}} \tag{3.36}$$

where

$$\lambda_g\left(y;S^{(0)}\right) \overset{\mathrm{def}}{=} \lambda_{1,g}\left(S_1^{(0)}\right)\mathbf{1}_1(y)+\lambda_{2,g}\left(S_2^{(0)}\right)\mathbf{1}_2(y), \tag{3.37}$$

and the values $S_\ell^{(0)}$ are defined by (3.6).

Proceeding now as in the previous section we obtain the following equation:

$$\int_Y \Phi(y)\frac{\partial\Theta^{(0)}}{\partial t}\,dy-\mathrm{div}_x\,\mathfrak{J}_g = 0, \tag{3.38}$$

where the function $\Theta^{(0)}$ is defined in (3.30) and

$$\mathfrak{J}_g \stackrel{\text{def}}{=} \int\limits_Y K(y)\lambda_g\left(y; S^{(0)}\right)\varrho_g\left(p_g^{(0)}\right)\left(\nabla_x p_g^{(0)} - \varrho_g\left(p_g^{(0)}\right)\vec{g} + \nabla_y p_g^{(1)}\right)\,dy. \tag{3.39}$$

Consider the first term on the left hand side of (3.38). We have:

$$\int\limits_Y \Phi(y)\frac{\partial\Theta^{(0)}}{\partial t}\,dy = \frac{\partial}{\partial t}\int\limits_Y \Phi(y)\,\varrho_g\left(p_g^{(0)}\right)\left\{\left(1 - S_1^{(0)}\right)\mathbf{1}_1(y) + \left(1 - S_2^{(0)}\right)\mathbf{1}_2(y)\right\}\,dy =$$

$$= \frac{\partial}{\partial t}\left(\varrho_g\left(p_g^{(0)}\right)\left[\left(1 - S_1^{(0)}\right)\int\limits_{Y_1}\Phi(y)\,dy + \left(1 - S_2^{(0)}\right)\int\limits_{Y_2}\Phi(y)\,dy\right]\right) = \langle\Phi\rangle\frac{\partial\Theta^*}{\partial t},$$

where

$$\Theta^*(x,t) = \Theta^*\left(S^{(0)}\right) \stackrel{\text{def}}{=} \varrho_g\left(p_g^{(0)}\right)(1 - S^*). \tag{3.40}$$

Here the function S^* is defined in (3.22).

Now from (3.38)–(3.40) we get the second homogenized equation:

$$\boxed{\langle\Phi\rangle\frac{\partial\Theta^*}{\partial t} - \operatorname{div}_x\left(\left(\Lambda_g(S^{(0)})\varrho_g\left(p_g^{(0)}\right)\left(\nabla_x p_g^{(0)} - \varrho_g\left(p_g^{(0)}\right)\vec{g}\right)\right)\right) = 0,} \tag{3.41}$$

where the homogenized mobility tensor Λ_g of the gas phase is given by:

$$\Lambda_g(S^{(0)}) \stackrel{\text{def}}{=} \left\langle K(y)\lambda_g\left(S^{(0)}; y\right)\left[\mathbb{I} + \nabla\xi_g(y; S^{(0)})\right]\right\rangle. \tag{3.42}$$

Here $\nabla\xi_g(y; S^{(0)})$ is the tensor with the components $\partial\xi_{j,g}/\partial y_i$.

3.3   Homogenized system

Thus the homogenized system has the following form:

$$\begin{cases} \langle\Phi\rangle\dfrac{\partial S^*}{\partial t} - \operatorname{div}_x\left(\Lambda_w(S^{(0)})\left(\nabla_x p_w^{(0)} - \vec{g}\right)\right) = 0 \quad \text{in } \Omega_T; \\[2mm] \langle\Phi\rangle\dfrac{\partial\Theta^*}{\partial t} - \operatorname{div}_x\left(\left(\Lambda_g(S^{(0)})\varrho_g\left(p_g^{(0)}\right)\left(\nabla_x p_g^{(0)} - \varrho_g\left(p_g^{(0)}\right)\vec{g}\right)\right)\right) = 0 \quad \text{in } \Omega_T; \\[2mm] P_c\left(y, S^{(0)}\right) = p_g^{(0)} - p_w^{(0)} \quad \text{in } \Omega_T \times Y. \end{cases} \tag{3.43}$$

In equations (3.43) the local variable y appears, at least formally, in the capillary pressure law. In order to write the system (3.43) exclusively in macroscopic variables, we will eliminate $S^{(0)}$ and replace it by the macro-scale saturation S^*. To do this it is sufficient to construct *an effective capillary pressure function* P_c^* such that

$$P_c^*(S^*) = p_g^{(0)} - p_w^{(0)} \quad \text{in } \Omega_T. \tag{3.44}$$

Then, for any given S^* the capillary pressure is known, and therefore $S^{(0)}$ is also known. We can then express all homogenized tensors as functions of the macro-scale saturation S^*.

4 Effective macroscale model and local problems

In order to construct the effective capillary pressure curve, for simplicity, we will use the following simplifying assumption that

$$\min_{S \in [0,1]} P_{1,c}(S) = \min_{S \in [0,1]} P_{2,c}(S) \quad \text{and} \quad \max_{S \in [0,1]} P_{1,c}(S) = \max_{S \in [0,1]} P_{2,c}(S). \tag{4.45}$$

Let us note that (4.45) is not necessary and serves only to simplify the presentation. We will set $\min_{S \in [0,1]} P_{\ell,c}(S) = P_{\ell,c}(1) = 0$ and $\max_{S \in [0,1]} P_{\ell,c}(S) = P_{\ell,c}(0) = \alpha$. Then, for any $u \in [0, \alpha]$ we can find $S^{(0)}(y) = S_1^{(0)} \mathbf{1}_1(y) + S_2^{(0)} \mathbf{1}_2(y)$ by solving the equations $u = P_{1,c}(S_1^{(0)}) = P_{1,c}(S_2^{(0)})$.
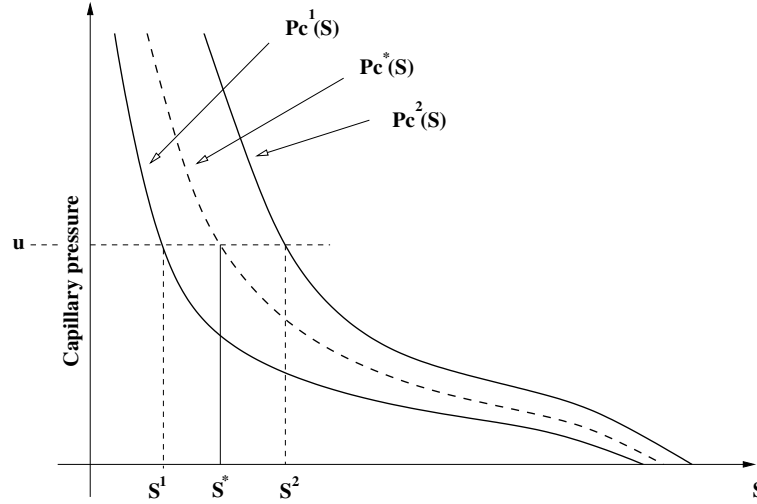
After the values $S_1^{(0)}$ and $S_2^{(0)}$ are obtained, we can compute S^* by the formula (3.22), that is

$$S^* = \frac{1}{\langle \Phi \rangle} \left[S_1^{(0)} \int_{Y_1} \Phi(y)\, dy + S_2^{(0)} \int_{Y_2} \Phi(y)\, dy \right]. \tag{4.46}$$

So we have defined a function $f \colon [0, \alpha] \to [0, 1]$, such that $S^* = f(u)$. It is easy to see that, due to strict monotonicity of both capillary pressure functions, the function $u \mapsto S^*$ is strictly decreasing, and thus have strictly decreasing inverse function: $u = f^{-1}(S^*)$, $f^{-1} \colon [0, 1] \to [0, \alpha]$. This function defines the effective capillary pressure function and it will be denoted P_c^*.

When the effective capillary pressure function is computed, we can calculate for any value of macro-scale saturation $S^* \in [0, 1]$ corresponding micro-scale repartition of the saturation $S_*^{(0)} = S_1^{(0)} \mathbf{1}_1(y) + S_2^{(0)} \mathbf{1}_2(y)$, where the values $S_1^{(0)}$ and $S_2^{(0)}$ are solutions to the equations $P_{1,c}(S_1^{(0)}) = P_{1,c}(S_2^{(0)}) = P_c^*(S^*)$. For the repartition $S_*^{(0)}$ we solve the cell problems:

$$\begin{cases} -\operatorname{div}_y \left(K(y)\lambda_w \left(y; S_*^{(0)} \right) \left(\nabla_y \xi_{j,w} + \vec{e}_j \right) \right) = 0 & \text{in } Y; \\ y \mapsto \xi_{j,w}(y) \quad Y - \text{periodic} \end{cases} \tag{4.47}$$

$$\begin{cases} -\operatorname{div}_y \left(K(y)\lambda_g \left(y; S_*^{(0)} \right) \left(\nabla_y \xi_{j,g} + \vec{e}_j \right) \right) = 0 & \text{in } Y; \\ y \mapsto \xi_{j,g}(y) \quad Y - \text{periodic}, \end{cases} \tag{4.48}$$

and calculate effective phase mobility tensors

$$\Lambda_w(S^*) \stackrel{\text{def}}{=} \left\langle K(y)\lambda_w\left(y; S_*^{(0)}\right)\left[\mathbb{I} + \nabla_y\xi_w\left(y; S_*^{(0)}\right)\right]\right\rangle, \tag{4.49}$$

$$\Lambda_g(S^*) \stackrel{\text{def}}{=} \left\langle K(y)\lambda_g\left(S_*^{(0)}; y\right)\left[\mathbb{I} + \nabla\xi_g(y; S_*^{(0)})\right]\right\rangle. \tag{4.50}$$

These tensors depend only on the macroscopic saturation S^*. Finally we have the macroscopic conservation laws in the form:

$$\begin{cases} \langle\Phi\rangle\dfrac{\partial S^*}{\partial t} - \text{div}_x\left(\Lambda_w(S^*)\left(\nabla_x p_w - \vec{g}\right)\right) = 0 \quad \text{in } \Omega_T; \\[3mm] \langle\Phi\rangle\dfrac{\partial}{\partial t}\left(\varrho_g\left(p_g\right)\left(1 - S^*\right)\right) - \text{div}_x\left(\left(\Lambda_g(S^*)\varrho_g\left(p_g\right)\left(\nabla_x p_g - \varrho_g\left(p_g\right)\vec{g}\right)\right) = 0 \quad \text{in } \Omega_T, \end{cases} \tag{4.51}$$

where we have dropped the index (0) from the macroscopic phase pressures p_w and p_g. To the equations (4.51) we have to add the macroscopic capillary pressure law:

$$p_g - p_w = P_c^*(S^*). \tag{4.52}$$

The structure of the macroscopic two-phase flow equations is the same as the structure of micro-scale equations. The only difference is in the effective phase mobilities $\Lambda_w(S^*)$ and $\Lambda_g(S^*)$, which are now generally full symmetric tensors, calculated by the resolution of the local cell problems, and are not naturally factorized into absolute (intrinsic) permeability and relative (phase) mobilities.

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